LINES. A point \( P \) and a vector \( \vec{v} \) define a line \( L \). It is the set of points
\[
L = \{ P + t\vec{v}, \text{where } t \text{ is a real number} \}
\]
The line contains the point \( P \) and points into the direction \( \vec{v} \).

EXAMPLE. \( L = \{ (x, y, z) = (1, 1, 2) + t(2, 4, 6) \} \).

This description is called the parameter equation for the line.

EQUATIONS OF A LINE. We can write \((x, y, z) = (1, 1, 2) + t(2, 4, 6)\) so that \(x = 1 + 2t, y = 1 + 4t, z = 2 + 6t\).
If we solve the first equation for \( t \) and plug it into the other equations, we get \(y = 1 + (2x - 2), z = 2 + 3(2x - 2)\).
We can therefore describe the line also as
\[
L = \{ (x, y, z) \mid y = 2x - 1, z = 6x - 4 \}.
\]

SYMMETRIC EQUATION. The line \( \vec{r} = P + t\vec{v} \) with \( P = (x_0, y_0, z_0) \) and \( \vec{v} = (a, b, c) \) satisfies the symmetric equations
\[
\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}\]
Indeed, every expression is equal to \( t \) because \( \vec{r} = (x, y, z) = (x_0, y_0, z_0) + t(a, b, c) \).

EXAMPLE. The symmetric equations of the above line are \((x - 1)/2 = (y - 1)/4 = (z - 2)/6\).

PROBLEM. Find the parametric and symmetric equations for the line through the points \( P = (0, 1, 1) \) and \( Q = (2, 3, 4) \).

SOLUTION. The parametric equations are \((x, y, z) = (0, 1, 1) + t(2, 2, 3)\) or \(x = 2t, y = 1 + 2t, z = 1 + 3t\).
Solving each equation for \( t \) gives the symmetric equations \( x/2 = (y - 1)/4 = (z - 1)/3 \).

PLANES. A point \( Q \) and two vectors \( \vec{v}, \vec{w} \) define a plane \( \Sigma \). It is the set of points
\[
\Sigma = \{ Q + t\vec{v} + s\vec{w}, \text{where } t, s \text{ are real numbers} \}
\]
The line contains the point \( P, P - Q, R - Q \).

EXAMPLE. \( \Sigma = \{ (x, y, z) \mid (1, 1, 2) + t(2, 4, 6) + s(1, 0, -1) \} \).

This is called the parameter description of a plane.

EQUATION OF PLANE. Given a plane as a parametric equation \( P = Q + t\vec{v} + s\vec{w} \). The vector \( \vec{n} = \vec{v} \times \vec{w} \) is perpendicular to both \( \vec{n} \) and \( \vec{w} \). Because also the vector \( PQ = Q-P \) is perpendicular to \( \vec{n} \) we have \((Q-P) \cdot \vec{n} = 0 \).
With \( Q = (x_0, y_0, z_0) \), \( P = (x, y, z) \), and \( \vec{n} = (a, b, c) \), this means \( ax + by + cz = ax_0 + by_0 + cz_0 = d \). The plane is therefore described by a single equation \( ax + by + cz = d \).

PROBLEM. Find the equation of a plane which contains the three points \( P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3) \).

SOLUTION. The plane contains the two vectors \( \vec{v} = (1, 2, 0) \) and \( \vec{w} = (2, 2, 2) \). We have \( \vec{n} = (4, -2, -2) \) and
equation is \( 4x - 2y - 2z = d \). The constant \( d \) is obtained by plugging in one point: \( 4(-1) - 2(-1) - 2(1) = -4 \).

