A surface integral is
\[ \iint_S f(x, y, z) \, dS = \iint_D f(r(u, v)) \left| r_u \times r_v \right| \, du \, dv, \]
where \( f \) is a function defined on the parametric surface \( r(u, v) \).

1. Evaluate the surface integral
\[ \iint_S (1 + z) \, dS, \]
where \( S \) is that part of the plane \( x + y + 2z = 2 \) in the first octant.

Suppose \( \mathbf{F} \) is a continuous vector field on an oriented surface \( S \) with unit normal vector \( \mathbf{n} \). The surface integral of \( \mathbf{F} \) over \( S \) is
\[ \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_D \mathbf{F} \cdot (r_u \times r_v) \, du \, dv \]
for a parametrically defined surface.

2. Evaluate the surface integral \( \iint_S \mathbf{F} \cdot dS \), where \( \mathbf{F} = yi - xj + zk \) and \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 4 \) in the first octant with inward orientation.
3. Evaluate the surface integral \( \iint_{S} \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F} = \langle x, y, 2z \rangle \) and \( S \) is the part of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the unit square \([0, 1] \times [0, 1]\) with the \textit{downward} orientation.

4. Evaluate the surface integral \( \iint_{S} \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F} = x\mathbf{i} + y\mathbf{j} + (2x + 2y)\mathbf{k} \) and \( S \) is the part of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the unit disk (centered at the origin) with upward orientation.

5. Evaluate the surface integral \( \iint_{S} \mathbf{F} \cdot d\mathbf{S} \), where \( \mathbf{F} = \langle -z, x, y \rangle \) and \( S \) is the full unit hemisphere (including the base!) on and above the \( xy \)-plane (so \( x^2 + y^2 + z^2 = 1 \) plus a disk) with the outward orientation.
Here we use the parameterization \( \mathbf{r}(x, y) = \langle x, y, 1 - \frac{1}{2}x - \frac{1}{2}y \rangle \). From this we find that

\[
\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle.
\]

Therefore

\[
\iint_S (1 + z) \, dS = \iint_D \left[ 1 + \left( 1 - \frac{1}{2}x - \frac{1}{2}y \right) \right] \, dA.
\]

The region \( D \) in our parameter space (the \( xy \)-plane) need to cover our surface is the triangle

\[ D = \{(x, y) : x \geq 0, \ y \geq 0, \ x + y \leq 2\}, \]

so we can write our limits as follows:

\[
\iint_S (1 + z) \, dS = \int_0^2 \int_0^{2-x} \left( 2 - \frac{1}{2}x - \frac{1}{2}y \right) \, dy \, dx
\]

\[
= \int_0^2 \left( \frac{1}{4}x^2 - 2x + 3 \right) \, dx
\]

\[
= \frac{1}{12} \cdot 2^3 - 2^2 + 3 \cdot 2 = \frac{8}{3}.
\]

This is based on Problem 23 from Section 13.6 of the textbook.

One common parameterization of the sphere of radius \( a \) is simply using spherical coordinates (with \( \rho = a \)):

\[ \mathbf{r}(\phi, \theta) = \langle a \sin(\phi) \cos(\theta), a \sin(\phi) \cos(\theta), a \cos(\phi) \rangle. \]

One could then compute \( \mathbf{r}_\phi \times \mathbf{r}_\theta \), but it is easier to just remember that we’ve done this before and the answer is:

\[ \mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin(\phi) \langle \sin(\phi) \cos(\theta), \sin(\phi) \cos(\theta), \cos(\phi) \rangle. \]

(See, for example, page 869 of the text.) The coefficient in front is simply the coefficient \( \rho^2 \sin(\phi) \) from the spherical volume element (with \( \rho = a \)) while the vector is simply the unit vector in the direction \( \langle x, y, z \rangle \) (the radial vector, which is perpendicular to the tangent plane to the sphere). Notice that the orientation is specified to be the inward normal, so we actually want \( \mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta \).

In any case, with this we proceed. In spherical coordinates (with \( \rho = a = 2 \) in this case), our vector field is

\[ \mathbf{F} = \langle y, -x, z \rangle = \langle 2 \sin(\phi) \sin(\theta), -2 \sin(\phi) \cos(\theta), 2 \cos(\phi) \rangle. \]
Thus

\[ \mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi) \]

\[ = -4 \sin(\phi) \left\langle 2 \sin(\phi) \sin(\theta), -2 \sin(\phi) \cos(\theta), 2 \cos(\phi) \right\rangle \cdot \left\langle \sin(\phi) \cos(\theta), \sin(\phi) \cos(\theta), \cos(\phi) \right\rangle \]

\[ = -8 \sin(\phi) \cos^2(\phi). \]

