1 Column and Null Spaces

1. Note that set (a) in Question 3 are sets we have seen before. The set of all linear combinations of the columns of a matrix \( A \) is the range of the linear transformation \( \mathbf{x} \mapsto A\mathbf{x} \). We call such a set a set the column space of \( A \).

2. **Definition:** The column space of a matrix \( A \) is the set of all linear combinations of the columns of \( A \). We denote this set by \( \text{Col} A \).

3. Our reasoning that set (a) is a subspace of \( \mathbb{R}^3 \) extends to any column space.

4. **Theorem:** The column space of a \( m \times n \) matrix is a subspace of \( \mathbb{R}^m \).

5. Our reasoning that set (a) is a subspace of \( \mathbb{R}^3 \) also extends to any set of the form \( \text{Span}\{v_1, v_2, \ldots, v_p\} \).

6. **Theorem:** For \( v_1, v_2, \ldots, v_p \) in \( \mathbb{R}^n \), the set \( \text{Span}\{v_1, v_2, \ldots, v_p\} \) is a subspace of \( \mathbb{R}^n \). We call this set the subspace spanned (or generated) by \( v_1, v_2, \ldots, v_p \).

7. Note we have also seen sets like set (b) in Question 3, the set of all solutions to a homogeneous matrix equation. We can view this set as the set of all vectors in the domain of the linear transformation \( \mathbf{x} \mapsto A\mathbf{x} \) that are mapped to the zero vector. We call such a set the null space of \( A \).

8. **Definition:** The null space of a matrix \( A \) is the set of all solutions to the homogeneous equation \( A\mathbf{x} = \mathbf{0} \). We denote this set by \( \text{Nul} A \).

9. Our reasoning that set (b) is a subspace of \( \mathbb{R}^3 \) extends to any null space.

10. **Theorem 12:** The null space of a \( m \times n \) matrix is a subspace of \( \mathbb{R}^n \).

2 Bases

1. **Definition:** A basis for a subspace \( H \) of \( \mathbb{R}^n \) is a linearly independent set in \( H \) that spans \( H \).

2. See Question 4 and Solution 4 at this point.

3. **Problem:** Find a basis for the null space of the matrix \( A = \begin{bmatrix} 1 & 8 & -2 \\ 2 & -8 & 4 \\ 4 & 8 & 0 \end{bmatrix} \).

4. In general, writing the solution set of \( A\mathbf{x} = \mathbf{0} \) in parametric vector form identifies a basis for \( \text{Nul} A \). See Example 5 on page 171 for a good example.

5. **Problem:** Find a basis for the column space of the matrix \( A = \begin{bmatrix} 1 & 8 & -2 \\ 2 & -8 & 4 \\ 4 & 8 & 0 \end{bmatrix} \).

6. Note that when \( A \) is row reduced to echelon form \( B \), the columns are drastically changed, but the equations \( A\mathbf{x} = \mathbf{0} \) and \( B\mathbf{x} = \mathbf{0} \) have the same set of solutions. That is, the columns of \( A \) have exactly the same linear dependence relationships as the columns of \( B \). This gives us the following theorem.

7. **Theorem 13:** The pivot columns of a matrix \( A \) form a basis for the column space of \( A \).