Problem 1 (fluid flow in a channel through a slit). Let $Q$ be a positive number. (a) Find a bounded harmonic function $u$ on the upper half-plane so that

(i) its boundary value on $-\infty < x < 0$ is $\frac{Q}{2}$,

(ii) its boundary value on $0 < x < 1$ is $Q$, and

(iii) its boundary value on $1 < x < \infty$ is 0.

Hint: Use the construction technique for the integrand of a Schwarz-Christoffel transformation.

(b) Find a bounded harmonic function $\varphi$ on the strip $\{0 < y < \pi\}$ such that

(i) its boundary value on $\{y = \pi\}$ is $\frac{Q}{2}$,

(ii) its boundary value on $\{x < 0, y = 0\}$ is $Q$, and

(iii) its boundary value on $\{x > 0, y = 0\}$ is 0.

Hint: Find $\varphi$ as the function $u$ in (a) composed with a linear fractional transformation and the exponential function.

(c) Show that the curves $\{\varphi(x, y) = \text{constant}\}$ are given by $\tan \frac{y}{2} = c \tanh \frac{x}{2}$ for some constant $c$. Hint: Use $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$.

(d) Interpret (b) and (c) as solving the following problem of fluid flow. There is a 2-dimensional steady irrotational incompressible fluid flow of constant density in the channel represented by the strip $\{0 < y < \pi\}$. The fluid enters through a slit represented by the origin at the rate of $Q$ units per unit time so that the flow exits at each end of the channel (represented by $x = -\infty$ and $x = \infty$) at the rate of $\frac{Q}{2}$ units per unit time. Show that the equation of a streamline is given by $\tan \frac{y}{2} = c \tanh \frac{x}{2}$ for some constant $c$.

Problem 2 (elliptic integrals and conformal mapping from the upper half-plane to a rectangle). Let $0 < k < 1$. Let

$$K = \int_{t=0}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$
\[ iK' = \int_{t=1}^{\frac{\pi}{2}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \]

Let \( R \) be the open rectangle with vertices at
\[ K, \quad K + iK', \quad -K + iK', \quad -K. \]

Let \( g(\zeta) \) be a branch for the function
\[ \sqrt{(1-\zeta^2)(1-k^2\zeta^2)} \]
on the upper half-plane. Consider the Schwarz-Christoffel transformation
\[ w = \int_0^z \frac{d\zeta}{g(\zeta)}. \]

(a) Verify that for some choice of the branch \( g(\zeta) \) the Schwarz-Christoffel transformation maps the upper half-plane in the \( z \) variable one-one onto the rectangle \( R \) in the \( w \) variable.

(b) Describe how that particular branch of \( g(\zeta) \) is defined (i.e., what cuts have to be made in \( \mathbb{C} \) and what the ranges of the numerical values of the angles in polar representations are).

(c) Describe the correspondence between the quadruple \( \{1, -1, \frac{1}{k}, -\frac{1}{k}\} \) of points in the \( z \) variable and the four vertices of \( R \) in the \( w \) variable (i.e., which point in the quadruple goes to which vertex of \( R \)).

The definite integrals \( K \) and \( K' \) are known as complete elliptic integrals of the first kind.

**Problem 3.** Let \( h > 0 \). Let \( \Omega \) be the domain in \( \mathbb{C} \) with variable \( w = u + iv \) obtained by removing the rectangle \( \{0 < v \leq h, u \leq 0\} \) from the open upper half-plane \( \{v > 0\} \). Consider the Schwarz-Christoffel transformation from the open upper half-plane in the \( z \) variable to the domain \( \Omega \) in the \( w \) variable whose derivative \( \frac{dw}{dz} \) is given by
\[ \frac{dw}{dz} = A \left( \frac{z + 1}{z - 1} \right)^{\frac{1}{2}}, \]
where $A$ is a nonzero complex number. Verify that for some nonconstant complex number $A$ the Schwarz-Christoffel transformation can be written in the following form

$$w = \frac{h}{\pi} \left((z + 1)^{\frac{1}{3}} (z - 1)^{\frac{1}{3}} + \log \left( z + (z + 1)^{\frac{1}{2}} (z - 1)^{\frac{1}{2}} \right) \right),$$

where

(i) the branch of $(z + 1)^{\frac{1}{3}}$ is chosen with $0 \leq \arg(z + 1) \leq \pi$,

(ii) the branch of $(z - 1)^{\frac{1}{3}}$ is chosen with $0 \leq \arg(z - 1) \leq \pi$, and

(iii) the branch of log is the principal branch with the argument defined between $-\pi$ and $\pi$.

Moreover, verify that that particular Schwarz-Christoffel transformation maps

(i) the interval $(-\infty, -1]$ in the $z$ variable to the line-segment

$$\{ -\infty < u \leq 0, \ v = h \}$$

in the $w$ variable.

(ii) the interval $[-1, 1]$ in the $z$ variable to the line-segment

$$\{ u = 0, \ 0 \leq v \leq h \}$$

in the $w$ variable, and

(iii) the interval $[1, \infty)$ in the $z$ variable to the line-segment $[0, \infty)$ in the $w$ variable.

**Problem 4.** Show that, if $\alpha$ and $\beta \neq 0$ are real numbers, the equation

$$z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$$

has $n - 1$ roots with positive real parts if $n$ is odd, and $n$ roots with positive real parts if $n$ is even.

**Hint:** Apply the argument principle to the right half-disk of radius $R$ and let $R \to \infty$. 