Definite Integrals Evaluated by Contour Integration Over a Half Circle.

We are going to use Cauchy’s residue theory over the boundary of a half disk to evaluate definite integrals of the following form.

\[ \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, dx \quad \text{with} \quad \deg P \leq \deg Q - 2, \]
\[ \int_{-\infty}^{\infty} \frac{P(x) \cos x}{Q(x)} \, dx \quad \text{with} \quad \deg P \leq \deg Q - 1, \]
\[ \int_{-\infty}^{\infty} \frac{P(x) \sin x}{Q(x)} \, dx \quad \text{with} \quad \deg P \leq \deg Q - 1, \]

Here the polynomials \( P(x), Q(x) \) have real coefficients and are relatively prime. For the first integral the polynomial \( Q(x) \) does not have a real zero. For the second integral the zeroes of the polynomial \( Q(x) \) are at most of order one and are contained in the zero-set of \( \cos x \). For the third integral the zeroes of the polynomial \( Q(x) \) are at most of order one and are contained in the zero-set of \( \sin x \). The integrals are computed by using the following residue theorem.

**Theorem (Residues).** Let \( D \) be a bounded domain in \( \mathbb{C} \) with piecewise smooth boundary \( C \). Let \( f(z) \) be a meromorphic function on \( D \) which near the boundary of \( D \) is continuous up to the boundary of \( D \). Then

\[ \oint_C f(z) \, dz = 2\pi i \sum_{a \in D} \text{Res}_a f, \]

where \( \text{Res}_a f \) is the residue of \( f \) at \( a \).

**Proof.** Let the poles of \( f \) in \( D \) be \( a_1, \cdots, a_k \). Let \( D_1, \cdots, D_k \) be disjoint closed disks inside \( D \) such that \( D_j \) is centered at \( a_j \) for \( 1 \leq j \leq k \). Let \( C_j \) be the boundary of \( D_j \) in the counterclockwise sense. By the theorem of Cauchy-Goursat

\[ \oint_C f(z) \, dz = \sum_{j=1}^{k} \oint_{C_j} f(z) \, dz = 2\pi i \sum_{j=1}^{k} \text{Res}_{a_j} f. \]

Q.E.D.

For the evaluation of the above definite integrals when \( \cos x \) or \( \sin x \) appear and \( Q(x) \) has some real zeroes, we need the following notion of a half-residue.
Definition. Let \( f(z) \) be a holomorphic function on the punctured disk

\[ \{ z \in \mathbb{C} \mid 0 < |z - a| < R \} \]

(where \( a \in \mathbb{C} \) and \( R > 0 \)) with a simple pole at \( a \). Let \( \alpha > 0 \) and \( C_{r,\alpha} \) be the half circle

\[ \{ z = a + re^{i\theta} \mid \alpha \leq \theta \leq \alpha \pi \} \]

in the counterclockwise sense for \( r > 0 \). The half-residue of \( f \) at \( a \) is defined as

\[ \frac{1}{2\pi i} \lim_{r \to 0} \int_{C_{r,\alpha}} f(z)dz \]

and is equal to \( \frac{1}{2} \text{Res}_a f \) which is independent of the choice of \( \alpha \).

The verification of

\[ \lim_{r \to 0} \frac{1}{2\pi i} \int_{C_{r,\alpha}} f(z)dz = \frac{1}{2} \text{Res}_a f \]

follows from writing

\[ f(z) = \frac{c_{-1}}{z - a} + g(z) \]

with \( g(z) \) holomorphic at \( z \) and from using the parametrization \( \theta \mapsto a + re^{i\theta} \) \((\alpha \leq \theta \leq \alpha + \pi)\) for \( C_{r,\alpha} \) to evaluate

\[ \int_{C_{r,\alpha}} f(z)dz = c_{-1} \int_{C_{r,\alpha}} \frac{dz}{z - a} + \int_{C_{r,\alpha}} g(z)dz. \]

From

\[ \int_{C_{r,\alpha}} \frac{dz}{z - a} = \int_{\theta = \alpha}^{\alpha + \pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = \pi i \]

and

\[ \lim_{r \to 0} \int_{C_{r,\alpha}} g(z)dz = 0 \]

it follows that

\[ \frac{1}{2\pi i} \int_{C_{r,\alpha}} f(z)dz = \frac{1}{2} c_{-1} = \frac{1}{2} \text{Res}_a f. \]

