Canonical Transformation

Recall that the canonical equations (or Hamiltonian equations)

\[
\begin{align*}
\frac{dy}{dx} &= \frac{\partial H}{\partial p} \\
\frac{dp}{dx} &= -\frac{\partial H}{\partial y},
\end{align*}
\]

where \( p = F_{y'} \), and \( H = -F + y'p \) come from the reduction of the second-order Euler-Lagrange equation

\[
F_y - \frac{d}{dx} F_{y'} = 0
\]

to a system of first-order equations and putting it in a form with certain symmetry in \( y \) and \( p \) via the Legendre transformation. The first equation of (\( * \)) is the dual for the definition of \( p = F_{y'} \) (which gives the definition of \( y' \) in terms of \( H \) and \( p \)). The second equation of (\( * \)) is the rewriting of the Euler-Lagrange equation with \( F_{y'} \) replaced by \( p \).

The canonical equations (\( * \)) can also be obtained from the usual Euler-Lagrange equation from the extremal problem for the functional

\[
\int p \, dy - H \, dx = \int (p \, y' - H) \, dx
\]

with dependent functions \( p \) and \( y \) of the independent variable \( t \) and the function \( H(y, p, x) \). The usual Euler-Lagrange equation for (\( b \)) with two dependent functions \( p \) and \( y \) is

\[
\begin{align*}
\frac{\partial}{\partial p} (p \, y' - H(x, y, p)) - \frac{d}{dx} \left( \frac{\partial}{\partial p'} (p \, y' - H(x, y, p)) \right) &= 0, \\
\frac{\partial}{\partial y} (p \, y' - H(x, y, p)) - \frac{d}{dx} \left( \frac{\partial}{\partial y'} (p \, y' - H(x, y, p)) \right) &= 0,
\end{align*}
\]

which can be rewritten as

\[
\begin{align*}
y' - \frac{\partial}{\partial p} H(x, y, p) &= 0, \\
- \frac{\partial}{\partial y} H(x, y, p) - \frac{dp}{dx} &= 0.
\end{align*}
\]
Canonical Transformation and its Generating Function. A canonical transformation is a change of variables from \( y, p \) to \( Y, P \) so that for some function \( H^*(x,Y,P) \) the canonical equations (*) are equivalent to the following system of equations in \( Y, P \) of the same form

\[
\begin{align*}
\frac{dY}{dx} &= \frac{\partial H^*}{\partial P} \\
\frac{dP}{dx} &= -\frac{\partial H^*}{\partial Y}.
\end{align*}
\]

One way to construct a canonical transformation is to make the two differentials \( p\,dy - H\,dx \) and \( P\,dY - H\,dx \) differ by an exact differential \( d\Phi \) so that

\[
p\,dy - H\,dx = P\,dY - H\,dx + d\Phi.
\]

By collecting the terms involving \( dx \), we can rewrite the equation as

\[
d\Phi = p\,dy - P\,dY + (H^* - H)\,dx
\]

and get

\[
p = \frac{\partial \Phi}{\partial y}, \quad P = -\frac{\partial \Phi}{\partial Y}, \quad H^* = H + \frac{\partial \Phi}{\partial x}.
\]

This gives the transformations between \((y, p, H)\) and \((Y, P, H^*)\). The function \( \Phi = \Phi(x, y, Y) \) (which depends only on \( x, y, Y \)) is called the generating function of this canonical transformation. A variant form of (†) is

\[
d\Psi = p\,dy + Y\,dP + (H^* - H)\,dx,
\]

where \( \Psi = \Phi + PY \) is the generating function so that the canonical transformation is given by

\[
p = \frac{\partial \Psi}{\partial y}, \quad Y = \frac{\partial \Psi}{\partial P}, \quad H^* = H + \frac{\partial \Psi}{\partial x}.
\]

The generating function \( \Psi = \Psi(x, Y, P) \) in this variant form depends on the variables \( x, Y, P \).

