(UNFINISHED NOTES ON) MORDELL'S CONJECTURE
AFTER FALTINGS AND LAWRENCE–VENKATESH

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Abstract. In this survey we briefly review Faltings’s proof of Mordell’s conjecture, and then sketch the new proof given by Lawrence–Venkatesh using $p$-adic and topological methods. This is an extended version of my talks in Mazur’s course on rational points. Warning: this set of notes is still incomplete, namely I have not included the monodromy arguments of Lawrence–Venkatesh.

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Plan. In Section 1 we give a brief sketch of (part of) Faltings’s proof. In Section 5 we give the outline of the proof of Lawrence–Venkatesh (which serves as an introduction of this survey). In the remaining of the article we explain their new proof in slightly more detail.

Conventions. For Faltings’s proof we use étale cohomology with $l$-adic coefficients, where $l$ is an (odd) prime. In the later part we switch to $p$-adic coefficients\(^1\). The reason for this intentional choice is that in Faltings’s proof, the key ingredient seems to be the analysis of certain height functions (in order to prove Theorem 1.1) while $l$-adic cohomology plays a somewhat intermediate role. On the other hand, for the proof by Lawrence–Venkatesh,\(^1\) and use $l$ to denote an auxiliary prime which serves a different purpose.
Part 1. The proof by Faltings

Notations (for Faltings’s proof).

- $K/\mathbb{Q}$ denotes a number field.
- $S$ always denotes a finite set of places of $K$, and is often assumed to contain all “bad/ramified” places when the setup is clear.
- $A, B$ are abelian varieties over $K$.
- $T_l(A)$ (resp. $V_l(A)$) denotes the integral (resp. rational) $l$-adic Tate module.

More notations will be introduced later for the other proof.

1. The outline of Faltings’s proof

The key statement is the so-called Faltings’s finiteness theorem, which says that each isogeny class over the number field $K$ only contains finitely many isomorphism classes. Equivalently:

**Theorem 1.1 (Finiteness A).** Let $A$ be an abelian variety over $K$. Then up to isomorphism, there are only finitely many abelian varieties $B$ over $K$ that are isogenous to $A$.

This is probably the deepest part of Faltings’s argument, where he makes some delicate analysis of his height functions over the boundary of certain Shimura varieties, which we unfortunately will not prove in these notes.

In what follows we outline the proof of the main theorem assuming Finiteness A (Theorem 1.1) as a blackbox. The line of Falting’s argument is then as follows: first he shows that Finiteness A implies Tate’s conjecture (Section 2), and then goes on to prove that together they imply Shafarevich’s conjecture (Section 3), from which one can deduce the main theorem using Parshin’s trick (Section 4), which was probably well-known around the 80s. To summarize, the structure of Faltings’s proof is

$$
\begin{aligned}
\text{Finiteness A} & \xrightarrow{\text{Sec.2}} \text{Tate} & \xrightarrow{3} & \text{III(weak)} & \xrightarrow{\text{Torelli}} & \text{III} & \xrightarrow{4} & \text{Mordell}
\end{aligned}
$$

Here by “III” we mean Shafarevich’s conjecture (and its weak form). We will sketch all the implications listed above in subsequent sections, though our focus will be on arrow (labeled by section) 2 and 3.

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2Here by an isogeny we implicitly mean an isogeny $\alpha : A \to B$ defined over $K$. 

- p-adic Hodge theory serves as an essential ingredient. Our switch in notation hopefully emphasizes this point without causing too much confusion.
We first show that Finiteness A implies

**Theorem 2.1** (Tate’s conjecture). *Notation as before,*

1. The action of $\text{Gal}_K$ on $V_l(A)$ is semisimple.
2. The $\mathbb{Q}_l$-linear morphism

$$\text{End}(A) \otimes \mathbb{Q}_l \xrightarrow{-} \text{End}(V_lA)^{\text{Gal}_K}$$

is an isomorphism.

2′ Slightly more generally, we have an isomorphism

$$\text{Hom}(A, B) \otimes \mathbb{Z}_l \xrightarrow{-} \text{Hom}(T_lA, T_lB)^{\text{Gal}_K}.$$
Lemma 2.3. Let \( W \subset V_l A \) be a subrepresentation of \( \text{Gal}_K \), then there exists an element \( u \in \text{End}(A) \otimes \mathbb{Q}_l \) such that \( uV_l A = W \).

Proof. It suffices to find such an element \( \mu \in \text{End}(A) \otimes \mathbb{Q}_l \) such that \( \mu \) projects \( T_l A \) to \( T_l A \cap W \). For this, let us consider a sequence

\[
\Lambda_n := T_l A \cap W + l^n T_l A
\]

of \( \text{Gal}_K \)-stable sub-lattices in \( T_l A \), each of finite index. Therefore, each \( \Lambda_n \) gives rise to an isogeny \( h_n : A_n \to A \). Now we apply Finiteness A (Theorem 1.1), which says that there exists an infinite index set \( I \) such that all the \( \{A_i\}_{i \in I} \) are isomorphic over \( K \). Let \( i_0 \) be the smallest index in \( I \) (i.e., \( \Lambda_{i_0} \) is the largest lattice among all the \( \Lambda_i, i \in I \)), and pick an isomorphism \( \gamma_i : A_{i_0} \cong A_i \) for each \( i \in I \). Now consider the following elements

\[
u_i := h_i \circ \gamma_i \circ h_{i_0}^{-1} \in \text{End}(A) \otimes \mathbb{Q}_l.
\]

Note that \( u_i \) only makes sense in \( \text{End}(A) \otimes \mathbb{Q}_l \). The idea is then to take a limit of these \( u_i \), for this we give a name to the following injection

\[
\psi : \text{End}(A) \otimes \mathbb{Q}_l \hookrightarrow \text{End}(V_l A)^{\text{Gal}_K}.
\]

Now observe that \( \psi(u_i) \) sends \( \Lambda_{i_0} \) isomorphically to \( \Lambda_i \subset \Lambda_{i_0} \) by the diagram

\[
\Lambda_{i_0} \xrightarrow{h_{i_0}} T_l(A_{i_0}) \xrightarrow{\gamma_i} T_l(A_i) \xrightarrow{h_i} \Lambda_i
\]

In other words each \( \psi(u_i) \in \text{End}(V_l A) \) in fact lives in the subspace \( \text{End}(\Lambda_{i_0}) \), which is a compact, therefore by passing to a subsequence if necessary, there exists a limit \( \mu \in \text{End}(\Lambda_{i_0}) \). Moreover, each \( u_i \in \text{End}(A) \otimes \mathbb{Q}_l \), so \( \psi(u_i) \) lies in the closed subspace \( \text{im}(\psi) \subset \text{End}(V_l A) \), hence \( \mu \) comes from an element \( \mu \in \text{End}(A) \otimes \mathbb{Q}_l \). Finally, it is easy to verify that \( uT_l A = T_l A \cap W \). □

Remark 2.4. Here is a cartoon for the proof of the lemma above.

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3or in \( \text{End}(A) \otimes \mathbb{Q} \), but not in \( \text{End}(A) \).
In the picture the outer square represents $V_l(A)$ and the middle black square represents $T_l(A)$ (viewing $Z_l \subset Q_l$ as a “closed and open ball”). The goal is to construct a projector to the vertical red line in the middle (which represents the subspace $W$). To do this we simply construct a sequence of projections which sends the purple square ($\Lambda_{w_0}$) closer and closer to the blue line (which represents $W \cap T_l(A)$) in the middle.

Now we are ready to prove Theorem 2.1.

Proof of part (1). Let $W \subset V_l A$ be a subrepresentation of $\text{Gal}_K$ and we look for $W'$ such that $V_l A \cong W \oplus W'$. \footnote{A naive guess would be to apply the previous lemma to find $u$, and then take $(1-u)V_l A$, but we do not know that $W \cap (1-u)V_l A = 0$ unless $u$ is idempotent. To find such an idempotent projector, we will consider the family of all these projectors and utilize the structure of $\text{End}(A) \otimes Q_l$ as a semisimple $Q_l$-algebra.} Consider the right ideal

$$I := \{ x \in \text{End}(A) \otimes Q_l : xV_l A \subset W \}. $$

Now let $\mu \in \text{End}(A) \otimes Q_l$ be an element such that $\mu V_l A = W$ (as guaranteed by Lemma 2.3), then $\mu \in I$ and therefore $I \cdot V_l A = W$. But all right ideals in $\text{End}(A) \otimes Q_l$ are generated by idempotents, so $I = (e)$ for some idempotents $e \in \text{End}(A) \otimes Q_l$, and we are done. \hfill $\square$

The proof of part (2) of Tate's conjecture uses a clever trick, the (rough) idea is to reformulate the commutativity of two operators on a vector space as the condition of preservation of the graph of one operator under the diagonal action of the other one.

Proof of part (2). Consider $\text{End}(A) \otimes Q_l \hookrightarrow E := \text{End}(V_l A)$ as a subalgebra, with centralizer

$$\mathfrak{Z} = Z_{\text{End}(V_l A)}(\text{End}(A) \otimes Q_l) \subset E.$$

By the double centralizer theorem (recalled above) $Z_E(\mathfrak{Z}) = \text{End}(A) \otimes Q_l$, therefore it suffices to show that, for any $\gamma \in \text{End}(V_l A)^{\text{Gal}_K}$ and $c \in \mathfrak{Z}$, we have $\gamma \cdot c = c \cdot \gamma \in \text{End}(V_l A)$. In other words, let $c \in \text{End}(V_l A)$ be an element that commutes with all $x \in \text{End}(A) \otimes Q_l$, then we need to show that $c$ commutes with $\gamma$ as well.

Now the clever part of the argument: consider the graph of $\gamma$

$$\Gamma_\gamma := \{(v, \gamma v) : v \in V_l A\} \subset V_l A \times V_l A.$$

The diagonal action of $c$ on $V_l A \times V_l A$ maps $\Gamma_\gamma$ to $\{(cv, c\gamma v) : v \in V_l A\}$, thus it is clear that $\gamma c = c\gamma \in \text{End}(V_l A)$ if and only if $c \cdot \Gamma_\gamma \subset \Gamma_\gamma$. For this we observe that $\Gamma_\gamma$ is clearly a $\text{Gal}_K$-subrepresentation of $V_l A \times V_l A \cong V_l(A \times A)$, therefore by Lemma 2.3 there exists $\mu \in \text{End}(A \times A) \otimes Q_l$ such that $\mu(V_l A \times V_l A) = \Gamma_\gamma$. In other words,

$$c \cdot \Gamma_\gamma = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \mu \cdot (V_l A \times V_l A) = \mu \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} (V_l A \times V_l A) \subset \Gamma_\gamma.$$

Here the commutativity of $\mu$ and the diagonal action of $c$ follows from the assumption that $c \in \mathfrak{Z}$.

\hfill $\square$
3. Shafarevich’s conjecture

First recall the statement of

**Theorem 3.1** (Shafarevich’s conjecture). Let $S$ be a finite set of places of $K$, and fix $g \geq 2$. There are only finitely many isomorphism classes of curves over $K$ of genus $g$ with good reduction outside $S$.

The key step is to prove a weaker version of Shafarevich’s conjecture (this is Theorem 5 in his original paper). The conjecture then follows by applying (a stronger version of) Torelli’s theorem $^5$.

**Theorem 3.2** (Weak Shafarevich). Let $S$ be a finite set of places of $K$, and fix $g \geq 2$. There are only finitely many isomorphism classes of curves over $K$ of genus $g$ with good reduction outside $S$.

Now we fix $S$ and $g$ as in the statement of Theorem 3.2. Clearly, if two abelian varieties $A, B$ are isogenous over $K$, then they have the same reduction type at a place $v$ of $K$. By Finiteness A (each isogeny class of abelian varieties over $K$ contains finitely many isomorphism classes), it suffices to prove Theorem 3.2 using “isogeny classes” instead of “isomorphism classes” in its formulation. This allows us to reformulate the theorem in the language of Galois representations. For this we introduce the following (non-standard) terminology

**Definition 3.3.** Fix $K$ as above, $S$ a finite set of places containing $S_l$, also fix integers $w, d \geq 0$. A permissible Galois representation of dimension $d$, level $S$ and pure of weight $w$ is a continuous representation $\rho : G_K \to \text{GL}_d(\mathbb{Q}_l)$ that is

- semisimple,
- unramified out side of $S$, and
- pure of weight $w$. $^6$

By the discussion above, and Remark 2.2, $^7$ the weak version of Shafarevich’s conjecture is equivalent to

**Proposition 3.4.** Fix $K, S, w, d = 2g$ as above, then up to isomorphism there are only finitely many permissible ($l$-adic) $\text{Gal}_K$-representations of dimension $d = 2g$, level $S$ and pure of weight $w$.

**Proof.** We first prove a lemma, under the same setup as the proposition.

**Lemma.** There exists a finite set of places $Q$ of $K$ disjoint from $S$, such that the isomorphism classes of permissible representations of dimension $d$ and level $S$ are completely determined by the trace of Frobenius for all $v \in Q$.

$^5$More precisely we need a version of Torelli over $\mathbb{Q}$, for which we need to assume that the genus $g \geq 2$. This accounts for the difference between the assumptions on $g$ in the statement of Theorem 3.1 and 3.2.

$^6$In the sense that for each $v \notin S$, all eigenvalues of Frobenius at $v$ are algebraic integers all of whose conjugates have complex absolute value $q^{w/2}$.

$^7$recall that this says that the isogeny class of abelian varieties are determined by the isomorphism class of the associated Galois representations.
The lemma clearly implies the proposition: for each $v \in Q$, there are finitely many possibilities for the trace of $\text{Frob}_v$, since its eigenvalues are Weil integers (i.e. algebraic integers such that all of its conjugates are) of pure weight $w$.\(^8\) It remains to prove this claim.\(^9\)

**Proof of the Lemma.** We first construct such a finite set $Q$. Let $\mathcal{L}$ be the set of all field extensions $K'/K$ unramified outside $S$ and of degree $\leq l^{2d^2}$ (in some fixed algebraic closure $\overline{\mathbb{Q}}$). $\mathcal{L}$ is a finite set by Hermite–Minkowski. We then get a finite Galois extension $L/K$ by taking the composite of all $K'/K$ in $\mathcal{L}$. Now choose a finite set $Q$ of finite places disjoint from $S$ such that the Frobenius conjugacy class $(v,L/K) = \{\text{Frob}_w : w | v\}$ of $v \in Q$ hits all conjugacy classes of $\text{Gal}(L/K)$, in other words, such that

$$\text{Gal}(L/K) = \bigcup_{v \in Q} (v,L/K).$$

Let $M,N$ be two $\mathbb{Z}_l$-valued permissible representations of $\text{Gal}_K$ (namely $\text{Gal}_K$-stable lattices inside permissible representations $V_M = M[\frac{1}{l}]$, resp. $V_N = N[\frac{1}{l}]$). Suppose that for any $v \in Q$, $\text{Tr}_{\rho_M}(\text{Frob}_v) = \text{Tr}_{\rho_N}(\text{Frob}_v)$, and we want to show that $V_M$ and $V_N$ are isomorphic. Let

$$R_{M,N} \subset \text{End}_{\mathbb{Z}_l}(M) \times \text{End}_{\mathbb{Z}_l}(N)$$

be the $\mathbb{Z}_l$-submodule generated by the image of $\text{Gal}_K$. Now we view $M$ and $N$ as $R_{M,N}$-modules, by construction, $M$ and $N$ are two semisimple $R_{M,N}$-modules. To conclude the lemma above, it suffices to show that the action of $R_{M,N}$ on $M$ and $N$ have the same trace for all elements $r \in R_{M,N}$. In other words, it suffices to prove the following

**Sublemma.** The image of $\text{Frob}_w$ in $\text{End}(M) \times \text{End}(N)$, where $w | v$ and $v \in Q$, generates $R_{M,N}$ as a $\mathbb{Z}_l$-submodule.

By Nakayama’s lemma, it suffices to show that $R_{M,N} \otimes \mathbb{Z}/l$ is generated by these Frobenii after reducing mod $l$. Now, since $R_{M,N}$ is a free (= torsion free) $\mathbb{Z}_l$-module of rank $\leq 2d^2$, we know that $|R_{M,N}/l| \leq l^{2d^2}$. Therefore the homomorphism

$$\text{Gal}_K \to (R_{M,N}/l)^\times$$

factors through $\text{Gal}(K'/K)$ for some $K' \in \mathcal{L}$ by construction of $\mathcal{L}$. Finally, the choice of $Q$ ensures that $R_{M,N}/l$ can be generated by the given Frobenii hence the claim follows. \(\square\)

Now that Theorem 3.2 is proven, we are almost ready to deduce the Shafarevich conjecture. We will need the following two facts.

