Recall from Monday that the gradient of a function $f$ is the vector $\nabla f$ defined by

$$\nabla f(x, y) = \text{grad } f = (f_x(x, y), f_y(x, y)) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

in the plane or

$$\nabla f(x, y, z) = \text{grad } f = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

in three-space.

1. For each of the following functions $f(x, y)$ in the plane, do the following:
   (i) Compute the gradient $\nabla f$ (or grad $f$).
   (ii) Identify the level curve $f(x, y) = \text{constant}$ through the point $(x, y) = (1, 1)$.
   (iii) Find a parameterization $r(t)$ of this curve.
   (iv) Verify that the tangent vector $r'(t)$ and the gradient vector $\nabla f$ are perpendicular at the point $(x, y) = (1, 1)$.
   (a) $f(x, y) = 3x - y$
   (b) $f(x, y) = 2x^2 + 3y^2$

2. For each of the following functions $F(x, y, z)$ in the plane, do the following:
   (i) Compute the gradient $\nabla F$ (or grad $F$).
   (ii) Identify the level surface $F(x, y, z) = \text{constant}$ through the point $(x, y, z) = (1, 1, 1)$.
   (iii) Find the tangent plane to the level surface from part (ii). Recall that the tangent plane to a surface $z = f(x, y)$ at the point $(x_0, y_0, z_0)$ is
   $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
   (iv) Verify that the gradient vector $\nabla F$ is perpendicular to (tangent plane of) the level surface at the point $(x, y, z) = (1, 1, 1)$.
   (a) $F(x, y, z) = 3x + 2y + z$
   (b) $F(x, y, z) = x^2 + y^2 - z^2$

These problems illustrate the following point:

The gradient of a function is always perpendicular to the level curves (or level surfaces) of the function!
There are two important facts about the gradient vector:

- $\text{grad } f$ (or $\nabla f$) is perpendicular to the level curves of $f$ (as we saw on page one of this handout)
- $|\text{grad } f|$ (or the magnitude of $\nabla f$) is the rate of change of $f$ in the direction of grad $f$

Here is an example sketch of the level curves of $f(x, y) = y^2 - x^2$ and the associated gradient vector field:

(The arrows shown here are in fact one-tenth the actual length of the gradient, but they’re shrunk to make the picture cleaner. Here $\nabla f = \langle f_x, f_y \rangle = \langle -2x, 2y \rangle$, so $|\nabla f| = \sqrt{4x^2 + 4y^2} = 2r$.)

Use the two facts shown above to sketch the gradient vector field given the following contour plots (pictures of level curves):
Gradients & Level Surfaces – Answers / Solutions

1 (a) (i) $\nabla f = \langle 3, -1 \rangle$ at every point, not just $(x, y) = (1, 1)$.

(ii) The curve $f(x, y) = f(1, 1)$ is $3x - y = 2$, a line.

(iii) A simple parameterization is $x = t$, so $y = 3t - 2$, or $r(t) = \langle t, 3t - 2 \rangle$.

(iv) The tangent vector in our parameterization is always $r'(t) = \langle 1, 3 \rangle$, so $\nabla f \cdot r'(t) = 0$ for all $t$, not just for $t = 1$ (the point $(x, y) = (1, 1)$).

(b) (i) $\nabla f = \langle 4x, 6y \rangle$, so $\nabla f(1, 1) = \langle 4, 6 \rangle$.

(ii) The curve $f(x, y) = f(1, 1)$ is $2x^2 + 3y^2 = 5$, an ellipse.

(iii) One parameterization is

$$r(t) = \langle x(t), y(t) \rangle = \left\langle \sqrt{\frac{5}{2}} \cos(t), \sqrt{\frac{5}{3}} \sin(t) \right\rangle.$$  

I found this by using the parameterization $\langle u, v \rangle = \langle \sqrt{5} \cos(t), \sqrt{5} \sin(t) \rangle$ for the circle $u^2 + v^2 = 5$, then writing our ellipse as $(\sqrt{2}x)^2 + (\sqrt{3}y)^2 = 5$ and making the substitutions $u = \sqrt{2}x$ and $v = \sqrt{3}y$.

