Homework Assignment # 7  
Due Thursday, November 29

This assignment will count as **two** homework assignments, and you will have two weeks to do it.

1. Find all residues at all singularities of:
   a. $z \cot(z)$
   b. $\frac{z^{-1}}{(z^4-1)^2}$
   c. $\sin(1/z)$
   d. $\frac{1}{e^z-1}$

2. Evaluate 
   \[
   \int_{|z|=2} (2z - 1) e^{(z-1)/z} \, dz.
   \]

3. a. Let $A, B$ be analytic at $z_0$ and suppose that $A(z)$ has a zero of order $k$ at $z_0$, and that $B(z)$ has a zero of order $k+1$ at $z_0$. Prove that $A(z)/B(z)$ has a simple pole at $z_0$, and that
   \[
   \text{Res}(\frac{A(z)}{B(z)}; z_0) = (k + 1) \frac{A^{(k)}(z_0)}{B^{(k+1)}(z_0)}.
   \]
   Use this to evaluate 
   \[
   \int_{|z|=1} \frac{z^3}{(1 - \cos(z))^2} \, dz.
   \]

   b. Let $A, B$ be analytic at $z_0$, with $A(z_0) \neq 0$ and $B(z)$ having a zero of order 2 at $z_0$. Prove that $A(z)/B(z)$ has a double pole at $z_0$, and that
   \[
   \text{Res}(\frac{A(z)}{B(z)}; z_0) = 2 \frac{A'(z_0)}{B''(z_0)} - \frac{2A(z_0)B'''(z_0)}{3(B''(z_0))^2}.
   \]
   Use this to evaluate 
   \[
   \int_{|z-1|=1} \frac{e^z}{(z - 1)^2} \, dz.
   \]
4. Evaluate the following definite integrals:
   a. \( \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} \, dx \)
   b. \( \int_{0}^{2\pi} \frac{d\theta}{1+\sin^2\theta} \)
   c. \( \int_{0}^{\infty} \frac{dx}{1+x^n}, \ n \in \mathbb{Z}, n \geq 2. \) (See p.147 of the book for a good contour to use.)

5. a. For any real number \( a > 0 \), show that
   \[
   \sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) + \frac{1}{2a^2},
   \]
   where \( \coth \) means hyperbolic cotangent.
   b. Evaluate \( \sum_{n=0}^{\infty} \binom{3n}{n} \frac{1}{8^n} \).

6. Prove the “Fractional Residue Theorem”: Suppose \( f \) is analytic with a simple pole at \( z_0 \), and let \( \gamma_r \) be a circular arc with an angle of \( \alpha \) radians on a circle of radius \( r \) centered at \( z_0 \). Show that
   \[
   \lim_{r \to 0} \int_{\gamma_r} f = \alpha i \text{Res}(f; z_0).
   \]

7. Suppose \( f(z) \) has an isolated singularity at \( \infty \). We define the residue of \( f \) at infinity to be
   \[
   \text{Res}(f(z); \infty) := \text{Res}(-\frac{1}{w^2} f(1/w); w = 0).
   \]

[Aside: The reason for this strange definition, if you’re curious, is that when working with manifolds (like the Riemann sphere), it does not make sense to take the residue of a function at point; the notion of residue only behaves well under changes of coordinates for differential forms. So the residue of \( f(z) \) at a point \( z_0 \) is really the residue of the differential form \( f(z)dz \) at \( z_0 \). And if we make the change of variables \( w = 1/z \), then \( f(z)dz \) is transformed into \( f(1/w)d(1/w) = -\frac{1}{w^2} f(1/w)dw \). Even if you don’t know about differential forms, this should give you a good mnemonic for remembering the definition of the residue at infinity.]
a. One has to be a bit careful with residues at infinity: Show by example that even if \( f \) has a removable singularity (i.e., is analytic) at \( \infty \), we might have \( \text{Res}(f(z); \infty) \neq 0 \).

[Remark: In terms of differential forms, the interpretation of this seemingly puzzling fact is that although \( f(z) \) is analytic at \( \infty \), the 1-form \( dz \) (and therefore also \( f(z)dz \) has a double pole at infinity.)]

b. Suppose \( f \) is analytic on the set \( \{|z| > R_0\} \), and let \( R > R_0 \). Prove that

\[
\int_{|z|=R} f(z)dz = -2\pi i \text{Res}(f; \infty).
\]

By considering the Riemann sphere, explain intuitively why one gets a minus sign in this formula.

c. Suppose that \( f \) is analytic on \( \mathbb{C} \) except at a finite number of isolated singularities. Prove that

\[
\sum_{z_0 \in \mathbb{C}} \text{Res}(f(z); z_0) = 0,
\]

where the sum runs over all points of the Riemann sphere \( \hat{\mathbb{C}} \) (i.e., \( \mathbb{C} \) plus \( \infty \)). [Hint: Consider the line integral of \( f \) over a large circle and use part (b).]

d. If \( \gamma \) is a regular closed curve in \( \mathbb{C} \) and \( f \) is analytic along \( \gamma \), with only finitely many isolated singularities outside \( \gamma \), show that

\[
\int_{\gamma} f(z)dz = -2\pi i \sum \{\text{residues of } f \text{ outside } \gamma \text{ including } \infty\}.
\]

e. If \( P(z) \) and \( Q(z) \) are polynomials and \( \deg(Q) - \deg(P) \geq 2 \), prove that the residue of \( f(z) = \frac{P(z)}{Q(z)} \) at infinity is zero.

f. Evaluate the contour integral

\[
\int_{|z|=1} \frac{dz}{(z - 2)(1 + 2z)^4(1 - 3z)^7}.
\]

[Hint: Use parts (d) and (e). Notice that the integrand has high-order poles inside the unit circle but low-order poles outside the unit circle, so using the residue at infinity is much simpler than doing this integral the “usual” way!]