General Variation Formula and Weierstrass-Erdmann Corner Condition

**General Variation Formula.** We take the variation of the functional

$$J = \int_{x_1}^{x_2} F(x, y, y') \, dx.$$  

with the two end-points \((x_1, y_1), (x_2, y_2)\) allowed freely to vary. Suppose we have a family curves \(y = y(x, t)\) parametrized by \(t\) so that the extremal corresponds to \(t = 0\). The new setting in our situation is that the end-points \(x_1\) and \(x_2\) are not independent of \(t\). In fact, \(x_1 = x_1(t)\) and \(x_2 = x_2(t)\) are both functions of \(t\). So the functional \(J\) as a function of \(t\) can be written as

$$J(t) = \int_{x_1(t)}^{x_2(t)} F\left(x, y(x, t), \frac{\partial}{\partial x} y(x, t)\right) \, dx.$$  

Before we differentiate the functional \(J(t)\) with respect to \(t\), we first discuss the two variations of \(y\) with respect to \(t\), one with \(x\) fixed and the other with \(x\) also varying as a function of \(t\). By the chain rule applied to \(y = y(x(t), t)\), we get

$$\frac{d}{dt} y(x(t), t) = \frac{\partial}{\partial t} y(x(t), t) + \frac{\partial}{\partial x} y(x(t), t) \frac{d}{dt} x(t).$$  

We denote the left-hand side by \(\delta y\), which represents the actual variation of an end-point so that if we require the end-point to lie on a prescribed curve it is this variation \(\delta y\) which will be used. The second factor \(\frac{d}{dt} x(t)\) of the second term on the right-hand side is \(\delta x\). The first term on the right-hand we will denote by \(\partial_t y\) so that

\[
\text{(†) } \partial_t y = \delta y - y' \delta x.
\]

By applying the Fundamental Theorem of Calculus to the variation of the upper limit and the lower limit of the integral functional, we get

\[
\text{(*) } \delta J = F(x, y, y') \delta x \bigg|_{x=x_1}^{x=x_2} + \int_{x_1}^{x_2} (\partial_t F(x, y, y')) \, dx.
\]

The second term on the right-hand side can be dealt with in the same way as the simpler situation of the end-points being fixed and we obtain

$$\int_{x_1}^{x_2} (\partial_t F(x, y, y')) \, dx = \int_{x_1}^{x_2} \left(F_y \partial_t y + F_{y'} \partial_t y'\right) \, dx.$$
which by integration by parts gives
\[ \int_{x_1}^{x_2} \left( F_y \partial_t y + F_y' (\partial_t y)' \right) dx = \int_{x_1}^{x_2} \left( F_y - \frac{d}{dx} F_y' \right) (\partial_t y) dx + F_y' \partial_t y \bigg|_{x=x_2}^{x=x_1}. \]

Now we combine together (†), (∗), and (♮) to get
\[ \delta J = \int_{x_1}^{x_2} \left( F_y - \frac{d}{dx} F_y' \right) (\partial_t y) dx + F_y' \delta y \bigg|_{x=x_1}^{x=x_2} + (F - y' F_y') \delta x \bigg|_{x=x_1}^{x=x_2}. \]

This is known as the general variation formula.

**Condition for End-Point to be on Prescribed Curve.** Suppose the end-point \((x_j, y_j) (j = 1, 2)\) is constrained to be on the prescribed curve \(g_j (x, y)\). First by considering the variation with the two end-points fixed we get the Euler-Lagrange equation to conclude from the general variation formula that
\[ \delta J = F_y' \delta y \bigg|_{x=x_1}^{x=x_2} + (F - y' F_y') \delta x \bigg|_{x=x_1}^{x=x_2}. \]

By differentiating the equation \(g_j (x_j, y_j) = 0\) with respect to the parameter \(t\) of the family of curves, we get
\[ \frac{\partial g_j}{\partial x} (\delta x \bigg|_{x=x_j}) + \frac{\partial g_j}{\partial y} (\delta y \bigg|_{x=x_j}) = 0 \]
which can be rewritten as
\[ (\delta y \bigg|_{x=x_j}) = - \frac{\partial g_j}{\partial x} (\delta x \bigg|_{x=x_j}). \]

When we put this into (‡), from the vanishing of \(\delta J\) we get
\[ 0 = F_y' \left( - \frac{\partial g_j}{\partial x} \delta x \bigg|_{x=x_j} \right) - F_y' \left. \left( - \frac{\partial g_j}{\partial x} \delta x \bigg|_{x=x_j} \right) \right|_{x=x_2}^{x=x_1} + (F - y' F_y') \delta x \bigg|_{x=x_1}^{x=x_2}. \]

Since \(\delta x\) is allowed to vary freely both at \(x = x_1\) and \(x = x_2\), it follows that
\[ F_y' \left( - \frac{\partial g_j}{\partial x} \right) + (F - y' F_y') = 0 \]
at \(x = x_j\) for \(j = 1, 2\), which can be written as
\[ \frac{\partial g_j}{\partial x} F_y' = \frac{\partial g_j}{\partial y} (F - y' F_y') \]
at \(x = x_j\) for \(j = 1, 2\).
Weierstrass-Erdmann Corner Condition. Suppose our functional $J$ is given as the sum of two integrals

$$J = \int_{x=x_1}^{x=\xi} F(x, y, y') \, dx + \int_{x=\xi}^{x=x_2} F(x, y, y') \, dx$$

so that the two end-points $y_1 = y(x_1)$ and $y_2 = y(x_2)$ but the ordinate $\eta = y(\xi)$ of the middle point with abscissa $x = \xi$ is allowed to vary freely.

