CROSS PRODUCT. The cross product of two vectors \( \vec{v} = (v_1, v_2, v_3) \) and \( \vec{w} = (w_1, w_2, w_3) \) is defined as the vector \( \vec{v} \times \vec{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \).

To compute it: multiply diagonally at the crosses.

<table>
<thead>
<tr>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( v_3 )</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>( X )</td>
<td>( X )</td>
<td>( X )</td>
<td>( w_2 )</td>
</tr>
</tbody>
</table>

DIRECTION OF \( \vec{v} \times \vec{w} \): \( \vec{v} \times \vec{w} \) is orthogonal to \( \vec{v} \) and orthogonal to \( \vec{w} \).

Proof. Check that \( \vec{v} \cdot (\vec{v} \times \vec{w}) = 0 \).

LENGTH: \( |\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin(\alpha) \)

Proof. The identity \( |\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2 \) can be proven by direct computation. Now, \( |\vec{v} \cdot \vec{w}| = |\vec{v}| |\vec{w}| \cos(\alpha) \).

AREA. The length \( |\vec{v} \times \vec{w}| \) is the area of the parallelogram spanned by \( \vec{v} \) and \( \vec{w} \).

Proof. Because \( |\vec{w}| \sin(\alpha) \) is the height of the parallelogram with base length \( |\vec{v}| \), the area is \( |\vec{v}| |\vec{w}| \sin(\alpha) \) which is by the above formula equal to \( |\vec{v} \times \vec{w}| \).

EXAMPLE. If \( \vec{v} = (a, 0, 0) \) and \( \vec{w} = (b \cos(\alpha), b \sin(\alpha), 0) \), then \( \vec{v} \times \vec{w} = (0, 0, ab \sin(\alpha)) \) which has length \( |ab \sin(\alpha)| \).

ZERO CROSS PRODUCT. We see that \( \vec{v} \times \vec{w} \) is zero if \( \vec{v} \) and \( \vec{w} \) are parallel.

ORIENTATION. The vectors \( \vec{v}, \vec{w} \) and \( \vec{v} \times \vec{w} \) form a right handed coordinate system. The right hand rule is: put the first vector \( \vec{v} \) on the thumb, the second vector \( \vec{w} \) on the pointing finger and the third vector \( \vec{v} \times \vec{w} \) on the third middle finger.

EXAMPLE. \( \vec{i}, \vec{j}, \vec{k} \times \vec{j} = \vec{k} \) forms a right handed coordinate system.

DOT PRODUCT (is a scalar) | CROSS PRODUCT (is a vector)
---|---
\( \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \) | commutative | \( \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \) | anti-commutative
\( |\vec{v} \cdot \vec{w}| = |\vec{v}| |\vec{w}| \cos(\alpha) \) | angle | \( |\vec{v} \times \vec{w}| = |\vec{v}| |\vec{w}| \sin(\alpha) \) | angle
\((\alpha \vec{v}) \cdot \vec{w} = \alpha (\vec{v} \cdot \vec{w}) \) | linearity | \((\alpha \vec{v}) \times \vec{w} = \alpha (\vec{v} \times \vec{w}) \) | linearity
\((\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \) | distributivity | \((\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w} \) | distributivity
\{1,2,3\},\{3,4,5\} | in Mathematica | \text{Cross}[\{1,2,3\},\{3,4,5\}] | in Mathematica
\[ \frac{\partial}{\partial t} (\vec{v} \cdot \vec{w}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{w} \] | product rule | \[ \frac{\partial}{\partial t} (\vec{v} \times \vec{w}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{w} \] | product rule

TRIPLE SCALAR PRODUCT. The scalar \( [\vec{u}, \vec{v}, \vec{w}] \) is the volume of the parallelepiped spanned by \( \vec{u}, \vec{v}, \vec{w} \) because \( h = \vec{u} \cdot \vec{n} / |\vec{n}| \) is the height of the parallelepiped if \( \vec{n} = (\vec{u} \times \vec{w}) \) is a normal vector to the ground parallelogram which has area \( A = |\vec{n}| = |\vec{v} \times \vec{w}| \). The volume of the parallelepiped is \( hA = \vec{u} \cdot \vec{n} |\vec{n}| / |\vec{n}| = |\vec{u} \cdot (\vec{v} \times \vec{w})| \).

PARALLELEPIPED. \([\vec{u}, \vec{v}, \vec{w}]\) is the volume of the parallelepiped spanned by \( \vec{u}, \vec{v}, \vec{w} \) because \( h = \vec{u} \cdot \vec{n} / |\vec{n}| \) is the height of the parallelepiped if \( \vec{n} = (\vec{u} \times \vec{w}) \) is a normal vector to the ground parallelogram which has area \( A = |\vec{n}| = |\vec{v} \times \vec{w}| \). The volume of the parallelepiped is \( hA = \vec{u} \cdot \vec{n} |\vec{n}| / |\vec{n}| = |\vec{u} \cdot (\vec{v} \times \vec{w})| \).

