Remarks on Solution of Problem 2
of Homework Assigned on October 26
2006 due November 7, 2006

Problem 2 (elliptic integrals and conformal mapping from the upper half-plane
to a rectangle). Let $0 < k < 1$. Let

$$K = \int_{t=0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$
$$K' = \int_{t=1}^{\frac{1}{k}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Let $R$ be the open rectangle with vertices at

$$K, \quad K + iK', \quad -K + iK', \quad -K.$$

Let $g(\zeta)$ be a branch for the function

$$\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}$$

on the upper half-plane. Consider the Schwarz-Christoffel transformation

$$w = \int_{z}^{\zeta} \frac{d\zeta}{g(\zeta)}.$$

(a) Verify that for some choice of the branch $g(\zeta)$ the Schwarz-Christoffel
transformation maps the upper half-plane in the $z$ variable one-one onto the
rectangle $R$ in the $w$ variable.

(b) Describe how that particular branch of $g(\zeta)$ is defined (i.e., what cuts
have to be made in $\mathbb{C}$ and what the ranges of the numerical values of the
angles in polar representations are).

(c) Describe the correspondence between the quadruple $\{1, -1, \frac{1}{k}, -\frac{1}{k}\}$ of
points in the $z$ variable and the four vertices of $R$ in the $w$ variable (i.e.,
which point in the quadruple goes to which vertex of $R$).

Remarks on Solution. The function $g(\zeta)$ can be written as

$$g(\zeta) = k \left( \zeta - \left( -\frac{1}{k} \right) \right)^{\frac{1}{2}} \left( \zeta - (-1) \right)^{\frac{1}{2}} \left( \zeta - 1 \right)^{\frac{1}{2}} \left( \zeta - 1 \frac{1}{k} \right)^{\frac{1}{2}}.$$
Let
\[ \zeta - \left( -\frac{1}{k} \right) = r_{-\frac{1}{k}}^\theta \, e^{-\frac{i\theta}{k}}, \]
\[ \zeta - (-1) = r_{-1}^\theta \, e^{i\theta_{-1}}, \]
\[ \zeta - 1 = r_1^\theta \, e^{i\theta_1}, \]
\[ \zeta - \frac{1}{k} = r_{\frac{1}{k}}^\theta \, e^{i\theta_{\frac{1}{k}}}. \]

For Part (b), the cuts will be given by
\[ 0 \leq \theta_{-\frac{1}{k}} \leq \pi, \]
\[ 0 \leq \theta_{-1} \leq \pi, \]
\[ 2\pi \leq \theta_1 \leq 3\pi, \]
\[ 0 \leq \theta_{\frac{1}{k}} \leq \pi. \]

Note the special choice of the range for the numerical value for the angle \( \theta_1 \) which is different from the other three. The reason is that we have to make sure that when \( \zeta \in (0, 1) \) the value of \( g(\zeta) \) should be positive. For our choice of the four ranges, when \( \zeta \in (0, 1) \) we get
\[ \theta_{-\frac{1}{k}} = 0, \]
\[ \theta_{-1} = 0, \]
\[ \theta_{-1} = 3\pi, \]
\[ \theta_{\frac{1}{k}} = \pi \]

and
\[ g(\zeta) = \sqrt{r_{-\frac{1}{k}} r_{-1} r_1} \, e^{\frac{i}{2} \left( \theta_{-\frac{1}{k}} + \theta_{-1} + \theta_1 + \theta_{\frac{1}{k}} \right)} \]
\[ = \left| \zeta - \left( -\frac{1}{k} \right) \right|^\frac{1}{2} \left| \zeta - (-1) \right|^\frac{1}{2} \left| \zeta - 1 \right|^\frac{1}{2} \left| \zeta - \frac{1}{k} \right|^\frac{1}{2} e^{i2\pi} > 0. \]