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\(^8\)In particular, for each such eigenvalue $e_v$, its minimal polynomial over $\mathbb{Q}$ is a polynomial of a bounded degree with integer coefficients, while each coefficient is also bounded.

\(^9\)The lemma is a bit surprising. Note that a given representation only factors through an infinite extension over $K$, so the finiteness statement a priori seems mysterious. The proof uses a clever trick: instead of consider the category of permissible $\text{Gal}_K$ representations altogether, one considers a pair of representations at a time.
Proposition 3.5. Fix $A_0/K$ and an integer $d \geq 1$, then up to isomorphisms, there are only finitely many polarized abelian varieties $(A, \lambda)$, where $A \cong A_0$ and $\lambda$ has degree $d$.

Proof.

Theorem 3.6 (Rational Torelli). Let $X$ be a curve over $K$ of genus $\geq 2$, then the isomorphism class of $X$ is completely determined by its Jacobian $(J_X, \lambda)$, as a (principally) polarized abelian variety over $K$.

Proof. See Milne’s notes on Jacobians (in [Cornell–Silverman]).

These two facts, together with Theorem 3.2 clearly implies Shafarevich’s conjecture (Theorem 3.1).

4. Mordell’s conjecture (Parshin’s trick)

In this section we review Parshin’s trick (which also serves as a preview of Section 8).
Part 2. The proof by Lawrence–Venkatesh

5. The structure of the proof of Lawrence–Venkatesh

5.1. Structure of Lawrence–Venkatesh’s argument. Now let us summarize what Faltings does to prove Mordell’s conjecture. The key step is to prove (both parts of) Tate’s conjecture, and then use it to deduce Shafarevich’s conjecture, from which he could then apply Parshin’s trick.

The approach of Lawrence–Venkatesh is different: they construct a suitable family of polarized abelian varieties over \( Y \) (strictly speaking over a finite étale cover of \( Y \)), and then analyze the corresponding \( p \)-adic period map from a residue disc to a “period domain”

\[
\Phi_v : \Omega_v \to \mathcal{H}(K_v),
\]

which is an analytic map from a local \( p \)-adic manifold (in the sense of Serre) to another. Note that the target is also the \( K_v \)-points of a scheme defined over \( K \). The overall structure of the argument takes the following form: one argues for one residue disc at a time (as there are finitely many of them), to this end we fix a point \( y_0 \in Y(K) \), and denote by

\[
\Omega_v = \Omega_v(y_0) := \{ y \in Y(K_v) = \mathcal{Y}(\mathcal{O}_v) : y \equiv y_0 \mod v \}
\]

the residue disc centered at \( y_0 \in Y(K) \). Consider the diagram

\[
\begin{array}{ccc}
Y(K, y_0) := \Omega_v \cap Y(K) & \xrightarrow{\Phi_v} & \mathcal{H}(K_v) \\
\Omega_v & \longleftarrow & \Phi_v
\end{array}
\]

For convenience let us denote the Zariski closure of the image of such a period map by \( \overline{\text{im}} \). Now, if we could show that

\[
\dim \overline{\text{im}}(\Phi_v) < \dim \overline{\text{im}}(\Phi_v),
\]

then \( Y(K, y_0) = \Omega_v \cap Y(K) \) lives inside a “proper” \( K_v \)-analytic submanifold of \( \Omega_v \) (meaning a subset cut out by an absolutely convergent \( p \)-adic power series over \( \Omega_v \)), which would force \( Y(K, y_0) \) to be finite.

How do we achieve this? Let us for the moment ignore the issue of possible failure of semisimplicity of the associated Galois representations, coming from the geometric fibers of the (carefully constructed) family over \( K \)-points of \( Y \). \(^{10}\) By basic \( p \)-adic Hodge theory, the image of \( Y(K, y_0) \) lives in a finite union of certain orbits of \( Z_\phi \) acting on \( \mathcal{H}(K_v) \), where \( Z_\phi \) is the centralizer of Frobenius in the endomorphism algebra of \( D_{\text{crys}}(\rho_{y_0}|_{\text{Gal}_{K_v}}) \). Therefore to prove the theorem, at least after ignoring the issue of semisimplicity, it suffices to construct a nice enough family and pick a suitable place \( v \) so that

\[
\dim \{Z_\phi\text{-orbit}\} < \dim \overline{\text{im}}(\Phi_v).
\]

\(^{10}\) We know a posteriori that this would not be an issue, by Faltings’s proof of Tate’s conjecture. But the whole point is to avoid using this.
It turns out (via an elementary argument) that the RHS can be computed and controlled using the complex period map (and hence by topological method). More precisely, if we consider the complex analytic period map $\Phi_C : \Omega_C \to H(C)$, then $\dim \overline{\text{im} \Phi_C} = \dim \overline{\text{im} \phi_v}$. We then arrange a family (over a large enough étale extension of $Y$) so that $\Phi_C$ has Zariski dense image. In particular this image is already as large as possible, and such that the dimension inequality holds. This step involves computation of the monodromy action.

The actual argument is in fact a bit more subtle, since we need to simultaneously do all of above and take care of the issue of semisimplicity, which lies in the technical heart of the method of Lawrence–Venkatesh.

5.2. Notations and a simplifying assumption. For this survey we make the following simplifying assumption for the sake of exposition:

From now on assume that $K$ does not contain any CM subfield.

This makes the argument of Lemma 7.7 simpler. It is not difficult to get rid of this assumption (one needs to introduce more restrictions on the place $v$ that is allowed – [1] call such $v$ friendly places, under the simplifying assumption, all unramified places are friendly).

Notations.

- $K/\mathbb{Q}$ denotes a number field.
- $v$ denotes an unramified prime of $K$ above a prime $p$, with residue field $k_v$.
- $S$ always denotes a finite set of places of $K$, and is often assumed to contain all “bad/ramified” places when the setup is clear;
- We use $\mathcal{O} = \mathcal{O}_K[\frac{1}{S}]$ to denote the $S$-integers of $K$.
- We use curly letters $\mathcal{X}, \mathcal{Y}$ to denote schemes over $\mathcal{O}$ or $\mathcal{O}_{K_v}$, and use $X, Y, Z$ to denote schemes over $K$ or $K_v$.
- Let $v$ be a place of $K$ above $p$, unramified over $\mathbb{Q}$.
- Let $S$ be a finite set of places, (disjoint from $S_p$), including all ramified/bad places.
- By $X \to Y' \xrightarrow{\pi} Y$ we usually mean an abelian scheme $X = (X, \omega)$ over $Y'$, which is finite étale over $Y$.
- Suppose that all families and all maps extend to $\mathcal{O}_K[\frac{1}{S}]$.
- Let $y_0 \in Y(K)$ and $\Omega_v = \Omega_v(y)$ be the residue disc centered at $y_0$
  \[\Omega_v = \{y \in Y(K_v) = \mathcal{Y}(\mathcal{O}_v) : y \equiv y_0 \mod v\},\]
  where $\mathcal{Y}$ is an integral model of $Y$ over $\mathcal{O}$.

