REMINDER: INTEGRATION POLAR COORDINATES.

\[ \int \int_R f(r, \theta) \, r \, dr \, d\theta . \]

EXAMPLE 1. Area of a disk of radius \( R \)

\[ \int_0^R \int_0^{2\pi} r \, d\theta \, dr = 2\pi R^2 \left( \frac{R^2}{2} \right) = R^2 \pi . \]

WHERE DOES THE FACTOR "\( r \)" COME FROM?

1. EXPLANATION. A small rectangle with dimensions \( dr \, d\theta \) in the \( (r, \theta) \) plane is mapped to a sector segment in the \( (x, y) \) plane. It has approximately the area \( r \, dr \, d\theta \). It is small for small \( r \).

2. EXPLANATION. The map \( (r, \theta) \mapsto (r \cos(\theta), r \sin(\theta)) \) which changes from Cartesian coordinates to polar coordinates. The Jacobian is defined as the matrix

\[
\begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
r \sin(\theta) & r \cos(\theta)
\end{pmatrix}
\]

It contains the gradient vectors of \( f(r, \theta) \) and \( g(r, \theta) \) as rows. The determinant \( f_r g_\theta - f_\theta g_r = r \) is the crossproduct of these two gradient vectors and gives the area of the parallelepiped.

CYLINDRICAL COORDINATES. Use polar coordinates in the \( x-y \) plane and leave the \( z \) coordinate. Take \( T(r, \theta, z) = (r \cos(\theta), r \sin(\theta), z) \).

The integration factor \( r \) is the same as in polar coordinates.

\[
\int \int \int_{T(R)} f(x, y, z) \, dx \, dy \, dz = \int \int \int_R f(r, \theta, z) \, r \, dr \, d\theta \, dz
\]

COORDINATES OF CAMBRIDGE. On the website \( \text{http://cello.cs.uiuc.edu/cgi-bin/slamm/ip2ll/} \) you can enter a host like \( \text{www.math.harvard.edu} \) and get latitude and longitude of the host: \( (\text{lat, lon}) = (42.365, -71.1) \). Using \( (r, \theta, \phi) \) coordinates, we obtain the position \( (r, 90 - 42.365, -71.1) \) of the host in spherical coordinates. The site does not give the height, but we are about on see-level, so that \( r = 6365 \text{km} \).

EXAMPLE. Calculate the volume bounded by the parabolic \( z = 1 - (x^2 + y^2) \) and the \( x-y \) plane. In cylindrical coordinates, the paraboloid is \( z(r, \phi) = 1 - r^2 \):

\[
\int_0^1 \int_0^{2\pi} \int_0^{1-r^2} r^2 \, dz \, d\phi \, dr = \int_0^1 \int_0^{2\pi} (r - r^3) \, d\phi \, dr = 2\pi \left( \frac{r^2}{2} - \frac{r^4}{4} \right) \bigg|_0^1 = \pi .
\]
SPHERICAL COORDINATES. Spherical coordinates use the radius $\rho$ as well as two angles: $\theta$ the polar angle and $\phi$, the angle between the vector and the $z$ axis. The coordinate change is

$$T : (x, y, z) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) .$$

The integration factor can be seen from the dimensions of a spherical wedge with dimensions $d\rho, \rho \sin(\phi) \, d\theta, \rho d\phi = \rho^2 \sin(\phi) \, d\theta d\phi d\rho$.

$$\int \int \int_{V(R)} f(x, y, z) \, dx \, dy \, dz = \int \int \int_{V} f(\rho, \theta, z) \, \rho^2 \sin(\phi) \, d\rho d\theta d\phi$$

VOLUME OF SPHERE. A sphere of radius $R$ has the volume

$$\int_{0}^{R} \int_{0}^{2\pi} \int_{0}^{\pi} \rho^2 \sin(\phi) \, d\phi d\theta d\rho .$$

The most inner integral $\int_{0}^{\pi} \rho^2 \sin(\phi) d\phi = -\rho^2 \cos(\phi)|_{0}^{\pi} = 2\rho^2$. The next layer is, because $\phi$ does not appear: $\int_{0}^{2\pi} 2\rho^2 \, d\phi = 4\pi\rho^2$. The final integral is $\int_{0}^{R} 4\pi\rho^2 \, d\rho = 4\pi R^3/3$.

MOMENT OF INERTIA. The moment of inertia of a body $G$ with respect to an axis $L$ is the triple integral $\int \int \int_{V} r(x, y, z)^2 \, dx \, dy \, dz$, where $r(x, y, z) = R \sin(\phi)$ is the distance from $L$. Problem: calculate the moment of inertia of a sphere of radius $R$ with respect to the $z$-axis:

$$I = \int_{0}^{R} \int_{0}^{2\pi} \int_{0}^{\pi} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \, d\phi d\theta d\rho = \left(\frac{1}{3} \sin^3(\phi) \right)(\int_{0}^{R} \rho^4 \, dr)(\int_{0}^{2\pi} d\theta) = \frac{4}{3} \cdot \frac{R^5}{5} \cdot 2\pi = \frac{8\pi R^5}{15} = \frac{VR^2}{5} .$$

If a sphere spins around the $z$-axis with angular velocity $\omega$, then $I\omega^2/2$ is the kinetic energy of that sphere. Example: the moment of inertia of the earth is $810^{37} km^2$. The angular velocity is $\omega = 1/day = 1/(86400s)$ so that the energy of the earth rotation is $810^{37} km^2/(74649600000 s^2) \sim 10^{28} J = 10^{25} kJ \sim 2.510^{24} kcal$.

DIAMOND. Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.

Solution: we use spherical coordinates to find the center of mass $(\bar{x}, \bar{y}, \bar{z})$:

$$V = \int_{0}^{1} \int_{0}^{2\pi} \int_{0}^{\pi/6} \rho^2 \sin(\phi) \, d\phi d\theta d\rho = \left(\frac{1}{3} - \frac{\sqrt{3}}{3}\right) 2\pi$$

$$\\bar{x} = \int_{0}^{1} \int_{0}^{2\pi} \int_{0}^{\pi/6} \rho^3 \sin^2(\phi) \cos(\theta) \, d\phi d\theta d\rho / V = 0$$

$$\\bar{y} = \int_{0}^{1} \int_{0}^{2\pi} \int_{0}^{\pi/6} \rho^3 \sin^2(\phi) \sin(\theta) \, d\phi d\theta d\rho / V = 0$$

$$\\bar{z} = \int_{0}^{1} \int_{0}^{2\pi} \int_{0}^{\pi/6} \rho^3 \cos(\phi) \sin(\phi) \, d\phi d\theta d\rho / V = \frac{2\pi}{32V}$$