1D CHAIN RULE. If \( f \) and \( g \) are functions of one variable \( t \), then \( d/dt f(g(t)) = f'(g(t))g'(t) \). For example, \( d/dt \sin(5t) = \cos(5t) \cdot 5 \).

FINDING DERIVATIVES. Also the 1D chain rule was useful. For example, to find the derivative of \( \log(x) \) we write \( 1 = d/dx \exp(\log(x)) = d/dx \exp(\log(x)) = \log'(x) \) so that \( \log'(x) = 1/x \). An other example: to find \( \arccos(x) \), we write \( 1 = d/dx \cos(\arccos(x)) = -\sin(\arccos(x)) \arccos'(x) = -\sqrt{1 - \sin^2(\arccos(x))} \arccos'(x) = \sqrt{1 - x^2} \arccos'(x) \) so that \( \arccos(x) = -1/\sqrt{1 - x^2} \).

GRADIENT. Define \( \nabla f(x, y) = (f_x(x, y), f_y(x, y)) \). It is called the gradient of \( f \). It is the natural derivative of a function of several variables and a vector.

THE CHAIN RULE. If \( \vec{r}(t) \) is curve in space and \( f \) is a function of three variables, we get a function of one variable \( t \mapsto f(\vec{r}(t)) \). The chain rule is

\[
\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)
\]

WRITING IT OUT. The chain rule is, when written out

\[
\frac{df(x(t), y(t))}{dt} = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t).
\]

EXAMPLE. Let \( z = \sin(x + 2y) \), where \( x \) and \( y \) are functions of \( t \): \( x = e^t, y = \cos(t) \). What is \( \frac{dz}{dt} \)?

Here, \( z = f(x, y) = \sin(x + 2y) \), \( z_x = \cos(x + 2y) \), and \( z_y = 2 \cos(x + 2y) \) and \( \frac{dz}{dt} = e^t \). \( \frac{dz}{dt} = -\sin(t) \) and \( \frac{dz}{dt} = \cos(x + 2y) e^t - 2 \cos(x + 2y) \sin(t) \).

EXAMPLE. Olivers spider “Nabla” moves along a circle \( \vec{r}(t) = (\cos(t), \sin(t)) \) on a table with temperature distribution \( T(x, y) = x^2 - y^2 \). Find the rate of change of the temperature, “Nabla” experiences.

\[
\nabla T(x, y) = (2x, -2y), \quad \nabla T(\vec{r}(t)) = \nabla T(x, y) \cdot \vec{r}'(t) = (2 \cos(t), -2 \sin(t)) \cdot (-\sin(t), \cos(t)) = 2 \cos^2(t) + 2 \sin^2(t) = 2 \sin^2(t).
\]

APPLICATION ENERGY CONSERVATION. If \( H(x, y) \) is the energy of a particle with position \( x \) and velocity \( y \), the system moves satisfies the equations \( x'(t) = H_y, \quad y'(t) = -H_x \). For example, if \( H(x, y) = y^2/2 + V(x) \) is a sum of kinetic and potential energy, then \( \vec{r}'(t) = \vec{V}(x) = \nabla V(x) \) is equivalent to \( x''(t) = -\nabla V(x) \). THEOREM: The energy \( H \) is conserved. Proof. The chain rule shows that \( d/dt H(x(t), y(t)) = H_x(x(t), y(t)) \cdot x'(t) + H_y(x(t), y(t)) \cdot y'(t) = H_x(x(t), y(t)) - H_y(x(t), y(t)) = 0 \).

APPLICATION: IMPLICIT DIFFERENTIATION. From \( f(x, y) = 0 \) one can express \( y \) as a function of \( x \). From \( d/dx (f(x, y(x)) = \nabla f \cdot (1, y'(x)) = f_x + f_y y' = 0 \), we obtain

\[
y' = -f_y/f_x.
\]

Even so, we do not know \( y(x) \), we can compute its derivative!

EXAMPLE. \( f(x, y) = x^3 + x \sin(xy) = 0 \) defines \( y = g(x) \). If \( f(x, g(x)) = 0 \), then \( g_x(x) = -f_x/f_y = -3x^2 + \sin(xy) + x \cos(xy)/(\cos^2(xy)) \).

IMPLICIT DIFFERENTIATION IN THREE VARIABLES. The equation \( f(x, y, z) = 0 \) defines a surface. Near a point where \( f_3 \) is not zero, the surface can be described as a graph \( z = z(x, y) \). We can compute the derivative \( z_y \) without actually knowing the function \( z(x, y) \). To do so, we consider \( y \) a fixed parameter and compute using the chain rule

\[
f_x(x, y, z(x, y)) + f_y(x, y, z(x, y)) y(x, y, z) = 0
\]

so that \( z_y(x, y) = -f_y(x, y, z(x, y))/f_x(x, y, z(x, y)) \).