5.3. The Gauss–Manin connection. Fix $K \hookrightarrow \mathbb{C}$, fix $y_0 \in Y(K)$. Suppose that we are given a family $\pi : X \to Y$ where $\pi$ is smooth proper. Further assume that this morphism (in particular $X$ and $Y$) has good integral model over $\mathcal{O}$, namely we have a smooth proper map $\pi : \mathcal{X} \to \mathcal{Y}$ over $\mathcal{O}$. By enlarging $S$ if necessary, we assume that the relative de
Rham (resp. Hodge) cohomology $\mathcal{H}^1 := \mathbb{R}^q \pi_* \Omega^*_{X/Y}$ (resp. $\mathbb{R}^q \pi_* \Omega^p_{X/Y}$) are locally free $\mathcal{O}_Y$-modules, and that the Gauss–Manin connection over $K$ extends to

$$\nabla_{GM} : \mathcal{H}^1 \to \mathcal{H}^1 \otimes \Omega^1_{Y/K}.$$

The connection $\nabla_{GM}$ tells us how to identify de Rham cohomology of fibers over the $p$-adic disc $\Omega_v$ (resp. over some small enough complex analytic disc $\Omega_C$ near $y_0$). In other words, we obtain the identifications

$$\text{GM}_v : H^1_{dR}(X_{y_0}/K_v) \sim \to H^1_{dR}(X_y/K_v)$$

for any $y \in \Omega_v$ and likewise

$$\text{GM}_C : H^1_{dR}(X_{y_0}/C) \sim \to H^1_{dR}(X_y/C)$$

for any $y$ in a small enough $\Omega_C = \Omega_C(\epsilon)$. Moreover, after choosing a suitable local basis for $\mathcal{H}^q$ near $y_0$, both maps are given by the evaluation of a square matrix $A_{ij}$ at $y$, where $A_{ij}$ has entries in

$$\mathcal{O}_Y \subset \mathcal{O}_v[[z_1, \ldots, z_m]] \subset \mathcal{O}_v[[z_1, \ldots, z_m]] \cong \widehat{\mathcal{O}}_{Y, y_0},$$

and is convergent in both $\Omega_v$ and $\Omega_C$. The upshot is that both GM maps are given by analytic functions with $K$-coefficients.

5.4. Period maps over $K_v$. Let $\mathcal{H}_K$ be the flag variety over $K$ that describes the Hodge filtration (and its behavior under polarization) on the de Rham cohomology $H^1_{dR}(X_{y_0}/K)$, with the base point $b_0$ given by

$$\text{Fil}^1 H^1_{dR}(X_{y_0}/K).$$

For us the relevant flag variety will be the Lagrangian grassmannian $\text{LGr} = \text{LGr}(V_0, \lambda)$, which parametrizes Lagrangian subspaces of $V_0 = H^1_{dR}(X_{y_0}/K)$, under the symplectic pairing $\lambda$ given by the data of polarization coming from $X$ (this is part of the initial input). The Gauss-Manin connection gives rise to a power series in $K$, which induces the following period mappings (by transporting the Hodge filtration of $H^1_{dR}(X_{y}/K_v)$ to a filtration on $V_0$ by the Gauss–Manin connection, i.e., the isomorphism GM):

- a $K_v$-analytic map
  $$\Phi_v : \Omega_v \to \mathcal{H}(K_v).$$
- a $\mathbb{C}$-analytic map
  $$\Phi_C : \Omega_C \to \mathcal{H}(\mathbb{C})$$

where $\Omega_C$ is some small complex disk.

As mentioned earlier, both maps come from some power series over $K$ by looking at the Gauss–Manin connection near the point $y_0$.

\textsuperscript{11}note that the last identification is non-canonical but determines the first inclusion.
Let us focus on the $p$-adic period mapping first. To this end, we introduce the Frobenius automorphism into the picture. Since $v$ is unramified above $p$, we have functorial isomorphisms

$$H^1_{dR}(X_{y_0}/K_v) \xrightarrow{\sim} H^1_{crys}(X_{K_v}/\mathcal{O}_{K_v})[\frac{1}{p}] \xrightarrow{\sim} H^1_{dR}(X_y/K_v)$$

for each $y \in \Omega_v$, which provides $V_0$ (resp. $V_{dR,y} := H^1_{dR}(X_y/K_v)$) the Frobenius automorphism $\phi_0$ (resp. $\phi_v$), coming from the functorial Frobenius $\phi$ of crystalline cohomology. Furthermore, the composition above in fact agrees with the map GM given by the Gauss–Manin connection, in other words, GM sends the filtered $\phi$-module $(V_0, \phi_0, \Phi_v(y))$ to $(V_{dR,y}, \phi_y, \Fil^1 V_{dR,y})$.

Let us denote by $\text{MF}^\phi_{/K_v}$ the category of filtered $\phi$-modules over $K_v$. As before, let $Y(K, y_0) = Y(K) \cap \Omega_v$ be the set of rational points in the residue disc $\Omega_v$, and define

$$Y(K, y_0)^{ss} := \{ y \in Y(K, y_0) : \rho_y \text{ is semisimple} \}$$

where $\rho_y$ is the Gal$_K$ representation coming from the $p$-adic cohomology of the (geometric) fiber $X^\text{y,R}$. Now consider the following map of sets

$$Y(K, y_0)^{ss} \xrightarrow{\gamma} \left\{ \text{Permissible Gal}_K\text{-reps of level S, pure wt 1, crystalline at } v \right\}$$

$$\xrightarrow{\theta} \left\{ \text{Crystalline Gal}_{K_v}\text{-reps} \right\}$$

$$\xrightarrow{\text{D}_{crys}} \text{MF}^\phi_{/K_v}$$

By construction $\theta$ sends $y$ to $(\text{D}_{dR}(V_y), \phi_{\text{D}_{crys}(V_y)}, \Fil^1 \text{D}_{dR})$ by definition, where $V_y = H^1(X_{y,R}, \mathbb{Q}_p)$. Now by the crystalline ($\text{B}_{crys}$) comparison theorem, we have

$$\theta(y) = (V_{dR,y}, \phi_y, \Fil^1).$$

**Remark 5.1.** This comparison (not completely trivial), together with the (not so difficult) fact that $\text{D}_{crys}$ is fully faithful $^{12}$, are the only $p$-adic Hodge theory inputs used in the proof.

Finally, for any $\phi$-module $(V, \phi)$ over $K_v$, we use $Z_\phi$ (the centralizer of Frobenius) to denote the $K_v$-linear endomorphisms $\text{End}_\phi(V)$ in the category of $\phi$-modules. Clearly $Z_\phi$ is a $\mathbb{Q}_p$ subspace of $\text{End}(V)$.

The following lemma reveals the structure of the arguments of [1]. Let $\kappa_v = [K_v : \mathbb{Q}_p]$, and recall that $\text{im}(\Phi_v)$ denotes the Zariski closure of the image of $\Phi_v$ in the $K_v$-scheme $\mathcal{H}_v$.

$^{12}$It is a much less trivial fact that one can describe the essential image of this functor, but we will not need it.
Lemma 5.2. Write $\phi_v = \phi_0$ as above. Assume the following

$$(\text{Hyp } \star) \quad \dim_{K_v} Z_{\phi_v^{\infty_v}} < \dim \overline{\text{im}}(\Phi_v).$$

Then $Y(K, y_0)^{ss}$ is finite.

**Proof.** It is convenient to introduce a (rigidified) intermediate set

$$\tilde{\text{MF}}_{K_v}^\phi := \left\{ (V, \phi, \text{Fil}, \iota) \mid \begin{array}{l}
(V, \phi) \mapsto (V_0, \phi_0) \text{ is an isom of } \phi-
\text{modules.}
\end{array} \right\}$$

From our discussion above on Gauss–Manin connections, the period map $\Phi_v$ on $Y(K, y_0)^{ss}$ factors through $\tilde{\text{MF}}_{K_v}^\phi$

$$\begin{array}{c}
Y(K, y_0)^{ss} \\
\Omega_v \xrightarrow{\Phi_v} \mathcal{H}(K_v)
\end{array}$$

where $\tilde{\theta}$ sends $y \in Y(K, y_0)^{ss}$ to the tuple $(V_{dR,y}, \phi_y, \text{Fil}^1, \iota_v)$ where $\iota_v$ is given by $\text{GM}^{-1}$. The map $\delta$ is easy to describe: for any $(V, \phi, \text{Fil}, \iota)$, we use $\iota$ to transport the filtration $\text{Fil}$ on $V$ to a filtration $\iota(\text{Fil})$ on $V_0$, which gives the image of $\delta$. The commutativity of the diagram above is clear, as $\Theta_v(y)$ is defined by transporting the filtration $\text{Fil}^1$ using the Gauss–Manin connection. The upshot is that $\text{im}(\tilde{\theta})$ only contains finitely many isomorphism classes of filtered $\phi$-modules, since the image of $\gamma$ (in the diagram above the lemma) only contains finitely isomorphism classes by Proposition 3.4, and the vertical arrows in that diagram can be upgraded to functors.

