1D INTEGRATION IN 100 WORDS. If \( f(x) \) is a continuous function, then \( \int_a^b f(x) \, dx \) can be defined as a limit of the Riemann sum \( f_n(x) = \frac{1}{n} \sum_{x_k \in [a,b]} f(x_k) \) for \( n \to \infty \) with \( x_k = k/n \). The integral divided by \((b-a)\) is the average of \( f \) on \([a,b]\). The integral can be interpreted as a signed area under the graph of \( f \). If \( f(x) = 1 \), the integral is the length of the interval. The function \( F(x) = \int_a^x f(y) \, dy \) is called an anti-derivative of \( f \). The fundamental theorem of calculus states \( F'(x) = f(x) \). Unlike the derivative, anti-derivatives can not always be expressed in terms of known functions. An example is: \( F(x) = \int_0^x e^{-x^2} \, dx \). Often, the anti-derivative can be found: Example: \( f(x) = \cos^2(x) = \frac{1}{2} (\cos(2x) + 1) \), \( F(x) = \frac{x}{2} \sin(2x) = \frac{1}{4} \).

2D INTEGRATION. If \( f(x,y) \) is a continuous function of two variables on a region \( R \), the integral \( \int_R f(x,y) \, dxdy \) can be defined as the limit \( \frac{1}{n^2} \sum_{i,j \in R} f(x_i,y_j) \) with \( x_{i,j} = (i/n, j/n) \) when \( n \) goes to infinity. If \( f(x,y) = 1 \), then the integral is the area of the region \( R \). The integral divided by the area of \( R \) is the average value of \( f \) on \( R \). For many regions, the integral can be calculated as a double integral \( \int_a^b \int_{c(x)}^{d(x)} f(x,y) \, dy \, dx \). In general, the region must be split into pieces, then integrated separately.

EXAMPLE. Calculate \( \int_R f(x,y) \, dxdy \), where \( f(x,y) = 4x^2y^3 \) and where \( R \) is the rectangle \([0,1] \times [0,2]\).
\[
\int_0^1 \left[ \int_0^2 4x^2y^3 \, dy \right] \, dx = \int_0^1 [x^2y^4]_0^2 \, dx = \int_0^1 x^2(16 - 0) \, dx = 16x^3/3\big|_0^1 = \frac{16}{3}.
\]

FUBINI’S THEOREM. \( \int_a^b \int_c^d f(x,y) \, dxdy = \int_c^d \int_a^b f(y,x) \, dydx \).
To calculate this integral, we first determine the inner integral
\[ \int_0^1 e^{-x^2} \, dx = \int_0^1 xe^{-x^2} \, dx = -\frac{e^{-x^2}}{2} \bigg|_0^1 = \frac{1}{2} - \frac{1}{2e} \approx 0.316. \]
A special case of switching the order of integration is Fubini’s theorem.

**EXAMPLE.** Let \( R \) be the triangle \( 1 \geq x \geq 0, 1 \geq y \geq 0, y \leq x \). Calculate \( \int_R e^{-x^2} \, dxdy \).

**ATTEMPT.** \( \int_0^1 \int_y^1 e^{-x^2} \, dx \, dy \). We can not solve the inner integral because \( e^{-x^2} \)
has no anti-derivative in terms of elementary functions.

**IDEA.** Switch order: \( \int_0^1 \int_0^x e^{-x^2} \, dy \, dx = \int_0^1 xe^{-x^2} \, dx = -\frac{e^{-x^2}}{2} \bigg|_0^1 = \frac{1}{2} - \frac{1}{2e} \approx 0.316. \)

**QUANTUM MECHANICS.** In quantum mechanics, the motion of a particle (like an electron) in the plane is determined by a function \( u(x, y) \), the wave function. Unlike in classical mechanics, the position of a particle is given in a probabilistic way only. If \( R \) is a region and \( u \) is normalized so that \( \int |u|^2 \, dxdy = 1 \), then \( \int_R \left| u(x, y) \right|^2 \, dxdy \) is the probability, that the particle is in \( R \).

**EXAMPLE.** Unlike a classical particle, a quantum particle in a box \([0, \pi] \times [0, \pi]\) can have a discrete set of energies only. This is the reason for the name "quantum". If \(-u_{xx} + u_{yy} = \lambda u\), then a particle of mass \( m \) has the energy \( E = \lambda \hbar^2 / 2m \). A function \( u(x, y) = \sin(kx) \sin(ny) \) represents a particle of energy \( (k^2 + n^2)\hbar^2 / (2m) \).

Let us assume \( k = 2 \) and \( n = 3 \) from now on. Our aim is to find the probability that the particle with energy \( 13\hbar^2 / (2m) \) is in the middle 9th region \( R = [\pi/3, 2\pi/3] \times [\pi/3, 2\pi/3] \) of the box.

**SOLUTION:** We first have to normalize \( u^2(x, y) = \sin^2(2x) \sin^2(3y) \), so that the average over the whole square is 1:

\[ A = \int_0^\pi \int_0^\pi \sin^2(2x) \sin^2(3y) \, dxdy. \]

To calculate this integral, we first determine the inner integral
\[ \int_0^\pi \sin^2(2x) \sin^2(3y) \, dx = \sin^2(3y) \int_0^\pi \sin^2(2x) \, dx = \frac{\pi}{2} \sin^2(3y) \] (the factor
\( \sin^2(3y) \) is treated as a constant). Now, \( A = \int_0^\pi \left( \frac{\pi}{2} \sin^2(3y) \right) \, dy = \frac{\pi^2}{4} \) so that the probability amplitude function is \( f(x, y) = \frac{1}{\pi} \sin^2(2x) \sin^2(3y) \).

The probability that the particle is in \( R \) is slightly smaller than 1/9:

\[ \frac{1}{A} \int_R f(x, y) \, dxdy = \frac{4}{\pi^2} \int_{\pi/3}^{2\pi/3} \int_{\pi/3}^{2\pi/3} \sin^2(2x) \sin^2(3y) \, dxdy \]
\[ = \frac{4}{\pi^2} (4x - \sin(4x)) / 8(6x - \sin(6x)) / 12 \int_{\pi/3}^{2\pi/3} \]
\[ = 1/9 - 1/(4\sqrt{3}\pi). \]

MOMENT OF INERTIA. Compute the kinetic energy of a square iron plate \( R = [-1, 1] \times [-1, 1] \) of density \( \rho = 1 \) (about 10cm thick) rotating around its center with a 6000 rpm (rounds per minute). The angular velocity speed is \( \omega = 2\pi \times 6000/60 = 100 \times 2\pi \). Because \( E = \int I(\omega)^2 / 2 \, dxdy \), where \( I = \int (x^2 + y^2) \, dxdy \) is the moment of inertia. For the square, \( I = 4/3 \). Its energy of the plate is \( \omega^2/6 = 4\pi^2100/6 \text{ Joule} \approx 0.43 KWh \). You can run with this energy a 60 Watt bulb for 7 hours.

WHERE DO DOUBLE INTEGRALS OCCUR?
- compute areas.
- compute averages. Examples: average rain fall or average population in some area.
- probabilities. Expectation of random variables.
- quantum mechanics: probability of particle.
- find moment of inertia \( \int \int_R (x^2 + y^2) \rho(x, y) \, dxdy \)
- find center of mass (\( \int \int_R x\rho(x, y) \, dxdy/M, \int \int_R y\rho(x, y) \, dxdy/M \)), with \( M = \int \int_R \, dxdy. \)
- 1D integrals (see challenge problems).