**Integrals of Rational Functions over the Real Line.** For

\[ \int_{x=-\infty}^{\infty} \frac{P(x)dx}{Q(x)} , \]
the integral of \( \frac{P(z)}{Q(z)} \) over the contour of the boundary of the upper half disk of radius \( R \) centered at the origin as \( R \to \infty \) yields

\[
\int_{x=-\infty}^{\infty} \frac{P(x)dx}{Q(x)} = 2\pi i \sum_{\mathrm{Res}_z} \frac{P(z)}{Q(z)}.
\]

The integral \( \int_{C_R} \frac{P(z)}{Q(z)} dz \) over the half-circle

\[ C_R := \{ z \in \mathbb{C} \mid z = Re^{i\theta}, 0 \leq \theta \leq 2\pi \} \]

of the meromorphic function \( \frac{P(z)}{Q(z)} \) goes to zero as \( R \to \infty \), because

\[
\sup_{z \in C_R} \left| \frac{P(z)}{Q(z)} \right| = O \left( \frac{1}{R^2} \right)
\]

(where \( O(u) \) is the Landau symbol meaning that the quotient by \( u \) is bounded by a constant as \( R \to \infty \)) and the length of \( C_R \) is \( O(R) \) and as a consequence

\[
\left| \int_{C_R} \frac{P(z)}{Q(z)} dz \right| \leq \left( \sup_{z \in C_R} \left| \frac{P(z)}{Q(z)} \right| \right) \text{(length of } CR) \]

\[
= O \left( \frac{1}{R^2} \cdot R \right) = O \left( \frac{1}{R} \right) \to 0 \text{ as } R \to 0.
\]

Moreover, by the residue theorem

\[
\int_{C_R} \frac{P(z)}{Q(z)} dz + \int_{x=-R}^{R} \frac{P(x)}{Q(x)} = 2\pi i \sum_{\mathrm{Res}_z} \frac{P(z)}{Q(z)} = 2\pi i \sum_{\mathrm{Im} z > 0} \frac{P(z)}{Q(z)}
\]

which yields the formula (‡‡) as \( R \to \infty \).

**Example.** We now compute

\[
\int_{x=-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^3} \quad (a > 0).
\]
We use the meromorphic function
\[ f(z) := \frac{1}{(z^2 + a^2)^3} \]
for which there is only one point in the upper half-plane with nonzero residue. That point is \(i\) which is a pole of order 3 and the residue at it is given by
\[
\frac{1}{2!} \left( \frac{d^2}{dz^2} \frac{(z - ai)^3}{(z + ai)^3} \right)_{z=ai} = \frac{1}{2} \left( \frac{d^2}{dz^2} \frac{1}{(z + ai)^3} \right)_{z=ai} = \frac{1}{2} \left( \frac{(-3)(-4)}{(z + ai)^5} \right)_{z=ai} = \frac{3}{16a^5i}.
\]
We thus conclude from the above formula that
\[
\int_{x=-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^3} = \frac{3\pi}{8a^5}.
\]

