Evaluate the following triple integrals as iterated integrals.

(a) \( \int \int \int_E xy \, dV \), where \( E = [0,1] \times [0,2] \times [0,3] \).

(b) \( \int \int \int_E xy^3 z^2 \, dV \), where \( E = [-1,1] \times [-3,3] \times [0,3] \).

(c) \( \int \int \int_E y^2 z \cos(xyz) \, dV \), where \( E = [0,\pi] \times [0,1] \times [0,2] \).

**Hint:** Try using different orders of integration.

For each of the following regions \( E \), write the triple integral \( \int \int \int_E f(x,y,z) \, dV \) as an iterated integral. There may be up to six different ways to do this, depending on whether you write it with \( dx \, dy \, dz \) or \( dz \, dy \, dx \) or \( dx \, dz \, dy \) or... 

(a) The tetrahedron bounded by the planes \( x + y + z = 1 \), \( x = 0 \), \( y = 0 \), and \( z = 0 \).

(b) The (solid) sphere \( x^2 + y^2 + z^2 = a^2 \).

(c) The region between the paraboloid \( x = 1 - y^2 - z^2 \) and the \( yz \)-plane.
(d) The region bounded by the surface \( z = 3xy + 1 \) and the planes \( z = 0, \ x = 0, \ x = 1, \ y = 0, \) and \( y = x. \)

(e) The region bounded by the cylinder \( x^2 + z^2 = 1 \) and the planes \( y = 0 \) and \( y + z = 2. \)

(f) The pyramid whose base is the square \([−1, 1] \times [−1, 1]\) in the \(xy\)-plane and whose vertex is the point \((0, 0, 1)\).

3 Evaluate the following integrals.

(a) \( \iiint_E 1 \, dV, \) where \( E \) is the region in Problem 2(a).

(b) \( \iiint_E z \, dV, \) where \( E \) is the region in Problem 2(d).

4 Find the volume of the pyramid described in Problem 2(f).

5 Consider a brick in the region \([0, 1] \times [0, 2] \times [0, 1]\) whose density at a point \((x, y, z)\) is \( \rho(x, y, z) = 2 + xy - 2z. \) Find the mass of the brick.
Triple Integrals – Solutions

1. (a) The integral is
\[ \int_0^1 \int_0^2 \int_0^3 xy \, dz \, dy \, dx = \int_0^1 \int_0^2 3xy \, dy \, dx = \int_0^1 3x \left( \frac{2^2}{2} - 0 \right) \, dx = \int_0^1 6x \, dx = 3. \]

(b) The integral is
\[ \int_0^3 \int_{-3}^3 \int_{-1}^1 xy^3 z^2 \, dx \, dy \, dz = \int_0^3 \int_{-3}^3 y^3 z^2 \left( \frac{1^2}{2} - \frac{(-1)^2}{2} \right) \, dy \, dz = 0. \]

Note that we can put the three integrals in whatever order we want, and putting the integral with respect to \( x \) first makes the computation easier.

(c) If we write the integral as
\[ \int_0^\pi \int_0^1 \int_0^2 y^2 z \cos(\pi y z) \, dz \, dy \, dx, \]
we have to use integration by parts and the integral is lots of work. However, if we write it as
\[ \int_0^1 \int_0^2 \int_0^\pi y^2 z \cos(\pi y z) \, dx \, dz \, dy, \]
it is a lot easier. In the inner integral, we can note that
\[ \frac{\partial}{\partial x} y \sin(\pi y z) = y^2 z \cos(\pi y z) \]
so we get
\[ \int_0^1 \int_0^2 \int_0^\pi y^2 z \cos(\pi y z) \, dx \, dz \, dy = \int_0^1 \int_0^2 (y \sin(\pi y z) - y \sin(0)) \, dz \, dy = \int_0^1 \int_0^2 y \sin(\pi y z) \, dz \, dy. \]

We then similarly have
\[ \frac{\partial}{\partial z} \cos(\pi y z) = -y \sin(\pi y z) \]
so
\[ \int_0^1 \int_0^2 y \sin(\pi y z) \, dz \, dy = \frac{-1}{\pi} \int_0^1 (\cos(2\pi y) - \cos(0)) \, dy = \frac{1}{\pi} \]
and
\[ \int_0^1 (\cos(2\pi y) - 1) \, dy = \frac{1}{\pi}. \]
We only give one possible way to express each integral; there are others that are equally correct.