Action and Hamilton-Jacobi Equation. Recall that originally our problem is the extremal problem for the functional

\[
J = \int_{x=x_1}^{x_2} F(x, y, y') \, dx
\]
and then we change it to the equivalent problem of finding the extremal for the functional
\[ \int (py' - H) \, dx, \]
where \( H = py' - F \) so that \( py' - H = F \). Let \( S(x_1, y_1, x_2, y_2) \) be the integral of
\[ J = \int_{x_1}^{x_2} F(x, y, y') \, dx \]
along the extremal \( y = y(x) \) with \( y_1 = y(x_1) \) and \( y_2 = y(x_2) \). Recall that the general variation formula is
\[ \delta J = \int_{x_1}^{x_2} \left( F_y - \frac{d}{dx} F_{y'} \right) (\partial_{t} y) \, dx + F_{y'} \delta y \bigg|_{x=x_1}^{x=x_2} + (F - y'F_{y'}) \delta x \bigg|_{x=x_1}^{x=x_2}. \]
We consider the special case when the variation is for a family of extremals. From the Euler-Lagrange equation for the extremal the integral
\[ \int_{x_1}^{x_2} \left( F_y - \frac{d}{dx} F_{y'} \right) (\partial_{t} y) \, dx \]
on the right-hand side of the general variation formula vanishes and we are left with
\[ \delta J = F_{y'} \delta y \bigg|_{x=x_1}^{x=x_2} + (F - y'F_{y'}) \delta x \bigg|_{x=x_1}^{x=x_2}, \]
which we can rewrite as
\[ \delta S = p \delta y \bigg|_{x=x_1}^{x=x_2} - H \delta x \bigg|_{x=x_1}^{x=x_2}, \]
because \( p = F_{y'} \) and \( H = y'F_{y'} - F \). This means that
\begin{equation}
(5) \quad dS = (p_2 dy_2 - H(x_2, y_2, p_2) \, dx_2) - (p_1 dy_1 - H(x_1, y_1, p_1) \, dx_1).
\end{equation}
Thus
\[
\begin{cases}
\frac{\partial S}{\partial x_2} = -H(x_2, y_2, p_2), \\
\frac{\partial S}{\partial y_2} = p_2, \\
\frac{\partial S}{\partial x_1} = H(x_1, y_1, p_1), \\
\frac{\partial S}{\partial y_1} = -p_1.
\end{cases}
\]
We now replace \((x_2, y_2)\) by \((x, y)\) and replace \((x_1, y_1)\) by \((a, b)\) and regard \(S\) as a function of \((x, y, a, b)\). Then
\[
\frac{\partial S}{\partial x} + H \left( x, y, \frac{\partial S}{\partial y} \right) = 0,
\]
which is known as the Jacobi-Hamilton equation. We now fix \(a\) so that \(S\) is a function of \((x, y, b)\) and we can rewrite \((\#)\) as
\[
\tag{\#}'
dS = pdy - H(x, y, p)\,dx - p_1\,db.
\]
We can set \(Y = b\) and use \(S\) in \((\#)'\) as the generating function for a canonical transformation and get \(p_1 = P\) and \(H^* = 0\). The vanishing of \(H^*\) means that \(P = \text{constant}\) is a solution of the canonical equations. Thus
\[
\frac{\partial S}{\partial b} = \text{constant}
\]
is a solution of the canonical equations.

As a matter of fact, even if we do not know where \(S\) comes from so that we cannot use the general variation formula, as long as we have the Jacobi-Hamilton equation \((\#)\) with \(S\) depending on a parameter in a nondegenerate manner whose meaning we will specify later, we can always define
\[
p = \frac{\partial S}{\partial y}
\]
and set \(Y = b\) and define
\[
-P = \frac{\partial S}{\partial b}
\]
to get
\[
dS = pdy - H(x, y, p)\,dx - P\,dY.
\]
When we use \(S\) as the generating function for the canonical transformation, we get \(H^* = 0\) and the solution of the canonical equation is given by both \(Y\) and \(P\) being constant. In other words,
\[
\tag{\%}
\frac{\partial S}{\partial Y} = P
\]
with \(Y\) and \(P\) constant would give us the equation in \(x, y, p\) as the solution for the canonical equation. We now explain precisely what we mean by the
dependence of $S$ on $b$ being in a nondegenerate manner. It is to make sure that the equation (\%) can be solved for $y$ as a function for $x$. By the implicit function theorem it suffices to assume that the partial derivative of the left-hand side of (\%) with respect to the dependent variable $y$ is nonzero, which is the same as saying that

$$\frac{\partial^2 S}{\partial b \partial y} \neq 0,$$

because $Y$ is defined to be $b$. 