In Class MidTerm Exam for Math 113
10 - 11:30 am, March 20, 2008

- Please show ALL your work on this exam paper. Partial credit will be awarded where appropriate.

- Some problems will be asking you to prove results covered in class. For those that do not, you may quote without proof the theorems/propositions/lemmas given during lecture as long as you state each result clearly. Points may be deducted for incorrect or missing hypotheses.

- NO calculators are permitted.

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1. (15 points) Simplify the following expressions:

   (a) $e^{\log i}$
   
   (b) $\log i$
   
   (c) $i^{\log(-1)}$

**SOLUTIONS:** This was Exercise 2.R.13.

   (a) $e^{\log i} = i$
   
   (b) $\log i = i\frac{\pi}{2} + i2\pi n = i\frac{\pi}{2}(1 + 4n), \quad n = 0, \pm 1, \pm 2, \ldots$
   
   (c) $i^{\log(-1)} = e^{\log(-1) \log i} = e^{i\pi\{i\frac{\pi}{2}(1 + 4n)\}} = e^{-\frac{\pi^2}{2}(1 + 4n)}, \quad n = 0, \pm 1, \pm 2, \ldots$
2. (15 points)

(a) Prove that the $n$th roots of unity can be expressed as $1, w, w^2, w^3, \ldots, w^{n-1}$.

(b) What is $w$?

(c) Show that the sum of the $n$th roots of unity is zero.

**SOLUTIONS:** Parts (a) and (b) were Exercise 1.3.27.

(a) We are looking for solutions to the equation

$$1^{1/n} = w.$$  

Equivalently $w^n = 1$. It is known from elementary algebra that this equation has at most $n$ distinct solutions.

Define $w = e^{2\pi i/n}$. We notice that for any $k = 0, \pm 1, \pm 2, \ldots$

$$(w^k)^n = (e^{2\pi ik/n})^n = e^{2\pi ik} = 1.$$  

However $1 = w^0, w, w^2, w^3, \ldots, w^{n-1}$ are distinct since $w^k = w^j$ implies that $e^{2\pi i(k-j)/n} = 1$ which implies $k = j \mod n$.

(b) $w = e^{2\pi i/n}$.

(c) The sum of the roots is

$$1 + w + \cdots + w^{n-1}.$$  

Notice $w^n - 1 = 0$. And $w^n - 1 = (w - 1)(1 + w + \cdots + w^{n-1})$. Either $n = 1$ or $n > 1$ and $w \neq 1$. In both cases we achieve the desired conclusion.
3. (a) (7 points) Is \( f(z) = \overline{z} \) analytic? Prove or disprove.

(b) (7 points) Let \( A \) be an open subset of \( \mathbb{C} \) and \( A^* = \{ z : \overline{z} \in A \} \). Suppose \( f \) is analytic on \( A \) and define a function \( g \) on \( A^* \) by

\[
g(z) = f(\overline{z}).
\]

Show that \( g \) is analytic on \( A^* \).

**SOLUTIONS:** This was Example 1.5.17 and Example 1.5.19.

(a) The function does not satisfy the Cauchy-Riemann equations.

(b) The proof is on page 75 of Marsden and Hoffman.
4. (15 points)

(a) Define what it means for a function to be analytic (holomorphic) at a point $z \in \mathbb{C}$.
(b) State the Cauchy-Riemann equations.
(c) Prove that a function analytic at a point $z \in \mathbb{C}$ satisfies the Cauchy-Riemann equations at that point.

SOLUTIONS:

(a) See Definition 1.5.1 in Marsden and Hoffman.
(b) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
(c) See the first half of the proof of Theorem 1.5.8.
Evaluate the following integrals with $\gamma$ being the unit circle centered at the origin:

(a) (3 points) $\int_{\gamma} \sin z \, dz$.

(b) (3 points) $\int_{\gamma} \frac{\sin z}{z} \, dz$.

(c) (3 points) $\int_{\gamma} \frac{\sin z}{z^2} \, dz$.

(d) (3 points) $\int_{\gamma} \frac{\sin(e^z)}{z} \, dz$.

SOLUTIONS: This was Exercise 2.R.1.

(a) (3 points) $\int_{\gamma} \sin z \, dz = 0$. (Cauchy’s Thm)

(b) (3 points) $\int_{\gamma} \frac{\sin z}{z} \, dz = 2\pi i \sin 0 = 0$. (Cauchy’s Integral formula)

(c) (3 points) $\int_{\gamma} \frac{\sin z}{z^2} \, dz = \frac{2\pi i}{1!} \cos 0 = 2\pi i$. (Cauchy’s Integral formula)

(d) (3 points) $\int_{\gamma} \frac{\sin(e^z)}{z} \, dz = 2\pi i \cos(e^0) = 2\pi i \cos(1)$. (Cauchy’s Integral formula)
6. (14 points) Evaluate $\int_0^{2\pi} e^{-i\theta} e^{i\theta} d\theta$.

**SOLUTIONS:** This was Exercise 2.R.11. We have $\int_0^{2\pi} e^{-i\theta} e^{i\theta} d\theta = 2\pi$.

The way to see this is to reverse engineer Cauchy’s integral formula. Consider $\gamma = \{z : |z| = 1\}$. Then

$$\frac{1!}{2\pi i} \int_{\gamma} e^{z} z^2 dz = I(0, \gamma) \left. \frac{de^{z}}{dz} \right|_{z=0} = 1,$$

which is just Cauchy’s integral formula for derivatives.

On the other hand, with $z = e^{i\theta}$ we have the explicit integration

$$\int_{\gamma} e^{z} dz = \int_0^{2\pi} e^{e^{i\theta}} e^{i\theta} d\theta = i \int_0^{2\pi} e^{e^{i\theta}} e^{-i\theta} d\theta.$$

Putting these together yields the result.
7. (15 points) Let $f$ be entire and let $|f(z)| \leq M$ for $z$ on the circle $|z| = R$ with $R$ fixed. Prove that

$$\left| f^{(k)}(re^{i\theta}) \right| \leq \frac{k!M}{(R-r)^k},$$

for all $0 \leq r < R$ and for all $k = 0, 1, 2, 3, \ldots$

**SOLUTIONS:** This was Exercise 2.R.8. It is a nice generalization of Cauchy’s inequality.

Let $z_0 = re^{i\theta}$ and define the curve

$$\gamma = \{z : |z - z_0| = R - r\}.$$

Notice that $\gamma \subset \{z : |z| \leq R\}$. Therefore by the maximum modulus Principle $|f| \leq M$ on $\gamma$.

The Cauchy integral formula gives us

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta.$$

Now, since $z_0$ is the center of $\gamma$, Cauchy’s inequality grants us the conclusion:

$$\left| f^{(k)}(re^{i\theta}) \right| \leq \frac{k!M}{(R-r)^k}.$$

That proof uses the maximum modulus principle.
Extra space for work: