13.3: 8, 16, 20, 26, 29–32

8. \( \partial(1 + 2xy + \ln x)/\partial y = 2x = \partial(x^2)/\partial x \) and the domain of \( \mathbf{F} \) is \( \{(x, y) \mid x > 0\} \) which is open and simply-connected.

Hence \( \mathbf{F} \) is conservative, so there exists a function \( f \) such that \( \nabla f = \mathbf{F} \). Then \( f_x(x, y) = 1 + 2xy + \ln x \) implies \( f(x, y) = x^2 + xy^2 + x \ln x - x + g(y) \) and \( f_y(x, y) = x^2 + g'(y) \). But \( f_y(x, y) = x^2 \) so \( g'(y) = 0 \) \( \Rightarrow \) \( g(y) = K \).

Then \( f(x, y) = x^2 y + x \ln x + K \) and it is a potential function for \( \mathbf{F} \).

16. (a) \( f_x(x, y, z) = 2xz + y^2 \) implies \( f(x, y, z) = x^2 z + xy^2 + g(y, z) \) and so \( f_y(x, y, z) = 2xy + g_y(y, z) \). But \( f_y(x, y, z) = 2xy \) so \( g_y(y, z) = 0 \) \( \Rightarrow \) \( g(y, z) = h(z) \). Thus \( f(x, y, z) = x^2 z + xy^2 + h(z) \) and \( f_z(x, y, z) = x^2 + h'(z) \). But \( f_z(x, y, z) = x^2 + 3z^2 \), so \( h'(z) = 3z^2 \) \( \Rightarrow \) \( h(z) = z^3 + K \). Hence \( f(x, y, z) = x^2 z + xy^2 + z^3 \) (taking \( K = 0 \)).

(b) \( t = 0 \) corresponds to the point \((0, 1, -1)\) and \( t = 1 \) corresponds to \((1, 2, 1)\), so

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, -1) = 6 - (-1) = 7. \]

20. Here \( \mathbf{F}(x, y) = (1 - ye^{-x}) \mathbf{i} + e^{-x} \mathbf{j} \). Then \( f(x, y) = x + ye^{-x} \) is a potential function for \( \mathbf{F} \), that is, \( \nabla f = \mathbf{F} \) so \( \mathbf{F} \) is conservative and thus its line integral is independent of path. Hence

\[ \int_C (1 - ye^{-x}) \, dx + e^{-x} \, dy = \int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2) - f(0, 1) = (1 + 2e^{-1}) - 1 = 2/e. \]

26. \( \nabla f(x, y) = \cos(x - 2y) \mathbf{i} - 2 \cos(x - 2y) \mathbf{j} \)

(a) We use Theorem 2: \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(r(b)) - f(r(a)) \) where \( C_1 \) starts at \( t = a \) and ends at \( t = b \). So because \( f(0, 0) = \sin 0 = 0 \) and \( f(\pi, \pi) = \sin(\pi - 2\pi) = 0 \), one possible curve \( C_1 \) is the straight line from \((0, 0)\) to \((\pi, \pi)\); that is, \( r(t) = t\pi \mathbf{i} + t\pi \mathbf{j}, 0 \leq t \leq 1 \).

(b) From (a), \( \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = f(r(b)) - f(r(a)) \). So because \( f(0, 0) = \sin 0 = 0 \) and \( f\left(\frac{\pi}{2}, 0\right) = 1 \), one possible curve \( C_2 \) is \( r(t) = \frac{\pi}{2}t \mathbf{i}, 0 \leq t \leq 1 \), the straight line from \((0, 0)\) to \((\frac{\pi}{2}, 0)\).

29. \( D = \{(x, y) \mid x > 0, y > 0\} \) is the first quadrant (excluding the axes).

(a) \( D \) is open because around every point in \( D \) we can put a disk that lies in \( D \).

(b) \( D \) is connected because in the straight line segment joining any two points in \( D \) lies in \( D \).

(c) \( D \) is simply-connected because it’s connected and has no holes.

30. \( D = \{(x, y) \mid x \neq 0\} \) consists of all points in the \( xy \)-plane except for those on the \( y \)-axis.

(a) \( D \) is open.

(b) Points on opposite sides of the \( y \)-axis cannot be joined by a path that lies in \( D \), so \( D \) is not connected.

(c) \( D \) is not simply-connected because it is not connected.

31. \( D = \{(x, y) \mid 1 < x^2 + y^2 < 4\} \) is the annular region between the circles with center \((0, 0)\) and radii 1 and 2.

(a) \( D \) is open.

(b) \( D \) is connected.

(c) \( D \) is not simply-connected. For example, \( x^2 + y^2 = (1.5)^2 \) is simple and closed and lies within \( D \) but encloses points that are not in \( D \). (Or we can say, \( D \) has a hole, so is not simply-connected.)

32. \( D = \{(x, y) \mid x^2 + y^2 \leq 1 \text{ or } 4 \leq x^2 + y^2 \leq 9\} \) is the points on or inside the circle \( x^2 + y^2 = 1 \), together with the points on or between the circles \( x^2 + y^2 = 4 \) and \( x^2 + y^2 = 9 \).

(a) \( D \) is not open because, for instance, no disk with center \((0, 2)\) lies entirely within \( D \).

(b) \( D \) is not connected because, for example, \((0, 0)\) and \((0, 2.5)\) lie in \( D \) but cannot be joined by a path that lies entirely in \( D \).

(c) \( D \) is not simply-connected because, for example, \( x^2 + y^2 = 9 \) is a simple closed curve in \( D \) but encloses points that are not in \( D \).