(a) \[ \iiint_E f(x, y, z) \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x, y, z) \, dz \, dy \, dx. \]

(b) \[ \iiint_E f(x, y, z) \, dV = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x, y, z) \, dz \, dy \, dx. \]

(c) This region is defined by the inequality \( 0 \leq x \leq 1 - y^2 - z^2 \). For \( 1 - y^2 - z^2 \) to be nonnegative, we also need \((y, z)\) to lie on the disk \( D \) bounded by \( y^2 + z^2 = 1 \). Thus we get
\[
\iiint_E f(x, y, z) \, dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{1-y^2-z^2} f(x, y, z) \, dx \, dz \, dy.
\]

(d) This region is defined by the inequalities \( 0 \leq z \leq 3xy + 1 \), \( 0 \leq x \leq 1 \), and \( 0 \leq y \leq x \), so we get
\[
\iiint_E f(x, y, z) \, dV = \int_0^1 \int_0^{3xy+1} \int_0^x f(x, y, z) \, dz \, dy \, dx.
\]

(e) Being between the planes \( y = 0 \) and \( y + z = 2 \) says that \( 0 \leq y \leq 2 - z \), and being inside the cylinder says that \((x, z)\) is in the disk \( D \) bounded by \( x^2 + z^2 = 1 \). Thus we get
\[
\iiint_E f(x, y, z) \, dV = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-z} f(x, y, z) \, dy \, dz \, dx.
\]

(f) In the pyramid, \( z \) ranges from 0 at the base to 1 at the top. The cross-section of the pyramid given by a fixed value of \( z \) is the square \([-1+z, 1-z] \times [-1+z, 1-z]\). Thus we get
\[
\iiint_E f(x, y, z) \, dV = \int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} f(x, y, z) \, dx \, dy \, dz.
\]

(a) The integral is
\[
\int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} (1 - x - y) \, dy \, dx = \int_0^1 (1 - x)^2 - \frac{(1 - x)^2}{2} \, dx = \int_0^1 \frac{(1 - x)^2}{2} \, dx = \frac{1}{6}.
\]
(b) The integral is
\[ \int_0^1 \int_0^x \int_0^{3xy+1} z \, dz \, dy \, dx = \int_0^1 \int_0^x \frac{(3xy+1)^2}{2} \, dy \, dx \]
\[ = \frac{1}{2} \int_0^1 \int_0^x (9x^2y^2 + 6xy + 1) \, dy \, dx \]
\[ = \frac{1}{2} \int_0^1 (3x^5 + 3x^3 + x) \, dx = \frac{7}{8}. \]

4. The volume of a region \( E \) is given by the integral \( \iiint_E 1 \, dV \). In this case, that integral is
\[ \int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} 1 \, dx \, dy \, dz = \int_0^1 \int_{-1+z}^{1-z} (2 - 2z) \, dy \, dz \]
\[ = \int_0^1 (2 - 2z)^2 \, dz = \frac{4}{3}. \]

5. The mass is given by integrating the density over the region, so the mass is
\[ \int_0^1 \int_0^2 \int_0^1 (2 + xy - 2z) \, dz \, dy \, dx = \int_0^1 \int_0^2 (2 + xy - 1) \, dy \, dx \]
\[ = \int_0^1 (4 + 2x - 2) \, dy \, dx \]
\[ = 4 + 1 - 2 = 3. \]