In particular, for elements $(V, \phi, \text{Fil}, \iota)$ in the image of $\tilde{\theta}$, the resulting filtered $\phi$-modules $(V_0, \phi_0, \iota(\text{Fil}))$ only contains finitely many isomorphism classes. Pick a representative $M_i$ in each such isomorphism class, and denote their images under $\delta$ by $h_i \in \mathcal{H}(K_v)$. Unwinding definitions, the Zariski closure of $\Phi_v(Y(K, y_0)^{ss})$ in $\mathcal{H}(K_v)$ lies inside the closure of $Z_{\phi_v^{\infty_v}} \cap \{ \cup h_i \}$, which has dimension less or equal than $\dim_{K_v} Z_{\phi_v^{\infty_v}} < \dim \overline{\text{im}}(\Phi_v)$. This means that $Y(K, y_0)^{ss}$ is contained in a sub-analytic-manifold $Z_v \subset \Omega_v$ where $Z_v$ is cut out by some power series, hence finite. □

5.5. **Comparison between period maps over $K_v$ and over $\mathbb{C}$.** Now we immediately arrive at the following central question:

How do we verify Hypothesis $\star$?

Intuitively we should make the right hand side as big as possible (namely we will try to find a family so that $\Phi_v$ has dense image) and make the left hand side reasonably small.
To deal with right hand side, we observe the following crucial lemma. Let $m$ be a positive integer and $\epsilon \in \mathbb{R}_{>0}$, let $U_v(\epsilon)$ denote the $(m\text{-dimensional}) v$-adic open polydisk $U_v(\epsilon) = \{z : |z_1|_v < \epsilon\}$, and similarly define the $m$-dimensional complex polydisk $U_C(\epsilon)$.

**Lemma 5.3.** Suppose that we are given a sequence of power series

$$b_0, ..., b_n \in K[[t_1, ..., t_m]]$$

in $m$-variables, each converging absolutely on both $U_v(\epsilon)$ and $U_C(\epsilon)$ for some $\epsilon$. They give rise to maps $B_v : U_v(\epsilon) \to \mathbb{P}^n_K$ and $B_C : U_C(\epsilon) \to \mathbb{P}^n_C$. Then there exists a $K$-subscheme $Z \subset \mathbb{P}^n_K$ such that

$$\overline{\text{im}}(B_v) = Z_{K_v}, \quad \overline{\text{im}}(B_C) = Z_C.$$

**Proof.** Let $I$ be the ideal in $K[x_0, ..., x_n]$ generated by homogeneous polynomials $Q$ such that $Q(b_0, ..., b_n) = 0$. This gives rise to a subscheme $Z \subset \mathbb{P}^n_K$. Let us show that $\overline{\text{im}}(B_v) = Z_{K_v}$. Suppose that $Q_v \in K_v[x_0, ..., x_n]$ is a homogenous polynomial vanishing on $B_v(U_v(\epsilon))$, we want to show that $Q_v$ lies in the $K_v$-span of $I$. Since $Q_v(b_0, ..., b_n) = 0$ on $U_v(\epsilon)$, we know that $Q_v(b_0, ..., b_n)$ is identically 0 (for example by the Weierstrass preparation theorem for Tate algebras), which gives rise to an infinite system of $K$-linear equations on finitely many variables (given by the nonzero coefficients of $Q_v$). This cuts out a subscheme $A_{K_v}^r \subset A_K^k$ where $k$ is the number of the coefficients in $Q_v$, and each $K_v$-solution to this infinite system gives a $K_v$-point on the subscheme, which is clearly a $K_v$-linear combination of $K$-points. A similar argument shows that $\overline{\text{im}}(B_C) = Z_C$. \qed

As an important corollary, we have

**Corollary 5.4.** Retain the notations from the previous subsection,

$$\dim \overline{\text{im}}(\Phi_v) = \dim \overline{\text{im}}(\Phi_C)$$

The purpose of the following definition is now clear.

**Definition 5.5.** A family $X \to Y$ is said to have full monodromy at $y_0$ if $\Phi_v$ has dense image in $H_{K_v}$, or equivalently, if $\Phi_C$ has dense image.

**Remark 5.6.** The upshot is now the following: under the assumption of full monodromy, Hypothesis $\star$ becomes the following

$$\text{(Hyp } \star \star \text{)} \quad \dim K_v Z_{\Phi v} < \dim H_v$$

where we have made the image of $\Phi_v$ “as large as possible”. In other words, if we have full monodromy and Hypothesis $\star \star$, then $Y(K, y_0)^{ss}$ is finite.

6. The key reduction to monodromy I

6.1. The Frobenius centralizer. From now on let us assume we have a polarized abelian scheme $(X, \lambda) \to Y$ such that the family $X \to Y$ has full monodromy at $y_0 \in Y(K)$. Let us try to finish the argument using methods outlined in the previous section.
Define $C(K, y_0)^{ss}$ to be the complement of $Y(K, y_0)^{ss}$ in $Y(K, y_0)$. Of course it would be great if we could show that both $Y(K, y_0)^{ss}$ and $C(K, y_0)^{ss}$ are finite sets. Let us focus on $Y(K, y_0)^{ss}$ for the moment, which seems to be the easier part, at least by Remark 5.6. However, the following simple computation reveals that we are being a bit over optimistic. Let $d$ be the (relative) dimension of the abelian scheme $X 	o Y$, then LGr parametrizes $d$-dimensional Lagrangian subspaces of the $2d$-dimensional $K_v$-vector space $H^1_{dR}(X_{y_0}/K_v)$ (equipped with the symplectic pairing from the polarization $\lambda$ on $X$), so $H_v$ has dimension $d(d + 1)/2$. For the Frobenius centralizer, we know that $Z_{\phi^e_v}$ is a $K_v$-subspace of $\text{End}(V)$, so there is a trivial bound $\dim_{K_v} Z_{\phi^e_v} \leq (2d)^2 = 4d^2$. Clearly this is not enough to force Hyp **.

**Remark 6.1.** To use the trivial bound on $Z_{\phi^e_v}$, we need to make the RHS at least 8 times larger, since $4d^2 < 8 \cdot \frac{d(d+1)}{2}$.

To remedy the issue discussed above (which we do in the next subsection), we first observe the following simple lemma

**Lemma 6.2.** Let $\sigma : E_w \to E_w$ be a field automorphism of order $e$ and let $K_v = E_w^\sigma$. Let $V$ be a $E_w$ vector space of dimension $2d$ equipped with a $\sigma$-semilinear automorphism $\phi : V \to V$, then the natural map

$$Z_{\phi} \otimes_{K_v} E_w \hookrightarrow Z_{(\phi^\sigma)}$$

is injective. In particular we have

$$\dim_{K_v} Z_{\phi} \leq \dim_{E_w} Z_{(\phi^\sigma)} \leq (2d)^2.$$

**Proof.** Choose basis $\{f_1, \ldots, f_r\}$ for the $K_v$ vector space $Z_{\phi}$, assume for contradiction that there exist nontrivial linear relations $\sum \alpha_i f_i = 0$ with $\alpha_i \in E_w$. Pick one such relation that involves the minimum number of the basis $\{f_i\}$ and (by reordering if necessary) assume that $\alpha_1 = -1$, we can now write

$$f_1 = \sum_{2 \leq i \leq r} \alpha_i f_i.$$

By assumption we have $\phi(f_1 x) = f_1(\phi(x))$ for all $x \in V$, which implies that

$$\sum_{2 \leq i \leq r} \alpha_i f_i(\phi(x)) = \phi\left( \sum_{2 \leq i \leq r} \alpha_i f_i(x) \right) = \sum_{2 \leq i \leq r} \sigma(\alpha_i) f_i(\phi(x))$$

for all $\phi(x) \in V$, hence

$$\sum_{2 \leq i \leq r} \left( \sigma(\alpha_i) - \alpha_i \right) f_i = 0.$$

This would give a relation with less $f_i$’s involved unless all the coefficients are $0$, in other words, unless $\alpha_i \in K_v$ for each $i$, which would then contradict the assumption that $\{f_i\}$ form a $K_v$-basis for $Z_{\phi}$. \hfill $\square$

This is essentially Galois descent (in fact it is true that $Z_{\phi} \otimes_{K_v} E_w \xrightarrow{\sim} Z_{(\phi^\sigma)}$). Also note that we will use the lemma with $K_v$ being $K_v$ from the previous subsections, $E_w$ an unramified extension of $K_v$, $\sigma$ a generator of $\text{Gal}(E_w/K_v)$, and $\phi = \phi^e_v$.
6.2. A (semi)naive approach. Recall that to achieve Hypothesis ** we need to control the size of the centralizer of Frobenius while making the dimension of the (Lagrangian) grassmannian grow simultaneously. For this (and with the help of the lemma above), we are led to consider not family of abelian varieties over $Y$, but rather a family of the form

$$X \to Y' \to Y$$

where $X = (X, \lambda)$ is a polarized abelian scheme over $Y'$ and $Y'$ is a finite étale extension of $Y$. The goal of this section is to prove Proposition 6.4 – the most difficult part is actually setting up notations.