**Integrals of the Product of a Rational Function and Sine or Cosine Function over the Real Line.** For
\[
\int_{x=-\infty}^{\infty} \frac{P(x) \cos x}{Q(x)} \]
and
\[
\int_{x=-\infty}^{\infty} \frac{P(x) \sin x}{Q(x)},
\]
the integral of
\[
\frac{P(z)e^{iz}}{Q(z)}
\]
over the contour of the boundary of the upper half disk of radius \(R\) centered at the origin as \(R \to \infty\) yields
\[
\int_{x=-\infty}^{\infty} \frac{P(x) \cos x}{Q(x)} = \text{Re} \left( 2\pi i \sum_{\text{Im } z > 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\text{Im } z = 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} \right),
\]
\[
\int_{x=-\infty}^{\infty} \frac{P(x) \sin x}{Q(x)} = \text{Im} \left( 2\pi i \sum_{\text{Im } z > 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\text{Im } z = 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} \right).
\]
For this computation the following two new ingredients have to be incorporated.
(i) Since the degree of $Q(z)$ may only be one more than that of $P(z)$, to make sure that
\[
\int_{C_R} \frac{P(z)e^{iz}}{Q(z)} dz \to 0 \text{ as } R \to 0
\]
we have to do one integration by parts by integrating the factor to get $e^{iz}$ first
\[
\int_{C_R} \frac{P(z)e^{iz}}{Q(z)} dz
\]
\[
= \frac{P(z)e^{iz}}{iQ(z)} \bigg|_{z=-R}^{R} - \int_{C_R} \left( \frac{d}{dz} \frac{P(z)}{Q(z)} \right) e^{iz} dz
\]
\[
= \frac{P(z)e^{iz}}{iQ(z)} \bigg|_{z=-R}^{R} - \int_{C_R} \frac{(P'(z)Q(z) - P(z)Q'(z))}{Q(z)^2} e^{iz} dz
\]
and then use
\[
\left| \frac{(P'(z)Q(z) - P(z)Q'(z))}{Q(z)^2} \right| = O \left( \frac{1}{R^2} \right)
\]
(from the degree of $P'(z)Q(z) - P(z)Q'(z)$ no more than the degree of $Q(z)^2$ minus 2) and also use
\[
|e^{iz}| = e^{-\text{Im} z} \leq 1
\]
(from $\text{Im} z > 0$ on $C_R$).

(ii) For zero $x_0$ of $Q(x)$ on the real line $\mathbb{R}$ we have to modify the contour $\mathbb{R} + C_R$ by replacing $[x_0 - r, x_0 + r]$ by the lower half-circle
\[
C_{r,x_0} := \{ z \in \mathbb{C} \mid z = x_0 + re^{i\theta}, -\pi \leq \theta \leq \pi \}
\]
of radius $r > 0$ centered at $x_0$ in the counterclockwise sense. We label the real roots of $Q(x)$ as $\{x_j\}_j$ and choose the index $j$ such that
\[
\{ x \in \mathbb{R} \mid Q(x) = 0, -R \leq x \leq R \} = \{x_j\}_{j \in J_R}
\]
for $R > 0$. From the residue theorem
\[
\int_{C_{r,x_0}} \frac{P(z)e^{iz}}{Q(z)} dz + \int_{[-R,R] - \bigcup_{1 \leq j \leq J_R} [x_j - r, x_j + r]} \frac{P(x)e^{ix}}{Q(x)} dx + \sum_{j=1}^{J_R} \int_{C_{r,x_j}} \frac{P(z)e^{iz}}{Q(z)} dz
\]
is equal to
\[ 2\pi i \sum_{|z|<R, \ Im \ z \geq 0} \text{Res}_z \frac{P(z)}{Q(z)}. \]

Finally we get our two formulas
\[
\int_{x=-\infty}^{\infty} \frac{P(x) \cos x \, dx}{Q(x)} = \text{Re} \left( 2\pi i \sum_{\text{Im} \ z > 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\text{Im} \ z = 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} \right),
\]
\[
\int_{x=-\infty}^{\infty} \frac{P(x) \sin x \, dx}{Q(x)} = \text{Im} \left( 2\pi i \sum_{\text{Im} \ z > 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} + \pi i \sum_{\text{Im} \ z = 0} \text{Res}_z \frac{P(z)e^{iz}}{Q(z)} \right)
\]

by letting \( R \to \infty \) and \( r \to 0 \) and using the half-residue theorem
\[
\lim_{r \to 0} \int_{C_r,x_j} \frac{P(z)e^{iz}}{Q(z)} \, dz = \pi i \text{Res}_{x_j} \frac{P(z)e^{iz}}{Q(z)}
\]
for every real root \( x_j \) of \( Q(x) \) and then taking the real and imaginary parts of both sides.

**Example.** We now compute
\[
\int_{x=-\infty}^{\infty} \frac{\sin x \, dx}{x}
\]
in the sense that it is the limit of
\[
\int_{x=-R}^{R} \frac{\sin x \, dx}{x}
\]
as \( R \to \infty \). The answer is, according to the above formula,
\[
\text{Im} \left( \pi i \text{Res}_{z=0} \frac{e^{iz}}{z} \right) = \pi.
\]