LINES AND PLANES IN MATHEMATICA.
Plotting a line: \( \text{ParametricPlot3D}[(1,1,1)+t\{3,4,5\},\{t,-2,2\}] \)
Plotting a plane: \( \text{ParametricPlot3D}[(1,1,1)+t\{3,4,5\}+s\{1,2,3\},\{t,-2,2\},\{s,-2,2\}] \)
Equation of a plane: \( P = \{4,5,1\}; Q = \{0,1,1\}; R = \{1,1,3\}; n = \text{Cross}[Q-P,R-P]; n.\{x,y,z\} = n.P \)
DISTANCE POINT-PLANE (3D). If \( P \) is a point in space and \( ax + by + cz = \vec{n} \cdot \vec{x} = d \) is a plane containing a point \( Q \), then
\[
d(P, L) = |(P - Q) \cdot \vec{n}|/|\vec{n}|
\]
is the distance between \( P \) and the plane.

You recognize that this is just the scalar projection of the vector \( \vec{QP} = P - Q \) onto the vector \( \vec{n} \).

DISTANCE POINT-LINE (3D). If \( P \) is a point in space and \( L \) is the line \( \vec{r}(t) = Q + t\vec{u} \), then
\[
d(P, L) = |(P - Q) \times \vec{u}|/|\vec{u}|
\]
is the distance between \( P \) and the line \( L \).

This formula is verified by writing \( (P - Q) \times \vec{u} = |P - Q||\vec{u}|\sin(\alpha) \), where \( \alpha \) is the angle between \( P - Q \) and \( \vec{u} \).

DISTANCE LINE-LINE (3D). \( L \) is the line \( \vec{r}(t) = Q + t\vec{u} \) and \( M \) is the line \( \vec{s}(t) = P + t\vec{v} \), then
\[
d(L, M) = |(P - Q) \cdot (\vec{u} \times \vec{v})|/|\vec{u} \times \vec{v}|
\]
is the distance between the two lines \( L \) and \( M \).

This formula is just the scalar projection of \( \vec{QP} = P - Q \) onto the vector \( \vec{n} = \vec{u} \times \vec{v} \) normal to both \( \vec{u} \) and \( \vec{v} \).

PLANE THROUGH 3 POINTS \( P, Q, R \): The vector \((a, b, c) = \vec{n} = (Q - P) \times (R - P)\) is normal to the plane. Therefore, the equation is \( ax + by + cz = d \), where \( d \) is a constant. The constant is \( d = ax_0 + by_0 + cz_0 \) because the point \( P = (x_0, y_0, z_0) \) is on the plane.

PLANE THROUGH POINT \( P \) AND LINE \( \vec{r}(t) = Q + t\vec{u} \). The vector \((a, b, c) = \vec{n} = \vec{u} \times (Q - P)\) is normal to the plane. Therefore the plane is given by \( ax + by + cz = d \), where \( d = ax_0 + by_0 + cz_0 \) and \( P = (x_0, y_0, z_0) \).

LINE ORTHOGONAL TO PLANE \( ax + by + cz = d \) THROUGH POINT \( P \). The vector \( \vec{n} = (a, b, c) \) is normal to the plane. The line is \( \vec{r}(t) = P + t\vec{n} \).

ANGLE BETWEEN PLANES. The angle between the two planes \( a_1x + b_1y + c_1z = d_1 \) and \( a_2x + b_2y + c_2z = d_2 \) is \( \arccos(\vec{n}_1 \cdot \vec{n}_2/|\vec{n}_1||\vec{n}_2|) \), where \( \vec{n}_i = (a_i, b_i, c_i) \). Alternatively, it is \( \arcsin(|\vec{n}_1 \times \vec{n}_2|/|\vec{n}_1||\vec{n}_2|) \).

INTERSECTION BETWEEN TWO PLANES. Find the line which is the intersection of two non-parallel planes \( a_1x + b_1y + c_1z = d_1 \) and \( a_2x + b_2y + c_2z = d_2 \). Find first a point \( P \) which is in the intersection. Then \( \vec{r}(t) = P + t(\vec{n}_1 \times \vec{n}_2) \) is the line, we were looking for.

LINES IN 2D.

A general point on a line is \( P = Q + t\vec{u} \). Eliminating \( t \) gives a single equation. For example, \( (x, y) = (1, 2) + t(3, 4) \) is equivalent to \( x = 1 + 3t, y = 2 + 4t \) and so \( 4x - 3y = -2 \). The general equation for a line in the plane is \( ax + by = d \).