5/5/2004 INTEGRAL THEOREM PROBLEMS

HOMEWORK. Problems 9,10,11 on the homeworksheet.

SUMMARY. This is a collection of problems on line integrals, Green’s theorem, Stokes theorem and the divergence theorem. Some of them are more challenging.

LINE INTEGRALS GREEN THEOREM.

The curve \( r(t) = (\cos^3(t), \sin^3(t)) \) is called a **hypocycloid**. It bounds a region \( R \) in the plane.

a) Calculate the line integral of the vector field \( F(x,y) = (x,y) \) along the curve.

b) Find the area of the hypocycloid.

   a) Because \( \text{curl}(F) = 0 \) the result is zero by Green’s theorem.

   b) Use the vector field \( F(x,y) = (0,x) \) which has \( \text{curl}(F) = 1 \). The line integral is

   \[ \int_0^{2\pi} F(r(t)) \cdot r'(t) \, dt = \int_0^{2\pi} \cos^3(t)3\sin^2(t)\cos(t) \, dt = \int_0^{2\pi} 3\cos^4(t)\sin^2(t) \, dt = 3\pi/8. \]

   (To compute the integral, use that

   \[ 8\cos^4(t)\sin^2(t) = \cos(2t)\sin^2(2t) + \sin^2(2t). \]

   )

LENGTH OF CURVE AND LINE INTEGRALS.

Assume \( C : t \mapsto r(t) \) is a closed path in space and \( F(r(t)) \) is the unit tangent vector to the curve (that is a vector parallel to the velocity vector which has length 1).

a) What is \( \int_C F \, dr \)?

b) Can \( F \) be a gradient field?

   Answer:

   a) \( F(r(t)) = r'(t)/|r'(t)| \). By definition of the line integral,

   \[ \int_C F(r(t)) \cdot r'(t) \, dt = \int_a^b \frac{r'(t)}{|r'(t)|} \cdot r'(t) \, dt = \int_a^b |r'(t)| \, dt, \]

   which is the length of the curve.

   b) No: If \( F \) were a gradient field, then by the fundamental theorem of line integrals, we would have that the line integral along a closed curve is zero. But because this is the length of the curve, this is not possible.

SURFACE AREA AND FLUX.

Assume \( S : (u,v) \mapsto r(u,v) \) is a closed surface in space and \( F(r(u,v)) \) is the unit normal vector on \( S \) (which points in the direction of \( r_u \times r_v \)).

a) What is \( \int_S F \cdot dS \)?

b) Is it possible that \( F \) is the curl of another vector field?

c) Is it possible that \( \text{div}(F) = (0,0,0) \) everywhere inside the surface.

   Answer:

   a) \( F(r(u,v)) = (r_u \times r_v)/|r_u \times r_v| \). By definition of the flux integral,

   \[ \int_S F \cdot dS = \int_R F(r(u,v)) \cdot r_u \times r_v = \int_R (\frac{r_u \times r_v}{|r_u \times r_v|}) \cdot r_u \times r_v = \int_R |r_u \times r_v| \, dudv \]

   which is the area of the surface.

   b) No, if \( F \) were the curl of another field \( G \), then the flux of \( F \) through the closed surface would be zero. But since it is the area, this is not possible.

   c) From the divergence theorem follows that \( \text{div}(F) \) is nonzero somewhere inside the surface.
Assume $R$ is a region in the plane and let $n$ denote the unit normal vector to the boundary $C$ of $R$. For any function $u(x,y)$, we use the notation $\partial f/\partial u = \text{grad}(u) \cdot \hat{n}$ which is the directional derivative of $u$ into the direction $\hat{n}$ normal to $C$. We also use the notation $\Delta u = u_{xx} + u_{yy}$.

Show that

$$\int_C \frac{\partial u}{\partial n} \, dr = \int \int_R u \, dA$$

Answer: Define $F(x,y) = (B;A)$ if $\partial u/\partial n = (A,B)$. The left integral is the line integral of $F$ along $C$. The right integral is the double integral over $\Delta u = \text{curl}(F)$.

**STOKES THEOREM OR DIVERGENCE THEOREM**

Find $\int_S \text{curl}(F) \cdot dS$, where $S$ is the ellipsoid $x^2 + y^2 + 2z^2 = 10$ and $F(x,y,z) = (\sin(xy), e^x, -yz)$.

Answer. The integral is zero because the boundary of $S$ is empty. This fact can be seen using Stokes theorem. It can also been seen by divergence theorem

$$\int \int_S \text{curl}(F) \cdot dS = \int \int \int \text{div}(\text{curl}(F)) \, dV$$

using $\text{div}(\text{curl})(F) = 0$.

**AREA OF POLYGONS.**

If $P_i = (x_i,y_i), i = 1, \ldots, n$ are the edges of a polygon in the plane, then its area is $A = \sum_i (x_i - x_{i+1})(y_{i+1} + y_i)/2$.

The proof is an application of Green’s theorem. The line integral of the vector field $F(x,y) = (-y,0)$ through the side $P_i, P_{i+1}$ is $(x_i - x_{i+1})(y_{i+1} + y_i)/2$, because $(x_{i+1} - x_i)$ is the projected area onto the x-axis and $(y_{i+1} + y_i)/2$ is the average value of the vector field on that side. Because $\text{curl}(F)(x,y) = 1$ for all $(x,y)$, the result follows from Greens theorem.

The result can also be seen geometrically: $(x_i - x_{i+1})(y_{i+1} + y_i)/2$ is the signed area of the trapezoid $(x_i,0), (x_{i+1},0), (x_{i+1},y_{i+1}), (x_i,y_i)$. In the picture, we see two of them. The second one is taken negatively.

**VOLUME OF POLYHEDRA.**

Verify with the divergence theorem: If $P_i = (x_i,y_i,z_i)$ are the edges of a polyhedron in space and $F_j = \{P_{i_1}, \ldots, P_{i_k}\}$ are the faces, then $V = \sum_j A_j \bar{z}_j$ where $A_j$ is the area of the $xy$-projection (*) of the polygon $F_j$ and $\bar{z}_j = (z_{i_1} + \ldots + z_{i_k})/k_j$ is the average $z$ value of the face $F_j$.

Solution. The vector field $F(x,y,z) = z$ has divergence 1. The flux through a face $F$ is $|F_j|(z_{i_1} + \ldots + z_{i_k})/k_j$. Gauss theorem assures that the volume is the sum of the fluxes $A_j \bar{z}_j$ through the faces.

(*) The projection of a polygon is the "shadow" when projecting from space along the z-axes onto the xy-plane. A triangle $(1,0,1), (1,1,0), (0,1,2)$ for example would be projected to the triangle $(1,0), (1,1), (0,1)$. 
STOKES AND GAUSS TOGETHER.

Can you derive \( \text{div}(\text{curl}(F)) = 0 \) using Gauss and Stokes theorem? Consider a sphere \( S \) of radius \( r \) around a point \( (x, y, z) \). It bounds a ball \( G \). Consider a vector field \( F \). The flux of \( \text{curl}(F) \) through \( S \) is zero because of Stokes theorem. Gauss theorem tells that the integral of \( f = \text{div}(\text{curl}(F)) \) over \( G \) is zero. Because \( S \) was arbitrary, \( f \) must vanish everywhere.

FUNDAMENTAL THEOREM AND STOKES.

Can you derive the identity \( \text{curl}(\text{grad}(F)) = 0 \) from integral theorems?

To see that the vector field \( G = \text{curl}(\text{grad}(F)) = 0 \) is identically zero, it is enough to show that the flux of \( G \) through any surface \( S \) is zero. By Stokes theorem, the flux through \( S \) is \( \int_C \text{grad}(F) \cdot dr \). By the fundamental theorem of line integrals, this is zero.

VOLUME COMPUTATION WITH GAUSS.

Calculate the volume of the torus \( T(a, b) \) enclosed by the surface \( r(u, v) = ((a + b \cos(v)) \cos(u), (a + b \cos(v)) \sin(u), b \sin(v)) \) using Gauss theorem and the vector field \( F(x, y, z) = (x, y, 0) \).

The vector field \( F \) has divergence 1. The parameterization of the torus gives
\[
r_u \times r_v = (a + b \cos(v)) \cos(v) \sin(v), (a + b \cos(v)) \sin(v), b \sin(v) .
\]

The flux of this vectorfield through the boundary of the torus is
\[
\int_{0}^{2\pi} \int_{0}^{2\pi} b(a + b \cos(v))^2 \cos(v) \, dudv = 2\pi^2 ab^2.
\]

GAUSS OR STOKES?

You know that the flux of the vector field \( G = \text{curl}(F)(x, y, z) \) through 5 faces of a cube \( D \) is equal to 1 each. What is the flux of the same vector field \( G \) through the 6'th face?