EXAMPLE. Let \( f(x, y, z) = x^2 + y^2/4 + z^2/9 = 6 \) be an ellipsoid. Compute \( z_y(x, y) \) at the point \( (x, y, z) = (2.1.1) \).

Solution: \( z_y(x, y) = -f_y(2.1, 1)/f_x(2.1, 1, 1) = -4/(2/9) = -18 \).

APPLICATION: DIFFERENTIATION OF CHAIN RULES. One ring of the chain rules them all \( f(x, y) = x + y, \quad x = u(t), \quad y = v(t), \quad f(u(t) + v(t)) = f'(u(t)) + f'(v(t)) = u'(t) + v'(t) \).

\[
f(x, y) = xy, \quad u(t) = t, \quad v(t) = t, \quad f(u(t), v(t)) = f'(u(t)) + f'(v(t)) = 0.
\]

\[
f(x, y) = y, \quad u(t) = t, \quad v(t) = t, \quad f(u(t), v(t)) = f'(u(t)) + f'(v(t)) = 0.
\]

DIETERICI EQUATION. In thermodynamics the temperature \( T \), the pressure \( p \) and the volume \( V \) of a gas are related. One rebounds of the ideal gas law \( pV = RT \) is the Dieterici equation

\[
f(p, V, T) = p(V - b)e^{c/RT} - RT = 0.
\]

The constant \( b \) depends on the molecule and \( c \) depends on the interaction of the molecules. (A different variation of the ideal gas law is van der Waals law).

Problem: Compute \( V' \).

\[
V' = dV/T, \quad f'_V = -f'/f \quad \text{so that} \quad V'_T = -(aV'b)e^{c/RT} / (RT^2) - b/R(\rho e^{c/RT} - p(V - b)e^{c/RT}) / (RT^2)
\]

(\( a = 8/3 \)). This could be simplified to \( (R + \alpha V) / (RT(V - b - \alpha V)) \).

PROOF OF THE CHAIN RULE.

Near any point, we can approximate \( f \) by a linear function \( L \). It is enough to check the chain rule for linear functions \( f(x, y) = ax + by + c \) and it \( \vec{r}(t) = (x(t) + x_0, y(t) + y_0) \) is a line. \( \vec{f}'(\vec{r}(t)) = \vec{f}'(\vec{r}(t)) \cdot \vec{r}'(t) = \frac{\vec{r}'(t)}{\vec{r}'(t)} + \vec{r}'(t) = (a \vec{r}'(t) + b \vec{r}'(t)) = a \vec{r}'(t) + b \vec{r}'(t) \). This is the dot product of \( \vec{f}'(\vec{r}) \) with \( \vec{r}'(t) \).

WHERE IS THE CHAIN RULE NEEDED?

While the chain rule is useful in calculations using the composition of functions, the iteration of maps or in doing change of variables, it is also useful for understanding some theoretical aspects. Examples:

- In the proof of the fact that gradients are orthogonal to level surfaces. (see the Wednesday lecture).
- It appears in change of variable formulas.
- It will be used in the fundamental theorem for line integrals coming up later in the course.
- The chain rule illustrates also the Lagrange multiplier method which we will see later.
- In fluid dynamics, PDE’s often involve terms \( u_t + u \nabla u \) which give the change of the velocity in the frame of a fluid particle.
- In chaos theory, where one wants to understand what happens after iterating a map.

APPLICATION: If \( f(x, y, z) = 0 \), then \( x = x(y, z), \quad y = y(x, z) \). From \( y_z = -f_z/f_y, \quad x_z = -f_z/f_x \), we get the relation \( x_z z_x + y_z z_y = -1 \). This formula appears in thermodynamics.

EXAMPLE. GRADIENT IN POLAR COORDINATES. In polar coordinates, the gradient is defined as \( \nabla f = (f_r, f_\theta) \).

Using the chain rule, we can relate this to the usual gradient: \( d/dr f(r, \theta) = f_r(r, \theta) \cos(\theta) + f_\theta(r, \theta) \sin(\theta) \) and \( d/d\theta f(r(x, \theta), y(r, \theta)) = -f_x(r, y) \sin(\theta) + f_y(r, x) \cos(\theta) \) means that the length of \( \nabla f \) is the same in both coordinate systems.