Homework for Chapter 4. Extrema and Double integrals

Section 4.1: Extrema

1) (extrema) Find all the extrema of the function
\[ f(x, y) = 2x^3 + 4y^2 - 2y^4 - 6x \]
and determine whether they are maxima, minima or saddle points.

2) (extrema) Where on the parametrized surface
\[ \mathbf{r}(u, v) = \langle u^2, v^3, uv \rangle \]
is the temperature \( T(x, y, z) = 12x + y - 12z \) minimal? To find the minimum, look where the function \( f(u, v) = T(\mathbf{r}(u, v)) \) has an extremum. Find all local maxima, local minima or saddle points of \( f \).

Remark. After you have found the function \( f(u, v) \), you could replace the variables \( u, v \) again with \( x, y \) if you like and look at a function \( f(x, y) \).

3) (extrema) Find and classify all the extrema of the function
\[ f(x, y) = e^{-x^2 - y^2}(x^2 + 2y^2). \]

4) (global extrema) Find all extrema of the function
\[ f(x, y) = x^3 + y^3 - 3x - 12y + 20 \]
on the plane and characterize them. Do you find a global maximum or global minimum among them?

5) (extrema) The thickness of the region enclosed by the two graphs \( f_1(x, y) = 10 - 2x^2 - 2y^2 \) and \( f_2(x, y) = -x^4 - y^4 - 2 \) is denoted by \( f(x, y) = f_1(x, y) - f_2(x, y) \). Classify all critical points of \( f \) and find the global minimal thickness.

Section 4.2: Lagrange

1) (Lagrange) Find the cylindrical basket which is open on the top has the largest volume for fixed area \( \pi \). If \( x \) is the radius and \( y \) is the height, we have to extremize \( f(x, y) = \pi x^2 y \) under the constraint \( g(x, y) = 2\pi xy + \pi x^2 = \pi \). Use the method of Lagrange multipliers.

2) (global extrema with Lagrange) Find the extrema of the same function
\[ f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2) \]
as in problem 4.1.3 but now on the entire disc \( \{x^2 + y^2 \leq 4\} \) of radius 2. Besides the already found extrema inside the disk, you have to find extrema on the boundary.

3) Find and classify all the critical points of the function
\[ f(x, y) = 5 + 3x^2 + 3y^2 + y^3 + x^3. \]

Is there a global maximum or a global minimum for \( f(x, y) \)?

4) A solid bullet made of a half sphere and a cylinder has the volume \( V = \frac{2\pi r^3}{3} + \pi r^2 h \) and surface area \( A = 2\pi r^2 + 2\pi rh + \pi r^2 \). Doctor Manhatten designs a bullet with fixed volume and minimal area. With \( g = 3V/\pi = 1 \) and \( f = A/\pi \) he therefore minimizes
\[ f(h, r) = 3r^2 + 2rh \]
under the constraint
\[ g(h, r) = 2r^3 + 3r^2 h = 1. \]
Use the Lagrange method to find a local minimum of \( f \) under the constraint \( g = 1. \)

5) Minimize the material cost of an office tray
\[ f(x, y) = xy + x + 2y \]
of length \( x \), width \( y \) and height 1 under the constraint that the volume \( g(x, y) = xy \) is constant and equal to 4.
### Section 4.3: Double integrals

1) (double integral) Calculate the iterated integral \( \int_1^4 \int_0^2 (2x - \sqrt{y}) \, dx \, dy \).

2) (double integral) Find the area of the region \( R = \{(x, y) \mid 0 \leq x \leq 2\pi, \sin(x) - 1 \leq y \leq \cos(x) + 2\} \) and use it to compute the average value \( \int \int_R f(x, y) \, dxdy/\text{area}(R) \) of \( f(x, y) = y \) over that region.

3) (volume) Find the volume of the solid lying under the paraboloid \( z = x^2 + y^2 \) and above the rectangle \( R = [-2, 2] \times [-3, 3] = \{(x, y) \mid -2 \leq x \leq 2, -3 \leq y \leq 3\} \).

4) (switching order of integration) Calculate the iterated integral \( \int_0^1 \int_{x^2}^{x} (x^2 - y) \, dy \, dx \). Sketch the corresponding type I region. Write this integral as integral over a type II region and compute the integral again.

5) (double integral) Evaluate the double integral \( \int_0^2 \int_0^{x^2} \frac{x}{y^2} \, dy \, dx \).

### Section 4.4: Polar integration

1) (polar integrals) Integrate \( f(x, y) = x^2 \) over the unit disc \( \{x^2 + y^2 \leq 1 \} \) in two ways, first using Cartesian coordinates, then using polar coordinates.

2) (polar integrals) Find \( \int \int_R (x^2 + y^2)^{10} \, dA \), where \( R \) is the part of the unit disc \( \{x^2 + y^2 \leq 1 \} \) for which \( y > x \).

3) (polar integrals) What is the area of the region which is bounded by the following three curves, first by the polar curve \( r(\theta) = \theta \) with \( \theta \in [0, 2\pi] \), second by the polar curve \( r(\theta) = 2\theta \) with \( \theta \in [0, 2\pi] \) and third by the positive x-axis.

4) (polar integrals) Find the average value of \( f(x, y) = x^2 + y^2 \) on the annular region \( R : 1 \leq |(x, y)| \leq 2 \). The average is \( \int \int_R f \, dxdy/\int \int_R 1 \, dxdy \).

5) (surface area) Find the surface area of the part of the paraboloid \( x = y^2 + z^2 \) which is inside the cylinder \( y^2 + z^2 \leq 9 \).