Math 113, Fall 2001

Solutions to Problem Set 1

September 26, 2001

1a. Using the quadratic formula,
\[ z = \frac{-4\sqrt{2}i \pm \sqrt{-32 + 24i}}{2} \]
\[ = -2\sqrt{2}i \pm \sqrt{-8 + 6i} \]
\[ = -2\sqrt{2}i \pm (1 + 3i) \]
\[ = 1 + (3 - 2\sqrt{2})i, 1 + (-3 - 2\sqrt{2})i. \]

1b. We have \( z^8 = -1 = e^{(2n+1)i\pi} \) so \( z = e^{\frac{(2n+1)i\pi}{8}} \). Thus \( z \) can be any of \( e^{\frac{i\pi}{8}}, e^{\frac{3i\pi}{8}}, e^{\frac{5i\pi}{8}}, e^{\frac{7i\pi}{8}}, e^{\frac{9i\pi}{8}}, e^{\frac{11i\pi}{8}} \). 

2. If \( |z| = 1 \), then \( |\overline{z}| = 1 \), so
\[
\left| \frac{z - w}{1 - \overline{z}w} \right| = \left| \frac{\overline{z} - \overline{z}w}{1 - \overline{z}w} \right|
\]
\[
= \left| \frac{\overline{z}w - \overline{z}w}{\overline{z} - \overline{z}w} \right|
\]
\[
= 1.
\]

A similar technique shows that if \( |w| = 1 \), then the result holds.

3. If \( P \) is a polynomial in \( z \), then \( P \) satisfies the Cauchy-Riemann equation \( \frac{\partial P}{\partial x} = -i \frac{\partial P}{\partial y} \). But if \( P \) is a polynomial in \( \overline{z} \), one can show that \( P \) satisfies the analogous equation \( \frac{\partial P}{\partial x} = i \frac{\partial P}{\partial y} \). Hence \( \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = 0 \) and \( f \) is a constant.

4. Suppose \( P = a_0 + a_1 z + \ldots + a_n z^n \) with \( a_n \neq 0 \). Some manipulation of the triangle inequality yields \( |P(z)| \geq |a_n z^n| - |a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}| \). If \( M = \max\{a_0, a_1, \ldots, a_{n-1}\} \) and \( |z| > 1 \), then \( |a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}| \leq M |z|^{n-1} \). Hence for \( |z| \) large enough, \( |P(z)| \geq |a_n||z^n - M| z^{n-1} \). Clearly the right-hand side approaches infinity as \( |z| \) approaches infinity.

6. Note that \( 1 + \cos \theta + \ldots + \cos n \theta = \text{Re}(1 + e^{i\theta} + \ldots + e^{ni\theta}) \). If \( \theta \) is not a multiple of \( 2\pi \), then \( e^{i\theta} \neq 1 \), and the right-hand side equals \( \frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} \). Now
\[
\frac{1 - e^{(n+1)i\theta}}{1 - e^{i\theta}} = \frac{1 - e^{(n+1)i\theta}}{1 - e^{-i\theta}} \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}}
\]
\[ = \frac{1 - e^{-i\theta} - e^{i(n+1)\theta} + e^{ni\theta}}{2 - 2\cos\theta}. \]

The real part of this expression is \( \frac{1 - \cos\theta - \cos(n+1)\theta + \cos(n\theta)}{2 - 2\cos\theta} \), which is easily checked to be equal to \( \frac{\alpha \beta + \beta' \alpha'}{\alpha^2 + \beta^2} \).

7. For any \( \epsilon > 0 \), we know that there exists \( N \) such that for \( n \geq N \), \( L - \epsilon < \frac{a_{n+1}}{a_n} < L + \epsilon \). Hence for \( k > 0 \), \( a_{N+k} \) lies between \( a_N(L - \epsilon)^k \) and \( a_N(L + \epsilon)^k \), or between \( a_N(L - \epsilon)^{-N}(L - \epsilon)^{N+k} \) and \( a_N(L + \epsilon)^{-N}(L + \epsilon)^{N+k} \).

Now \( a_N(L - \epsilon)^{-N} \) and \( a_N(L + \epsilon)^{-N} \) are both positive constants, so
\[
\lim_{k \to \infty} \left( a_N(L - \epsilon)^{-k} \right)^{\frac{1}{N+k}} = \lim_{k \to \infty} \left( a_N(L + \epsilon)^{-k} \right)^{\frac{1}{N+k}} = 1. \]
Hence there exists some \( M > N \) such that for \( m \geq M \), \( (a_N(L - \epsilon)^{-N})^{\frac{1}{m}} \) and \( (a_N(L + \epsilon)^{-N})^{\frac{1}{m}} \) are both between \( 1 - \epsilon \) and \( 1 + \epsilon \).

Putting the two pieces together, we can conclude that for \( k \) large enough, \( (a_{N+k})^{\frac{1}{N+k}} \) lies between \( (1 - \epsilon)(L - \epsilon) \) and \( (1 + \epsilon)(L + \epsilon) \). If we take \( \epsilon < 1 \), we can say that for \( k \) large enough, \( (a_{N+k})^{\frac{1}{N+k}} \) lies between \( (L - 3\epsilon) \) and \( (L + 3\epsilon) \).

If \( \lim_{n \to \infty} (a_n)^{\frac{1}{n}} \neq L \), then there exists \( \epsilon > 0 \) such that for infinitely many \( n \), \( a_n^{\frac{1}{n}} \) lies outside of the interval \( (L - 3\epsilon, L + 3\epsilon) \), which cannot happen. Hence \( \lim_{n \to \infty} a_n^{\frac{1}{n}} = L \).

The ratio of successive terms in the infinite series \( \sum_{n=1}^{\infty} \frac{n \beta_n}{n} \) is \( z \left( \frac{n}{n+1} \right)^n \), and this limit as \( n \to \infty \) is \( \frac{\beta}{\epsilon} \). Hence the radius of convergence of this series is \( \epsilon \).