Section 6.2: Curl, Div and Flux

1) (Curl and Div) Find a nonzero vector field \( \mathbf{F} = \langle P(x, y), Q(x, y) \rangle \) in each of the following cases:
   a) \( \mathbf{F} \) is irrotational but not incompressible.
   b) \( \mathbf{F} \) is incompressible but not irrotational.
   c) \( \mathbf{F} \) is irrotational and incompressible.
   d) \( \mathbf{F} \) is not irrotational and not incompressible.

2) (Curl) The vector field \( \mathbf{F} = \langle x, y, -z \rangle \) satisfies \( \text{div}(\mathbf{F}) = 0 \). Can you find a vector field \( \mathbf{G} \) such that \( \text{curl}(\mathbf{G}) = \mathbf{F} \)? Such a field \( \mathbf{G} \) is called a vector potential.
   Hint. Write \( \mathbf{F} \) as a sum \( \langle x, y, 0 \rangle + \langle 0, -z, 0 \rangle \) and find vector potentials for each of the summand using a vector field you have seen in class.

3) (Flux integral) Evaluate the flux integral \( \int_{S} \mathbf{F} \cdot d\mathbf{S} \) for \( \mathbf{F}(x, y, z) = \langle xy, yz, zx \rangle \) that lies above the square \([0, 1] \times [0, 1]\) and has an upward orientation.

4) Let \( G \) be the region \( x^2 + y^2 \leq 1 \). Compute the line integral of the vector field \( \mathbf{F}(x, y) = \langle x^2 + y^2, x^2 - y^2 \rangle \) along the boundary.

5) Let \( \mathbf{F}(x, y) = \langle -y/(x^2 + y^2), x/(x^2 + y^2) \rangle \). Let \( C : \mathbf{r}(t) = \langle \cos^3(t), \sin^3(t) \rangle, t \in [0, 2\pi] \).
   a) Compute \( \int_{C} \mathbf{F} \cdot d\mathbf{r} \).
   b) Show that \( \text{curl}(\mathbf{F}) = 0 \) everywhere for \( (x, y) \neq (0, 0) \).
   c) Let \( f(x, y) = \arctan(y/x) \). Verify that \( \nabla f = \mathbf{F} \).
   d) Why do a) and b) not contradict the fact that a gradient field has the closed loop property? Why does a) and b) not contradict Green’s theorem?
Section 6.3: Stokes theorem

1) (Stokes theorem) Find \( \int_C \vec{F} \cdot d\vec{r} \), where \( \vec{F}(x, y, z) = (x^2y, x^3/3, xy) \) and \( C \) is the curve of intersection of the hyperbolic paraboloid \( z = y^2 - x^2 \) and the cylinder \( x^2 + y^2 = 1 \), oriented counterclockwise as viewed from above.

2) (Stokes theorem) If \( S \) is the surface \( x^2 + y^2 + z^2 = 1 \) and assume \( \vec{F} \) is a smooth vector field. Explain why \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0 \).

3) (Stokes theorem) Evaluate the flux integral

\[
\int_S \text{curl}(\vec{F}) \cdot d\vec{S},
\]

where \( \vec{F}(x, y, z) = (x e^y z^3 + 2xy e^z z^3, x + z^2 e^{x^2 z^2}, ye^{x^2 z^2} + xe^z) \) and where \( S \) is the part of the ellipsoid \( x^2 + y^2 + z^2 = 1 \) and \( z > 0 \) oriented so that the normal vector points upwards.

4) (Stokes theorem) Find the line integral \( \int_C \vec{F} \cdot d\vec{r} \), where \( C \) is the circle of radius 3 in the \( xz \)-plane oriented counter clockwise when looking from the point \((0, 0, 1)\) onto the plane and where \( \vec{F} \) is the vector field

\[
\vec{F}(x, y, z) = (2x^2 z + x^5, \cos(e^y), -2x z^2 + \sin(\sin(z)))
\]

Hint. Use a convenient surface \( S \) which has \( C \) as a boundary.

5) (Stokes theorem) Find the flux integral \( \iint_S \text{curl}(\vec{F}) \cdot d\vec{S} \), where

\[
\vec{F}(x, y, z) = (2 \cos(\pi y) e^{2x} + z^2, x^2 \cos(z/2) - \pi \sin(\pi y) e^{2x}, 2xz)
\]

and \( S \) is the thorn surface parametrized by

\[
\vec{r}(s, t) = ((1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s)
\]

with \( 0 \leq t \leq 2\pi, 0 \leq s \leq 1 \) and oriented so that the normal vectors point to the outside of the thorn.

Section 6.4:

1) (Divergence theorem) Compute using the divergence theorem the flux of the vector field \( \vec{F}(x, y, z) = (3y, xy, 2yz) \) through the unit cube \([0, 1] \times [0, 1] \times [0, 1] \).

2) (Divergence theorem) Find the flux of the vector field \( \vec{F}(x, y, z) = (xy, yz, zx) \) through the solid cylinder \( x^2 + y^2 \leq 1, 0 \leq z \leq 1 \).

3) (Divergence theorem) Use the divergence theorem to calculate the flux of \( \vec{F}(x, y, z) = (x^3, y^3, z^3) \) through the sphere \( S: x^2 + y^2 + z^2 = 1 \) where the sphere is oriented so that the normal vector points outwards.

4) (Divergence theorem) Assume the vector field

\[
\vec{F}(x, y, z) = (5x^3 + 12xy^2, y^3 + e^z \sin(z), 5z^3 + e^z \cos(z))
\]

is the magnetic field of the sun whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux \( \iint_S \vec{F} \cdot d\vec{S} \).

5) (Divergence theorem) Find \( \int_S \vec{F} \cdot d\vec{S} \), where \( \vec{F}(x, y, z) = (x, y, z) \) and \( S \) is the outwardly oriented surface obtained by removing the cube \([1, 2] \times [1, 2] \times [1, 2] \) from the cube \([0, 2] \times [0, 2] \times [0, 2] \).