Solution: the problem is best solved with the divergence theorem: because the flux of \( G \) through the entire surface is zero, the flux through the 6'th face must cancel the sum of the fluxes 5 through the other 5 surfaces. The result is \(-5\).

WORK COMPUTATION USING STOKES.

Calculate the work of the vector field \( F(x, y, z) = (x - y + z, y - z + x, z - x + y) \), along the path \( C \) which connects the points \((1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 0, 1) \rightarrow (1, 0, 0)\) in that order.

Answer. The line integral over each part is each 1. The total is 3. \( \text{curl}(F) = (2, 2, 2) \) and \( S : (u, v) \mapsto r(u, v) = (u, v, 1 - u - v) \) \( r_u \times r_v = (1, 1, 1) \) \( \int_S \text{curl}(F) \cdot r_u \times r_v \, dudv = 6 \) area of \( S = 3 \).
STOKES OR GAUSS?

Compute the flux of the vector field \( F(x, y, z) = (x - x \sin(\sin(z)), 2y, 3z + \sin(\sin(z))) \) through the upper hemisphere \( S = \{(x, y, z) | x^2 + y^2 + z^2 = 1, z \geq 0 \}. \)

Answer. We use Gauss: \( \text{div}(F) = 6 \) and \( \iint_B \text{div}(F) \, dV = 6 \text{Vol}(B) = 4\pi. \) We cannot easily compute the flux through the hemisphere. However, we can see that the flux through the floor of the region is zero because the normal component \( P \) of the vector field \( F = (M, N, P) \) is zero on \( z = 0. \) So: the result is \( 4\pi - 0 = 4\pi. \)

GREENS THEOREM.

Calculate the work of the vector field \( F(x, y) = \frac{1}{x^2+y^2} (-y, x) \) along the boundary of the ellipse \( r(t) = (3 \cos(t) + \sin(t), 5 \sin(t) + \cos(t)). \)

Solution. Take an other curve \( C : x^2 + y^2 \leq 4 \) and apply Green’s theorem to the region \( R \) bounded by the ellipse and the circle. Because \( \text{curl}(F) \) is zero in \( D, \) the line integral along the ellipse is the same as the line integral along the circle: \( t \mapsto r(t) = (2 \cos(t), 2 \sin(t)) \) with velocity \( r'(t) = (-2 \sin(t), 2 \cos(t)): \)

\[
\int F \, dr = \int_0^{2\pi} \left( -\frac{2 \cos(t), 2 \sin(t)}{4} \right) \cdot (-2 \sin(t), 2 \cos(t)) = \int_0^{2\pi} 1 \, dt = 2\pi.
\]

TRUE/FALSE QUESTIONS ON INTEGRAL THEOREMS.

(TF) The flux of the curl of a vector field through the unit sphere is zero.

(TF) The line integral of the curl of a vector field along a closed curve is zero.

(TF) The line integral \( \int_C F \cdot dr \) is independent of how a curve \( C : t \mapsto r(t) \) is parametrized.

(TF) The maximal speed of a curve is independent on how the curve is parametrized.

(TF) The flux integral \( \int_S F \cdot dS \) through a surface is independent on how the surface \( S \) is parametrized.

(TF) The area \( \int_S dS \) of a surface is independent on how the surface \( S \) is parametrized.

(TF) The maximal value of \( r_u \times r_v \) on a surface \( S \) is independent on how the surface is parametrized.

(TF) There exists a vector field in space which has zero divergence, zero curl but is not a constant field.

(TF) There exists a vector field in space which has zero gradient but is a constant vector field \( F(x, y, z) = (a, b, c). \)

(TF) There exists a function in space which has zero Laplacian \( f_{xx} + f_{yy} + f_{zz} = 0 \) but which is not constant.

(TF) \( \text{div} (\text{grad}(F)) = 0 \) and \( \text{div} (\text{curl}(F)) = 0 \) and \( \text{curl}(\text{grad}(F)) = 0. \)

(TF) The line integral of a gradient field along any part of a level curve \( F = \text{const} \) is zero.

(TF) If \( \text{div}(F) = 0, \) then the line integral along any closed curve is zero.

(TF) If \( \text{curl}(F) = 0, \) then the line integral along any closed curve is zero.

(TF) If \( \text{div}(F) = 0 \) then the flux integral along any sphere in space is zero.

(TF) If \( \text{curl}(F) = 0 \) then the flux integral along any sphere in space is zero.