First let us briefly redo the analysis of Subsection 5.3 in the new setup. Again without loss of generality we assume that we have an integral model $X \to Y' \to Y$ and the Gauss–Manin connection extends integrally. For a point $y : \text{Spec} \ K \to Y$ in $Y(\mathbb{K})$, let $E_y = \prod_{y' \in \pi^{-1}(y)} \mathbb{K}_{y'}$ be the étale $\mathbb{K}$-algebra which corresponds to $\pi^{-1}(y) \subset Y'$. The polarization $\lambda$ on $X$ induces a $E_y$-bilinear symplectic pairing on $H^1_{dR}(X_y / E_y) \cong H^1_{dR}(X_y / \mathbb{K})$ (as $E_y$ has no differentials over $\mathbb{K}$). Write $E_{y,v} := E_y \otimes_{\mathbb{K}} \mathbb{K}_v$, which splits as a product of local fields

$$E_{y,v} = \prod_{(y',w)} \mathbb{K}_{y',w}$$

where $y' \in \pi^{-1}(y)$, and $w$ is a place of the global field $\mathbb{K}$ over $v$. Write $X_{y',w} = X_y \otimes_{E_y} \mathbb{K}_{y',w}$, then we have a splitting of de Rham cohomology

$$H^1_{dR}(X_y / \mathbb{K}) \cong H^1_{dR}(X_y / E_y) = \prod_{(y',w)} H^1_{dR}(X_{y',w} / \mathbb{K}_{y',w}),$$

which (as $E_{y,v}$-modules) is compatible with the splitting $E_{y,v} = \prod \mathbb{K}_{y',w}$. At the base point $y_0$, write $E_0$ for $E_{y_0}$. Similarly we have $E_{0,v} = \prod \mathbb{K}_{y'_0,w_0}$ and a splitting for $H^1_{dR}(X_{y_0} / E_{0,v})$.

The relevant flag variety for us is now

$$\mathcal{H}_v := \text{Res}_{K_v}^E \text{LGr}(H^1_{dR}(X_{y_0} / \mathbb{K}_v), \lambda).$$

This scheme also splits as a product

$$\mathcal{H}_v = \prod_{(y'_0,w_0)} \mathcal{H}_{y'_0,w_0},$$

where the product is taken over all $(y'_0, w_0)$ lying above $(y_0, v)$, and

$$\mathcal{H}_{y'_0,w_0} = \text{Res}_{K_v}^K \text{LGr}(H^1_{dR}(X_{y'_0, w_0} / \mathbb{K}_{y'_0,w_0}), \lambda).$$

\footnote{It is almost miraculous that this can actually be made to work (and although the proof given below is completely elementary, I have no intuition of why this should work \textit{a priori}).}
Now we get to the period map $\Phi_v$. For $y \in Y(K, y_0)$, the Gauss–Manin connection gives rise to an isomorphism $E_{0,v} \xrightarrow{\sim} E_{y,v}$ (viewed as de Rham cohomology of $\pi^{-1}(y_0)$ and $\pi^{-1}(y)$ for example), which induces a bijection

$$(y', w) \longleftrightarrow (y_0', w_0)$$

between the set of $(y', w)$ that lie above $(y, v)$ and $(y_0', w_0)$ that lie above $(y_0, v)$, and is compatible with

$$GM_v : H^1_{dR}(X_{y_0}/K_v) \xrightarrow{\sim} H^1_{dR}(X_y/K_v).$$

To proceed, we make the following definition.

**Definition 6.3.** Define $Y(K, y_0)^{naive-ss}$ to be the set of points $y \in Y(K, y_0)$ such that there exists $(y', w)$ lying over $(y, v)$ satisfying that $\rho_{y'}$ is a semisimple Gal$_{K_y'}$-representation, and that $[K_{y', w_0} : K_v] \geq 8$. To wit (this somewhat complicated definition), let us denote $F_0 := \{ y_0' \in \pi^{-1}(y_0) \mid \exists w_0 \text{ over } v \text{ such that } [K_{y_0', w_0} : K_v] \geq 8 \}$.

For each $y_0' \in F_0$, fix once and for all such a place $w_0$ (satisfying $[K_{y_0', w_0} : K_v] \geq 8$). For each $y \in Y(K, y_0)$, we get a corresponding finite set $F_y$, and a fixed place $w$ lying over $v$ for each $y' \in F_y$. Now unwinding the definition above, we have

$$Y(K, y_0)^{naive-ss} := \left\{ y \in Y(K, y_0) \mid \text{there exists } y' \in F_y \text{ s.t. } \rho_{y'} \text{ is semisimple.} \right\}.$$

Assuming full monodromy at $y_0$ as usual, we have the following

**Proposition 6.4.** $Y(K, y_0)^{naive-ss}$ is finite (under the assumption of full monodromy).

The proof is now very similar to the proof of Lemma 5.2, with mostly notational differences.

**Proof.** Note that in the definition above, we have fixed a place $w_0$ over $v$ for each $K_{y_0'}$ with $y_0' \in F_0$. Define

$$Y(K, y_0)^{(y_0',w_0)} := \left\{ y \in Y(K, y_0) \mid \rho_{y'} \text{ is semisimple, where } y' \text{ is determined by the bijection } (y_0', w_0) \longleftrightarrow \right\}.$$

So $Y(K, y_0)^{naive-ss}$ is finite union

$$Y(K, y_0)^{naive-ss} = \bigcup_{y_0' \in F_0} Y(K, y_0)^{(y_0',w_0)}.$$
and it suffices to show that each \( Y(K, y_0)(y'_0, w_0) \) is finite. To this end we consider the following commutative diagram

\[
\begin{array}{ccc}
Y(K, y_0)(y'_0, w_0) & \overset{\theta = \theta_{y'_0, w_0}}{\longrightarrow} & \tilde{\text{MF}}_{K'_{y'_0, w_0}} \\
\Omega_v & \Phi_v & \mathcal{H}_v \\
\downarrow & \downarrow & \downarrow \delta = \delta_{y'_0, w_0} \\
\mathcal{H}_{y'_0, w_0} & & \end{array}
\]

In this diagram \( \mathcal{H}_v \) only plays an auxiliary role so we ignore it. By the same arguments as Lemma 5.2, it suffices to check that

\[
\dim_{K_v} Z_{\phi_v^{K_v}} < \dim \mathcal{H}_{y'_0, w_0}
\]

where the latter is the dimension as a \( K_v \)-scheme. Note that now the Frobenius centralizers live inside the \textit{a priori} quite large space

\[
\text{End}_{K_{y'_0, w_0}} \left( H^1_{dR}(X, K_{y'_0, w_0}) \right)
\]

which has \( K_{y'_0, w_0} \) dimension \( 4d^2 \). However, (now the miracle comes!) by Lemma 6.2, we know that

\[
\dim_{K_v} Z_{\phi_v^{K_v}} \leq 4d^2.
\]

Therefore we win because

\[
\dim_{K_v} \mathcal{H}_{y'_0, w_0} = [K_{y'_0, w_0} : K_v] \cdot \dim_{K_{y'_0, w_0}} \text{LGr} \geq 8 \cdot \frac{d(d + 1)}{2}.
\]

Now we have taken care of “half” of \( Y(K, y_0) \). Let us turn to the complement of \( Y(K, y_0)^{\text{naive-ss}} \)

\[
C(K, y_0)^{\text{naive}} := Y(K, y_0) \setminus Y(K, y_0)^{\text{naive-ss}}.
\]

We would be happy if we could show that \( C(K, y_0)^{\text{naive}} \) is finite – in other words, that failure of semisimplicity could only occur finitely many times. This turns out to be difficult(?), at least according to [1]. Instead we will consider a certain subset

\[
Y(K, y_0)^* \subset Y(K, y_0),
\]

and define \( Y(K, y_0)^{*, \text{ss}} \) (resp. \( C(K, y_0)^* \)) to be the intersection of \( Y(K, y_0)^* \) with \( Y(K, y_0)^{\text{naive-ss}} \) (resp. with \( C(K, y_0)^{\text{naive}} \)). We now arrive at the following diagram

\[
\begin{array}{ccc}
Y(K, y_0)^{*, \text{ss}} & \longrightarrow & Y(K, y_0)^* \\
\downarrow & & \downarrow \\
Y(K, y_0)^{\text{naive-ss}} & \longrightarrow & Y(K, y_0) \\
& & \longrightarrow \end{array}
\]

\[
\begin{array}{ccc}
& & C(K, y_0)^* \\
& & \downarrow \\
& & C(K, y_0)^{\text{naive}}
\end{array}
\]
Proposition 6.4 tells us that both sets on the left are finite.