Then we can first fix the middle point and vary the two integrals separately to get one Euler-Lagrange equation on each of the two intervals $[x_1, \xi]$ and $[\xi, x_2]$. Then we consider the condition imposed by the free variation of the middle point $(\xi, \eta)$ to get

$$\delta J = F_y' \delta y \bigg|_{x=\xi} + (F - y' F_y') \delta x \bigg|_{x=\xi} + F_y' \delta y \bigg|_{x=\xi} + (F - y' F_y') \delta x \bigg|_{x=\xi}$$

which, on account of the fixing of the two end-points $(x_1, y_1)$ and $(x_2, y_2)$, becomes

$$\delta J = \left( F_y' \bigg|_{x=\xi-0} \right) \left( \delta y \bigg|_{x=\xi} \right) + \left( (F - y' F_y') \bigg|_{x=\xi-0} \right) \left( \delta x \bigg|_{x=\xi} \right).$$

If both $\delta y \bigg|_{x=\xi}$ and $\delta x \bigg|_{x=\xi}$ are allowed to vary freely, the vanishing of $\delta J$ implies the following two statements which are known as the Weierstrass-Erdmann corner condition

$$F_y' \bigg|_{x=\xi-0} = F_y' \bigg|_{x=\xi+0} \quad \text{and} \quad (F - y' F_y') \bigg|_{x=\xi-0} = (F - y' F_y') \bigg|_{x=\xi+0}.$$

In other words, the two canonical coordinates $p = F_y'$ and $H = y' F_y' - F$ are continuous at the corner if the middle point is allowed to vary freely.

If we have the free variation of only $\delta y \bigg|_{x=\xi}$, then we have only the partial Weierstrass-Erdmann corner condition

$$F_y' \bigg|_{x=\xi-0} = F_y' \bigg|_{x=\xi+0}.$$
On the other hand, if we have the free variation of only \( \delta x \bigg|_{x=\xi} \), then we have only the partial Weierstrass-Erdmann corner condition

\[
(F - y' F_{y'}) \bigg|_{x=\xi-0} = (F - y' F_{y'}) \bigg|_{x=\xi+0}.
\]

In general, if we have the free variation of only \( \delta x \bigg|_{x=\xi} \) and \( \delta y \bigg|_{x=\xi} \) on a prescribed curve \( g(x, y) = 0 \), then we have

\[
\delta x \bigg|_{x=\xi} g_x + \delta y \bigg|_{x=\xi} g_y = 0
\]

and the partial Weierstrass-Erdmann corner condition

\[
-g_x F_{y'} + g_y (F - y' F_{y'}) \bigg|_{x=\xi-0} = -g_x F_{y'} + g_y (F - y' F_{y'}) \bigg|_{x=\xi+0}.
\]

**Snell’s Law as Application of the Weierstrass-Erdmann Corner Condition.**

Consider the problem of light traveling in two different media where the velocity of light is \( c_1 \) and \( c_2 \) respectively. The light trajectory is the extremal of the functional

\[
J = \int_{x=x_1}^{x=\xi} F_1(x, y, y') \, dx + \int_{x=\xi}^{x=x_2} F_2(x, y, y') \, dx,
\]

where

\[
F_j = \frac{\sqrt{1 + y'^2}}{c_j}
\]

for \( j = 1, 2 \). First we consider the case of fixed the middle point (where the corner occurs) to get the two Euler-Lagrange equations

\[
(F_j)_y - \frac{d}{dx} (F_j) y' = 0.
\]

Since each \( F_j \) is independent of \( x \), we have the first integral which is the analog of the conservation of energy

\[
H_j = y' (F_j)_{y'} - F_j = \text{constant}.
\]

Explicit computation gives

\[
H_j = y' (F_j)_{y'} - F_j = \frac{1}{c_j} \frac{y'^2}{\sqrt{1 + y'^2}} - \frac{\sqrt{1 + y'^2}}{c_j} = \frac{-1}{c_j} \frac{1}{c_j} \frac{1}{\sqrt{1 + y'^2}}.
\]
This means that light travels in a straight line in each of the two media. We now assume that the interface is given by $x = \xi$ with $\xi$ fixed. Then only $\delta y$ is allowed to vary freely at $x = \xi$. At the interface, the canonical coordinate $p$ is continuous. That is,

$$(F_1)_{y'} \big|_{x=\xi-0} = (F_2)_{y'} \big|_{x=\xi+0},$$

which means

$$\frac{1}{c_1} \frac{y'}{\sqrt{1 + y'^2}} \bigg|_{x=\xi-0} = \frac{1}{c_2} \frac{y'}{\sqrt{1 + y'^2}} \bigg|_{x=\xi+0}. $$

Introduce the angle of incidence $\theta_1$ and the angle of refraction $\theta_2$ given by

$$y' \big|_{x=\xi-0} = \tan \theta_1 \quad \text{and} \quad y' \big|_{x=\xi+0} = \tan \theta_2.$$

Then we can write the above condition in the form of Snell’s law for the refraction of light which states that

$$\frac{\sin \theta_1}{c_1} = \frac{\sin \theta_2}{c_2}.$$