(iv) The tangent vector in our parameterization is $r'(t) = \left\langle -\sqrt{\frac{5}{2}} \sin(t), \sqrt{\frac{5}{3}} \sin(t) \right\rangle$. You might think we need to find $t_0$ when $(x, y) = (1, 1)$, but in reality we only need to find $\cos(t_0)$ and $\sin(t_0)$. We know that $r(t_0) = \langle 1, 1 \rangle$, so

$$\left\langle \sqrt{\frac{5}{2}} \cos(t_0), \sqrt{\frac{5}{3}} \sin(t_0) \right\rangle = \langle 1, 1 \rangle \quad \text{or} \quad \cos(t_0) = \sqrt{\frac{2}{5}} \quad \text{and} \quad \sin(t_0) = \sqrt{\frac{3}{5}}.$$  

Thus $r'(t_0) = \left\langle -\sqrt{\frac{5}{2}} \cdot \sqrt{\frac{2}{5}}, \sqrt{\frac{5}{3}} \cdot \sqrt{\frac{2}{5}} \right\rangle = \left\langle -\sqrt{\frac{3}{2}}, \sqrt{2/3} \right\rangle$. Thus

$$\nabla f(1, 1) \cdot r'(t_0) = \langle 4, 6 \rangle \cdot \left\langle -\sqrt{\frac{3}{2}}, \sqrt{2/3} \right\rangle = -4\sqrt{\frac{3}{2}} + 6\sqrt{2/3} = 0.$$  

Thus the two vectors are perpendicular.

2 (a) (i) $\nabla F = \langle 3, 2, 1 \rangle$

(ii) The level surface $F(x, y, z) = F(1, 1, 1)$ is the plane $3x + 2y + z = 6$.

(iii) To use this formula, we solve for $z$: $z = f(x, y) = 6 - 3x - 2y$. Thus the tangent line is

$$z - 1 = -3(x - 1) - 2(y - 1) \quad \text{or} \quad 3x + 2y + z = 6.$$  

Fancy that! The tangent plane to a plane is the plane itself!

(iv) The gradient $\nabla F = \langle 3, 2, 1 \rangle$ is the same as the normal to the tangent plane (and the level surface itself); hence the gradient is perpendicular to the tangent plane of the level surface.
(b) (i) $\nabla F = \langle 2x, 2y, -2z \rangle$, so $\nabla F(1, 1, 1) = \langle 2, 2, -2 \rangle$.

(ii) The level surface $F(x, y, z) = F(1, 1, 1)$ is the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

(iii) To use this formula, we solve for $z$: $z = f(x, y) = \sqrt{x^2 + y^2 - 1}$ (we want the positive square root since $z = 1$ at our point). At $(x, y) = (1, 1)$, the derivative $f_x(1, 1)$ is easy to find:

$$f_x = \frac{1}{2} \left(x^2 + y^2 - 1\right)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2 - 1}},$$

so $f_x(1, 1) = \frac{1}{\sqrt{1^2 + 1^2 - 1}} = 1$ at the point $(x, y) = (1, 1)$. Thus the tangent line is

$$z - 1 = 1(x - 1) + 1(y - 1) \quad \text{or} \quad x + y - z = 1.$$ 

Note that the normal to this tangent plane is $n = \langle 1, 1, -1 \rangle$.

(iv) The gradient $\nabla F(1, 1, 1) = \langle 2, 2, -2 \rangle$ is parallel to the normal $n = \langle 1, 1, -1 \rangle$ to the tangent plane. As before, therefore, the gradient is perpendicular to the tangent plane of the level surface.

**Back Page:** Here are the two graphs with some gradient vectors drawn in:

In both cases the gradient vectors have been scaled to make sure the picture is not overwhelmed (or underwhelmed) with arrows.