**Remark 6.5.** The strategy to prove Mordell’s conjecture is the following:

- Introduce a suitable condition \(^*\) so that \(C(K, y_0)^*\) is always finite.
- Then carefully construct a family \(X \to Y' \to Y\), such that for each residue disc \(Y(K, y_0)\) we have
  
  1. \(Y(K, y_0)^* = Y(K, y_0)\);
  2. The family has full monodromy at \(y_0\).

We take care of the first bullet point in the next section, and briefly comment on the second bullet point in the remaining of the article.

### 7. The key reduction to monodromy II

#### 7.1. Modified approach using the size of Frobenius orbit.

The \(^*\) condition alluded to above is given by a certain “size” measuring the size of Frobenius orbits of \(\pi^{-1}(y)\) at each \(y\).

**Definition 7.1.** For a finite set \(T\) together with a continuous \(G_K\)-action unramified at \(v\), we define

\[
\text{Size}_v(T) = \frac{\#\{t \in T : \#\text{Orb}_{\text{Frob}_v}(t) < 8\}}{\#T},
\]

where the numerator is the number of elements in \(T\) that belong to “small” Frobenius orbits (namely \(\text{Frob}_v\)-orbits of size < 8).

**Definition 7.2.** Let \(d\) be the relative dimension of \(X \to Y'\) in the family \(X \to Y' \to Y\) as usual. Define

\[
Y(K, y_0)^* := \left\{ y \in Y(K, y_0) \mid \text{Size}_v(\pi^{-1}(y)) < \frac{1}{d + 1} \right\}.
\]

This also defines \(C(K, y_0)^*\) as in the diagram before Remark 6.5.

As promised, the goal of this section is to prove the following

**Proposition 7.3.** \(C(K, y_0)^*\) is finite.

#### 7.2. Another proper subscheme of the flag variety.

The basic idea of proving Proposition 7.3 is the same as before: we want to find a proper subscheme of \(H_v\) that contains \(\Phi_v(C(K, y_0)^*)\). It turns out that the assumption on \(\text{Size}_v\) ensures that \(C(K, y_0)^*\) is contained in the union of all \(C(K, y_0)^*,(y_0',w_0)\). Hence it suffices to show that \(C(K, y_0)^*,(y_0',w_0)\) is finite. It turns out that the condition of non-semisimplicity, together with the requirement that local extension degree is large while \(\text{Size}_v\) is small, is enough to cut out a proper subscheme of \(H_v\). Let us make this more precise.

For each \(K\)-point \(y \in C(K, y_0)^*\), we will need to consider all \((y', w)\) lying over \((y, v)\) at the same time, where the local extension has large degree (not just the fixed \(w\) of each \(y' \in E_y\)). Let \((y_0', w_0)\) be such a pair over \((y_0, v)\), recall that the image of \(y\) under the following composition

\[
\Phi_{y_0',w_0} : C(K, y_0)^*,(y_0',w_0) \hookrightarrow \Omega_v \xrightarrow{\Phi_v} H_v \xrightarrow{\text{proj}} H_{y_0,w_0}
\]
is given by transporting the Hodge filtration $\text{Fil}^1$ using the Gauss–Manin connection from the (filtration given by the) filtered $\phi$-module

$$\left(H^1_{\text{dR}}(X_{y',w}/K_{y',w}), \phi_{y',w}, \text{Fil}^1\right),$$

here $(y', w) \leftrightarrow (y'_0, w_0)$ as usual. For notational convenience we introduce the following set. **For each pair $(y'_0, w_0)$ such that** $[K_{y'_0,w_0} : K_v] \geq 8$, we define

$$C(K, y_0)^*(y'_0, w_0) := \left\{ y \in C(K, y_0)^* \mid \text{there is a nonzero proper } \phi\text{-stable subspace } W_{y',w} \subset H^1_{\text{dR}}(X_{y',w}/K_{y',w}) \text{ s.t. } \dim_{K_{y',w}} \text{Fil}^1 W_{y',w} \geq \frac{1}{2} \dim_{K_{y',w}} W_{y',w} \right\}. $$

Where, in the definition, $(y', w)$ is determined via $(y', w) \leftrightarrow (y'_0, w_0)$

**Proposition 7.4.** Notation as above, we have

$$C(K, y_0)^* \subset \bigcup_{[K_{y'_0,w_0} : K_v] \geq 8} C(K, y_0)^*(y'_0, w_0).$$

In other words, for each $y \in C(K, y_0)^*$, there exists a $(y', v)$ over $(y', v)$ and a nonzero proper Frobenius stable subspace $W_{y',w} \subset H^1_{\text{dR}}(X_{y',w}/K_{y',w})$ satisfying $[K_{y',w} : K_v] \geq 8$ and that

$$\dim_{K_{y',w}} \text{Fil}^1 W_{y',w} \geq \frac{1}{2} \dim_{K_{y',w}} W_{y',w}.$$ 

Here $\text{Fil}^1 W_{y',w} := \text{Fil}^1 \cap W_{y',w} \subset H^1_{\text{dR}}(X_{y',w}/K_{y',w})$.

We prove this proposition in the next subsection. First let us deduce Proposition 7.3 from it. For this, let $L_w$ be a finite unramified extension of $K_v$ of degree $\geq 8$, with Frobenius automorphism $\sigma$ being the a generator for $L_w/K_v$. Let $(V, \lambda)$ be a symplectic vector space over $L_w$ of $L_w$-dimension $2d$, where $V$ is equipped with a bijective $\sigma$-linear endomorphism $\phi$ (hence $(V, \phi)$ becomes an isocrystal over $L_w$). Let $\mathcal{H}_w := \text{Res}_{L_w}^{K_v} \text{LGr}(V, \lambda)$, which is a $K_v$-scheme whose $K_v$ points parametrize Lagrangian $L_w$-subspaces $F \subset V$ of dimension $d$. Let $\mathcal{F} \subset \mathcal{H}_w$ denote the subscheme whose $K_v$-points parametrize $F \subset V$ such that there exists a $\phi$-stable subspace $W \subset V$ satisfying $\dim(F \cap W) \geq \dim W/2$.

**Lemma 7.5.** $\mathcal{F} \subset \mathcal{H}_w$ is a proper closed subscheme.

**Proof sketch.** Let $r = [L_w : K_v]$. By considering $V \otimes_{K_v} \overline{K}_v$, which splits as a direct sum $\otimes_{\tau} V_{\tau}$ where $\tau : L_w \to \overline{K}_v$ runs over the $r$ embeddings of $L_w$ into $\overline{K}_v$, we are reduced to show that the subfunctor

$$\mathcal{E} \subset \text{LGr}(V, \lambda)^r = \text{LGr}(V, \lambda) \times \cdots \times \text{LGr}(V, \lambda)$$

which parametrizes $(F_1, ..., F_r) \in \text{LGr}(V, \lambda)^r$ such that there exists a nontrivial proper subspace $W \subset V$ satisfying $\dim F_i \cap W \geq \dim W/2$ for all $i$, is representable by a proper
closed subscheme of $\text{LGr}(V, \lambda)^r$. Now consider the diagram

$$
\tilde{E} \subset \text{Gr}(V) \times \text{LGr}(V, \lambda)^r \\
\xrightarrow{pr} \\
\text{LGr}(V, \lambda)^r
$$

where $\tilde{E}$ parametrizes $(W, F_1, ..., F_r)$ such that $\text{dim}(F_i \cap W) \geq \text{dim} W/2$. Clearly $\tilde{E}$ is a closed subscheme by upper semicontinuity, and $E$ is its scheme theoretic image under the projection map (which is proper), so $\tilde{E}$ is a closed subscheme. Finally, since $\text{LGr}(V, \lambda)$ is irreducible, it suffices to exhibit a point in the complement of $\tilde{E}$.

