The partial differential equation $u_t = au_x$ is known as the transport equation (or sometimes the advection equation). In this problem we’ll find solutions $u(x, t)$ to this equation. In particular, we’ll include an initial condition and solve the initial value problem (IVP)

\[
(\star) \quad \begin{cases} 
\frac{\partial u}{\partial t} = a \frac{\partial u}{\partial x} \\
u(x, 0) = f(x).
\end{cases}
\]

(a) Here is a graph of $u = f(x)$ (an example initial condition for the IVP):

\[
\text{This graph is of } u(x, 0) = f(x). \text{ Is the value of } u_x(x, 0) \text{ positive, negative, or zero at the points } x = A, x = B, \text{ and } x = C? \\
\]

(b) If $u(x, t)$ is a solution of the IVP labeled $(\star)$ for $a > 0$, what is the sign of $u_t(x, 0)$ at the points $x = A, x = B, \text{ and } x = C$?

(c) One solution to the IVP $(\star)$ is $u(x, t) = f(x + at)$. We explore this solution in the next two questions.

(i) Show that $u(x, t) = f(x + at)$ really is a solution to the IVP $(\star)$ by calculating $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$.

(ii) Sketch the graph of $u(x, t)$ (which, recall, is $f(x + at)$) when $t = 1$.

**Hint:** This is the graph of $f(x + a)$. How does that relate to the graph of $f(x)$ (shown above)?

The partial differential equation $u_{tt} = c^2 u_{xx}$ is known as the wave equation. (You might also call this the one-dimensional wave equation as there is one spatial dimension $x$ with the one temporal dimension $t$.) Here we’ll find some solutions $u(x, t)$ to this equation. In particular, we’ll include an initial condition and solve the initial value problem (IVP)

\[
(\star\star) \quad \begin{cases} 
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \\
u(x, 0) = f(x) \\
u_t(x, 0) = 0.
\end{cases}
\]

One solution to the IVP $(\star\star)$ is $u(x, t) = \frac{1}{2} (f(x + ct) + f(x - ct))$. Again we ask two questions:

(i) Show that $u(x, t) = \frac{1}{2} (f(x + ct) + f(x - ct))$ really is a solution to the IVP $(\star\star)$. 
(ii) Sketch the graph of \( u(x,t) \) for \( t = 1 \) assuming that \( f(x) \) is the same as in the first problem.

**Hint:** This is just \( u(x,1) = \frac{1}{2}(f(x+c) + f(x-c)) \), the average of two functions. What do the graphs of these two functions look like?

(b) The partial differential equation \( u_t = c^2 u_{xx} \) is known as the **heat equation** (or the **diffusion equation** as it covers the diffusion of heat). As before, we’ll include an initial condition and consider the **initial value problem** (IVP)

\[
\begin{cases}
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \\
u(x,0) = g(x).
\end{cases}
\]

(i) Show that one solution to the differential equation is

\[
u(x,t) = e^{-c^2 \omega^2 t}(A \cos(\omega x) + B \sin(\omega x)),\]

where \( c \) is the constant from the differential equation and \( A, B, \) and \( \omega \) are additional constants.

(ii) If this \( u(x,t) \) is a solution of the IVP labeled (†), what is the function \( g(x) \)?

(iii) What happens to \( u(x,t) \) as \( t \) grows without bound? That is, what is \( \lim_{t \to \infty} u(x,t) \)