EXAMPLE. Find the volume of the parallelepiped which has the one corner \( O = (1,1,0) \) and three corners \( P = (2,3,1), Q = (4,3,1), R = (1,4,1) \) connected to it.

ANSWER: The parallelepiped is spanned by \( \vec{u} = (1,2,1), \vec{v} = (3,2,1), \) and \( \vec{w} = (0,3,2) \). We get \( \vec{v} \times \vec{w} = (1, -6, 9) \) and \( \vec{u} \cdot (\vec{v} \times \vec{w}) = -2 \). The volume is 2.
The cross product appears in many different applications:

**DISTANCE POINT-LINE (3D).** If $P$ is a point in space and $L$ is the line which contains the vector $\vec{u}$, then

$$d(P, L) = |\vec{PQ} \times \vec{u}|/|\vec{u}|$$

is the distance between $P$ and the line $L$.

**PLANE THROUGH 3 POINTS $P, Q, R$:**
The vector $\vec{n} = \vec{PQ} \times \vec{PR}$ is orthogonal to the plane. We will next week that $\vec{n} = (a, b, c)$ defines the plane $ax + by + cz = d$, with $d = ax_0 + by_0 + cz_0$ which passes through the points $P = (x_0, y_0, z_0), Q, R$.

The cross product appears in many different applications:

**ANGULAR MOMENTUM.** If a mass point of mass $m$ moves along a curve $\vec{r}(t)$, then the vector $\vec{L}(t) = m\vec{r}(t) \times \vec{r}'(t)$ is called the **angular momentum** of the point. It is coordinate system dependent.

**ANGULAR MOMENTUM CONSERVATION.**

$$\frac{d}{dt}\vec{L}(t) = m\vec{r}'(t) \times \vec{r}'(t) + m\vec{r}(t) \times \vec{r}''(t) = \vec{r}(t) \times \vec{F}(t)$$

In a central field, where $\vec{F}(t)$ is parallel to $\vec{r}(t)$, we get $d/dt L(t) = 0$ which means $L(t)$ is constant.

**TORQUE.** In physics, the quantity $\vec{r}(t) \times \vec{F}(t)$ is also called the **torque**. The time derivative of the **momentum** $m\vec{r}$ is the **force**, the time derivative of the **angular momentum** $\vec{L}$ is the **torque**.

**KEPLER’S AREA LAW.** (Proof by Newton)
The fact that $\vec{L}(t)$ is constant means first of all that $\vec{r}(t)$ stays in a plane spanned by $\vec{r}(0)$ and $\vec{r}'(0)$. The experimental fact that the vector $\vec{r}(t)$ sweeps over **equal areas in equal times** expresses angular momentum conservation: $|\vec{r}(t) \times \vec{r}'(t)dt/2| = |\vec{L}dt/m/2|$ is the area of a small triangle. The vector $\vec{r}(t)$ sweeps over an area $\int_{0}^{T} |\vec{L}|dt/(2m) = |\vec{L}|T/(2m)$ in time $[0, T]$.

**MORE PLACES IN PHYSICS WHERE THE CROSS PRODUCT OCCURS:**

The **top**, the motion of a rigid body is describe by the angular momentum $L$ and the angular velocity vector $\Omega$ in the body. Then $L = L \times \Omega + M$, where $M$ is an external **torque** obtained by external forces.

**Electromagnetism:** (informal) A particle moving along $\vec{r}(t)$ in a **magnetic field** $\vec{B}$ for example experiences the force $\vec{F}(t) = q\vec{r}'(t) \times \vec{B}$, where $q$ is the charge of the particle. In a constant magnetic field, the particles move on circles: if $m$ is the mass of the particle, then $mr''(t) = q\vec{r}'(t) \times \vec{B}$ implies $mr''(t) = q\vec{r}'(t) \times \vec{B}$. Now $d/dt |r|^2 = 2\vec{r} \cdot \vec{r}' = \vec{r} \cdot q\vec{r}'(t) \times \vec{B} = 0$ so that $|\vec{r}|$ is constant.

**Hurricanes** are powerful storms with wind velocities of 74 miles per hour or more. On the northern hemisphere, hurricanes turn counterclockwise, on the southern hemisphere clockwise. This is a feature of all low pressure systems and can be explained by the Coriolis force. In a rotating coordinate system a particle of mass $m$ moving along $\vec{r}(t)$ experience the following forces: $m\vec{r}' \times \vec{r}'$ (inertia of rotation), $2m\vec{\omega} \times \vec{r}'$ (Coriolis force) and $m\vec{\omega} \times (\vec{\omega} \times \vec{r}')$ (Centrifugal force). The Coriolis force is also responsible for the circulation in Jupiter’s Red Spot.