One way to do this is the following. Pick basis $e_1, ..., e_d, e'_1, ..., e'_d$ so that $\lambda$ takes the standard form $\begin{pmatrix} -1_d & 1_d \end{pmatrix}$. Consider

$$
F_1 = \langle e_1, e_2, ..., e_d \rangle \\
F_2 = \langle e'_1, e'_2, ..., e'_d \rangle \\
F_3 = \langle e_1 + e'_1, e_2 + e'_2, ..., e_d + e'_d \rangle \\
F_4 = \langle e_1 - e'_1, e_2 - e'_2, ..., e_d - e'_d \rangle \\
F_5 = \langle 2e_1 + e'_1, 3e_2 + e'_2, ..., (d + 1)e_d + e'_d \rangle \\
F_6 = \langle e_2 + e'_1, e_3 + e'_2, ..., e_1 + e'_d \rangle
$$

Suppose $W$ is a proper subspace satisfying $\text{dim} F_i \cap W \geq \text{dim} W/2$ for each $i$, then $W = (W \cap F_i) \oplus (W \cap F_j)$ for all $1 \leq i \neq j \leq 6$, by the dimension restriction and the fact that $F_i$'s pairwise intersect trivially. In particular, $W$ is of dimension $2\delta$, with $0 < \delta < d$. From this it is not difficult to deduce that $F_1 \cap W$ is $\delta$-dimensional subspace of $V$ stable under the reflection $e_i \mapsto -e_i$, the “expansion” $e_i \mapsto (i + 1)e_i$, and stable under the rotation $e_i \mapsto e_{i+1}$. The only such subspaces are 0 and $F_1$, which is a contradiction as $0 < \delta < d$.

**Corollary 7.6.** Each $C(K, y_0)^*, (y_0', w_0)$ is finite.

**Proof.** By construction, $\Phi_{y_0', w_0} \left( C(K, y_0)^*, (y_0', w_0) \right) \subset \mathcal{F}_{y_0', w_0} \subset \mathcal{H}_{y_0', w_0}$. □

### 7.3. Frobenius and Hodge-Tate weights

Now we prove Proposition 7.4, which concludes the finiteness of $C(K, y_0)^*$. The key is to understand the relation between Hodge–Tate weights at places above $p$ and Frobenius weights (in the sense of pure weight $w$) for a global de Rham representation. For the setup, let $V$ be a geometric $\mathbb{Q}_p$-valued representation of $\text{Gal}_K$ which is crystalline above $p$ (note that the assumption of being crystalline is not actually needed, being de Rham is enough). Assume that $K$ contains no CM subfield as before (now we finally use this assumption) and $v$ an unramified place. The assumption on $v$ ensures that $D = D_{\text{cris}}(V)$ is equipped with a filtration $\text{Fil}$ coming from $B_{\text{dR}}^+$. The Hodge number $t_H(D) := \sum i \dim \text{Gr}_i(D)$ computes the sum of Hodge–Tate weights of $V$.
counted with multiplicities. Recall that slope of (the endpoint of the) Hodge polygon is given by

$$\mu_H(D) := \frac{t_H(D)}{\dim D} = \frac{\sum_i i \dim \text{Gr}^i(D)}{\dim D}$$

Further assume that $V$ is pure of weight $w$.

**Lemma 7.7.** $\mu_H(D_{\text{cris}}(V)) = w/2$.

**Proof.** By taking $\text{det}(V)$ the lemma is equivalent to the following claim: let $\eta : \text{Gal}_K \to \mathbb{Q}_p^\times$ be a geometric character, pure of weight $\tilde{w}$, then for each $\tau : K_v \hookrightarrow \overline{K}_v$, the Hodge–Tate weight $HT_\tau(\eta) = \tilde{w}/2$. This in turn follows from a lemma of Artin–Weil, which says that (in terms of Hecke characters $\psi : \mathbb{A}_K^\times / \mathbb{Q}^\times \to \mathbb{C}^\times$ associated to $\eta$), the infinity type of an algebraic Hecke character descends to the maximal CM subfield. More precisely, in our case (since the maximal CM subfield is $\mathbb{Q}$), up to a finite order character, $\psi$ is the base-change of a power of the absolute value character $|\cdot| : \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times \to \mathbb{C}^\times$. Now use the fact that $\eta$ is pure of weight $\tilde{w}$, we know that $\psi \mid_{K_v^\times} = \psi_0 \cdot \text{BC}_{K_v / \mathbb{Q}_p} |\cdot|^{\tilde{w}/2}$ where $\psi_0$ is a finite order character (if $\eta$ is crystalline – in our case it is – then $\psi_0$ is trivial). This finishes the proof of the lemma. \qed

Finally we prove Proposition 7.4, hence Proposition 7.3.

**Proof of Proposition 7.4.** Let $y \in C(K, y_0)^\ast$. Assume for contradiction that, for each $(y', w)$ with $[K_{y_0, w_0} : K_v] \geq 8$, and each $\phi$-invariant subspace $W \subset H^1_{\text{dR}}(X_{y', w}/K_{y', w})$, we have

$$2 \cdot \dim \text{Fil}^1 W \leq \dim W - 1.$$ 

Now for each $y' \in \pi^{-1}(y)$, let $V^\text{min}_{y'} \subset \rho_{y'}$ be a nonzero subrepresentation of $\rho_{y'}$ of minimal dimension (could be all of $\rho_{y'}$). Let

$$D^\text{min}_{y', w} := D_{\text{cris}}(V^\text{min}_{y'}) \subset H^1_{\text{dR}}(X_{y', w}/K_{y', w}) = D_{\text{cris}}(\rho_{y'})$$

as a Frobenius stable subspace. By assumption, we know that whenever $[K_{y_0, w_0} : K_v] \geq 8$, then $2 \dim \text{Fil}^1 D^\text{min}_{y', w} \leq \dim D^\text{min}_{y', w} - 1$. Moreover, by definition, in this case $\rho_{y'}$ is not semisimple, hence $\dim D^\text{min}_{y', w} \leq d$ (by minimality assumption, since the dual of $V^\text{min}_{y'}$ under the symplectic pairing is a sub-representation of complementary dimension). This means that

$$\frac{\dim \text{Fil}^1 D^\text{min}_{y', w}}{\dim D^\text{min}_{y', w}} \leq \frac{1}{2} \frac{\dim D^\text{min}_{y', w} - 1}{\dim D^\text{min}_{y', w}} \leq \frac{1}{2} - \frac{1}{2d}.$$ 

Now by Lemma 7.7 (applied to the representation of $\text{Gal}_K$ induced from $V^\text{min}_{y'}$), we have

$$\dim_{w|v}[K_{y', w} : K_v] \cdot \frac{\dim \text{Fil}^1 D^\text{min}_{y', w}}{\dim D^\text{min}_{y', w}} = \frac{1}{2}[K_{y'} : K]$$
where the sum is taken over all places $w$ of $K_{y'}$ that lie above $v$. Now summing over all of $y' \in \pi^{-1}(y)$, and apply the bound we obtained above, we get

$$
\sum_{[K_{y',w}:K_v]<8} [K_{y',w} : K_v] \geq \frac{1}{d} \sum_{[K_{y',w}:K_v]\geq8} [K_{y',w} : K_v].
$$

This is impossible, as it implies that there too many Frobenius orbits of size $\geq 8$, in other words, it contradicts the condition $\text{Size}_v(\pi^{-1}(y)) < \frac{1}{d+1}$.

□
8. Construction of a suitable family

In this section we construct a suitable family $X \to Y' \to Y$, such that $C(K,y_0) = C(K,y_0)^*$ at each $y_0$, hence the former set is finite. It remains to check full monodromy, which we briefly sketch in the last section.

8.1. The Hurwitz space.

8.2. Bounding Size.

9. Full monodromy

Now we turn to the final (and probably the most interesting) part of the paper, namely using topological argument to show that the construction given in Sections 8.1 and 8.2 have full monodromy in the sense of Definition 5.5.

9.1. Mapping class groups and Dehn twists.

9.2. The key reduction.

References