We integrate this over the domain \( \{ (\phi, \theta) : 0 \leq \phi \leq \pi/2, \ 0 \leq \theta \leq \pi/2 \} \), so we have

\[
\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\pi/2} \int_0^{\pi/2} -8 \sin(\phi) \cos^2(\phi) \ d\theta \ d\phi
\]

\[ = -4\pi \int_0^{\pi/2} \sin(\phi) \cos^2(\phi) \ d\phi \]

\[ = -4\pi \left[ -\frac{1}{3} \cos^3(\phi) \right]_0^{\pi/2} = -\frac{4\pi}{3}. \]

Another approach is to use the parameterization by \( x \) and \( y \)

\[ \mathbf{r}(x, y) = \langle x, y, \sqrt{4 - x^2 - y^2} \rangle \]

then integrating over the quarter circle in the \( xy \)-plane. Let’s see how this goes:

\[
\mathbf{r}_y \times \mathbf{r}_x = \begin{vmatrix} i & j & k \\ 0 & 1 & - \frac{y}{\sqrt{4-x^2-y^2}} \\ 1 & 0 & - \frac{x}{\sqrt{4-x^2-y^2}} \end{vmatrix} = \left\langle -\frac{x}{\sqrt{4-x^2-y^2}}, -\frac{y}{\sqrt{4-x^2-y^2}}, -1 \right\rangle.
\]

(Notice we’ve used \( \mathbf{r}_y \times \mathbf{r}_x \) to get the inward-pointing normal.) Thus

\[
\mathbf{F} \cdot (\mathbf{r}_y \times \mathbf{r}_x) = \langle y, -x, z \rangle \cdot \left\langle -\frac{x}{\sqrt{4-x^2-y^2}}, -\frac{y}{\sqrt{4-x^2-y^2}}, -1 \right\rangle
\]

\[ = \langle y, -x, \sqrt{4-x^2-y^2} \rangle \cdot \left\langle -\frac{x}{\sqrt{4-x^2-y^2}}, -\frac{y}{\sqrt{4-x^2-y^2}}, -1 \right\rangle
\]

\[ = -\sqrt{4-x^2-y^2} \]

and so

\[
\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^2 \int_0^{\sqrt{4-x^2}} -\sqrt{4-x^2-y^2} \ dy \ dx
\]

\[ = -\int_0^2 \left[ \frac{y}{2} \sqrt{4-x^2-y^2} + \frac{4-x^2}{2} \sin^{-1}\left( \frac{y}{\sqrt{4-x^2}} \right) \right]_0^{\sqrt{4-x^2}} \ dx
\]

\[ = -\frac{\pi}{4} \int_0^2 (4-x^2) \ dx = -\frac{\pi}{4} \cdot \frac{16}{3} = -\frac{4\pi}{3}, \]

as before.
A simple parameterization of this surface is \( \mathbf{r}(x, y) = \langle x, y, 4 - x^2 - y^2 \rangle \). Thus

\[
\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = \langle 2x, 2y, 1 \rangle.
\]

(Note that this is not properly oriented. We’re told to use the “downward orientation” but here the \( \mathbf{k} \) component is positive. Thus we should be using \( \mathbf{r}_y \times \mathbf{r}_x = -\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, -1 \rangle \).)

Thus

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle x, y, 2z \rangle \cdot \langle -2x, -2y, -1 \rangle \, dx \, dy
\]

\[
= \iint_D (-2x^2 - 2y^2 - 2z) \, dx \, dy.
\]

Now we notice that (according to our parameterization) \( z = 4 - x^2 - y^2 \). The region \( D \) in our parameter space is a unit square, so we get

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left[ -2x^2 - 2y^2 - 2(4 - x^2 - y^2) \right] \, dx \, dy
\]

\[
= \int_0^1 \int_0^1 (-8) \, dx \, dy = -8 \text{ Area}(D) = -8.
\]
One way to do this is to use the parameterization \( r(x, y) = (x, y, 4 - x^2 - y^2) \), so \( r_x = (1, 0, -2x) \), \( r_y = (0, 1, -2y) \), and so

\[
\begin{vmatrix}
i & j & k \\
1 & 0 & -2x \\
0 & 1 & -2y
\end{vmatrix} = (2x, 2y, 1).
\]

Thus we can write

\[
F \cdot (r_x \times r_y) = (x, y, 2x + 2y) \cdot (2x, 2y, 1) = 2x^2 + 2y^2 + 2x + 2y.
\]

