2D INTEGRALS FOR SIMPLE REGIONS

Y-SIMPLE REGIONS: A class of regions is what is bound between the graphs of two functions \( c(x) \) and \( d(x) \). Such regions are sometimes called \( y \)-**simple regions**. One can write the region as

\[
R = \{(x, y) \mid c(x) \leq y \leq d(x) \}
\]

An integral over such a region is an iterated integral which is:

\[
\iint_R f(x, y) \, dA = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx
\]

**EXAMPLE 2)** Integrate \( f(x, y) = y \) over the region bound by the \( x \)-axes, the lines \( y = x + 1 \) and \( y = 1 - x \). The problem is best solved as a \( x \)-simple integral. As you can see from the picture, we would have to compute 2 different integrals as a type I integral. To do so, we have to write the bounds as a function of \( y \): they are \( x = y - 1 \) and \( x = 1 - y \). An integral over such a region is an iterated integral:

\[
\iint_R f(x, y) \, dA = \int_0^1 \left( \int_{y-1}^{1-y} y \, dx \right) \, dy = 2 \int_0^1 y^2(1-y) \, dy = 2(1/4 - 1/3) = 1/10
\]

X-SIMPLE REGIONS. It is defined by two functions \( a(y) \) and \( b(y) \) which are functions of \( y \). One can write the region as

\[
R = \{(x, y) \mid a(y) \leq x \leq b(y) \}
\]

An integral over such a region is an iterated integral:

\[
\iint_R f(x, y) \, dA = \int_a^d \left( \int_a^b f(x, y) \, dx \right) \, dy
\]

**EXAMPLE 1)** Integrate \( f(x, y) = x^2 \) over the region bounded above by \( \sin(x^2) \) and bounded below by the graph of \( -\sin(x^2) \) for \( 0 \leq x \leq \pi \). The value of this integral has a physical meaning. It is a moment of inertia. We will come back to that next week.

\[
\int_0^{\pi} \int_{-\sin(x^2)}^{\sin(x^2)} x^2 \, dy \, dx = 2 \int_0^{\pi} \sin(x^2)^{3/2} \, dx
\]

Now, we have an integral, which we can solve by substitution

\[
= -(2/3) \cos(x^2)^{1/2} \Big|_0^\pi = 4/3
\]

**EXAMPLE 3**. Let \( R \) be the triangle \( 1 \geq x \geq 0, \ 0 \leq y \leq x \). What is

\[
\int \int_R e^{-x^2} \, dx \, dy?
\]

The \( x \)-simple integral \( \int_0^1 \int_{x}^{1} e^{-x^2} \, dx \, dy \) can not be solved because \( e^{-x^2} \) has no anti-derivative in terms of elementary functions. The \( y \)-simple integral \( \int_{0}^{1} \int_{x}^{1} e^{-x^2} \, dx \, dy \) however can be solved:

\[
= \int_{0}^{1} x e^{-x^2} \, dx = -\frac{e^{-x^2}}{2} \big|_0^1 = \frac{(1 - e^{-1})}{2} \approx 0.316
\]

**EXAMPLE 4**: THE AREA OF A DISC OF RADIUS \( R \):

\[
\int_{-R}^{R} \int_{\sqrt{R^2 - x^2}}^{0} 1 \, dy \, dx = \int_{-R}^{R} \sqrt{R^2 - x^2} \, dx
\]

This integral can be solved with the substitution \( x = R \sin(u), \, dx = R \cos(u) \)

\[
\int_{-\pi/2}^{\pi/2} R^2 - R^2 \sin^2(u) \, R \cos(u) \, du = \int_{-\pi/2}^{\pi/2} R^2 \cos^2(u) \, du
\]

Now continue with a trigonometric identity to get

\[
R^2 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos(2u)}{2} \, du = R^2 \pi
\]

This is too complicated. We will see how to do that better in polar coordinates.

**WORDS OF WISDOM**:
If a double integral you can not solve, the order of integration change you must.

For solving double integrals, a picture at hand must be.