Hence our flux is

\[
\iint_S F \cdot dS = \iint_D (2x^2 + 2y^2 + 2x + 2y) \, dx \, dy,
\]

where \( D \) is the unit disk. Thus we make the change to polar coordinates, where \( 2x^2 + 2y^2 + 2x + 2y = 2r^2 + 2r \cos(\theta) \sin(\theta) \) and \( dx \, dy = r \, dr \, d\theta \). We get

\[
\iint_S F \cdot dS = \int_0^{2\pi} \int_0^1 \left[ 2r^2 + 2r \cos(\theta) \sin(\theta) \right] r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 \left[ 2r^3 + 2r^2 \cos(\theta) + 2r^2 \sin(\theta) \right] \, dr \, d\theta
\]

\[
= \left[ \frac{1}{2} \theta + \frac{2}{3} \left( \cos(\theta) + \sin(\theta) \right) \right]_0^{2\pi} = \pi.
\]

Another approach is to use polar coordinates to parameterize the surface from the very beginning. That is, we could use the parameterization \( r(r, \theta) = (r \cos(\theta), r \sin(\theta), 4 - r^2) \), in which case \( r_r = (\cos(\theta), \sin(\theta), -2r) \) and \( r_\theta = (-r \sin(\theta), r \cos(\theta), 0) \), so

\[
r_r \times r_\theta = \begin{vmatrix}
i & j & k \\
\cos(\theta) & \sin(\theta) & -2r \\
-r \sin(\theta) & r \cos(\theta) & 0
\end{vmatrix} = (2r^2 \cos(\theta), 2r^2 \sin(\theta), r).
\]

In these coordinates our vector field is \( F = \langle r \cos(\theta), r \sin(\theta), 2r \cos(\theta) + 2r \sin(\theta) \rangle \), and therefore our flux is

\[
\iint_S F \cdot dS = \int_0^{2\pi} \int_0^1 \langle r \cos(\theta), r \sin(\theta), 2r \cos(\theta) + 2r \sin(\theta) \rangle \cdot \langle 2r^2 \cos(\theta), 2r^2 \sin(\theta), r \rangle \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^1 \left[ 2r^3 + 2r^2 \left( \cos(\theta) + \sin(\theta) \right) \right] \, dr \, d\theta,
\]

which is identical to an integral computed above.
Here we need to compute two integrals:

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_{S_1} \mathbf{F} \cdot d\mathbf{S} + \int \int_{S_2} \mathbf{F} \cdot d\mathbf{S},$$

where $S_1$ is the hemisphere (with outward-pointing normal) and $S_2$ is the unit disk in the $xy$-plane (with downward-pointing normal).

The integral over $S_1$ is very similar to the previous problem. We’ll use the first parameterization of Problem 3, namely

$$\mathbf{r}(\phi, \theta) = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle.$$

This gives us an outward-pointing normal

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin(\phi) \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle,$$

so

$$\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)$$

$$= \langle - \cos(\phi), \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta) \rangle \cdot \langle \sin^2(\phi) \cos(\theta), \sin^2(\phi) \sin(\theta), \sin(\phi) \cos(\phi) \rangle$$

$$= - \sin^2(\phi) \cos(\phi) \cos(\theta) + \sin^3(\phi) \sin(\theta) \cos(\theta) + \sin^2(\phi) \cos(\phi) \sin(\theta).$$

Thus

$$\int \int_{S_1} \mathbf{F} \cdot d\mathbf{S}$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \left( - \sin^2(\phi) \cos(\phi) \cos(\theta) + \sin^3(\phi) \sin(\theta) \cos(\theta) + \sin^2(\phi) \cos(\phi) \sin(\theta) \right) d\theta d\phi$$

$$= \int_0^{\pi/2} \left( - \sin^2(\phi) \cos(\phi) \sin(\theta) - \sin^3(\phi) \cdot \frac{1}{2} \sin^2(\theta) - \sin^2(\phi) \cos(\phi) \cos(\theta) \right) \bigg|_\theta=0 d\phi$$

$$= 0.$$

The second integral is even simpler. Here $S_2$ is the unit disk in the $xy$-plane, with the unit normal $\mathbf{n} = -\mathbf{k}$ (straight down). Thus

$$\int \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int \int_D \langle -z, x, y \rangle \cdot \langle 0, 0, -1 \rangle \, dA = \int \int_D -y \, dA,$$

where $D$ is the unit disk in the $xy$-plane. We use polar coordinates to compute this integral:

$$\int \int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_D -y \, dA = \int_0^1 \int_0^{2\pi} -r \sin(\theta) \cdot r \, d\theta \, dr$$

$$= \int_0^1 r^2 \cos(\theta) \bigg|_0^{2\pi} dr = 0.$$
Thus the full integral is

\[ \iiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iiint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 0 + 0 = 0. \]

We’ll see a simpler way of computing this integral on Wednesday when we learn about the Divergence Theorem.