1. INTRODUCTION

The factorial function $n!$, usually defined as the number of permutations of $n$ distinct objects (or equivalently the product $n(n-1)\cdots 1$) where $n$ is a natural number, arises in many different combinatorial formulae and number-theoretic theorems. Most notably it satisfies the following properties:

1. For any natural numbers $k, l$, $(k + l)!$ is divisible by $k!l!$;
2. If $f(x) \in \mathbb{Z}[X]$ has relatively prime coefficients and is of degree $k$, then $d(\mathbb{Z}, f) := \gcd\{f(a) : a \in \mathbb{Z}\}$ divides $k!$; moreover there exists such a polynomial $f$ that $d(\mathbb{Z}, f) = k!$;
3. The product $\prod_{i<j}(a_i - a_j)$, where $a_0, \ldots, a_n$ are integers, is divisible by $0!1!\cdots n!$; moreover there exist such integers making the product precisely $0!1!\cdots n!$;
4. The number of polynomial functions from $\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ is given by $\prod_{k=0}^{n-1} \frac{n}{\gcd(n, k!)}$.

For proofs one can refer to [1], in which original expositions of these results were listed in the reference.

There have been many attempts to generalize the factorial function, the most well-known probably being the gamma function, meromorphically defined on the complex plane with poles at the nonpositive integers. As outlined in [1], Bhargava made another attempt, defining “factorial functions” for all subsets of $\mathbb{Z}$, for which the above properties (1)-(4) still hold, mutatis mutandis. His generalization engendered to various open problems, many of which remains unsolved.

The formulation of this generalized factorial function depends on the so-called $p$-orderings. Let $S$ be an arbitrary subset of $\mathbb{Z}$, finite or infinite, and fix a prime number $p$. We construct an ordering inductively as follows:

0. Choose any element in $S$ and denote it by $a_0$;
1. Choose $a_1 \in S$ such that the highest power of $p$ dividing $a_1 - a_0$ is minimized;
   
   $\cdots$
   
   $k$. Choose $a_k \in S$ such that the highest power of $p$ dividing $(a_k - a_{k-1})\cdots(a_k - a_0)$ is minimized;
   
   $\cdots$

If we assume the convention that the highest power of any prime dividing zero is $\infty$, then no number can be repeated in $S$ unless every one of them has been chosen in previous steps. Such an ordering as constructed is called a $p$-ordering. Note that it is highly non-unique: every step involves a choice, usually non-unique. What is truly remarkable, however, is the following:

**Theorem 1.** Fix $S \subset \mathbb{Z}$ and a prime number $p$. Let $I = \{a_n\}$ be a $p$-ordering of $S$. Define $\nu_k(S, p, I)$ to be the highest power of $p$ dividing $(a_k - a_{k-1})\cdots(a_k - a_0)$. Clearly $\{\nu_k(S, p, I)\}$ is a non-decreasing sequence of powers of $p$. We have $\{\nu_k(S, p, I)\}$ is independent of $I$.

In view of the theorem, we denote the sequence instead by $\{\nu_k(S, p)\}$ and call it the associated $p$-sequence. We remark that it can be calculated using any $p$-ordering of $S$. Moreover, we note that for any fixed natural number $k$, $\nu_k(S, p) = 1$ for all but finitely many primes. It thus makes sense to define the following:

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**Definition 1.** Fix $S \subset \mathbb{Z}$. The $S$-factorial function is defined on the natural numbers by

\[
!_S = \prod_{p \in \mathfrak{P}} \nu_k(S, p),
\]

where $\mathfrak{P}$ is the set of all primes.

**Example 1.** We have to of course reconcile the generalization of the factorial function with the original one. Let $S = \mathbb{Z}$. We have that for any prime $p$, the natural ordering $0, 1, 2, \ldots$ is a $p$-ordering. Indeed, it is obvious that if we choose $a_0 = 0$ then $a_1 = 1$ does satisfy the requirement. Suppose the first $k$ numbers in the ordering are $0, \ldots, k - 1$. Then as $(a_k - 0)(a_k - 1) \cdots (a_k - k + 1)$ is the product of $k$ integers we know that it is divisible by $k!$. But this can actually be achieved with $a_k = k$. Now we have $\nu_k(\mathbb{Z}, p) = w_p((k - 0) \cdots (k - k + 1)) = w_p(k!)$, where $w_p(n)$ extracts the highest prime power of $p$ in $n \in \mathbb{Z}$. Thus

\[
!_S = \prod_{p \in \mathfrak{P}} \nu_k(S, p) = \prod_{p \in \mathfrak{P}} w_p(k!) = k!
\]

Hence the two definitions agree.

As mentioned before, the generalized factorial function satisfies the following properties for any $S \subset \mathbb{Z}$:

1. For any natural numbers $k, l$, $(k + l)!_S$ is divisible by $k!_S l!_S$;
2. If $f(x) \in \mathbb{Z}[X]$ has relatively prime coefficients and is of degree $k$, then $d(S, f) := \gcd\{f(a) : a \in S\}$ divides $k!_S$; moreover there exists such a polynomial $f$ that $d(S, f) = k!_S$;
3. The product $\prod_{i<j} (a_i - a_j)$, where $a_0, \ldots, a_n \in S$, is divisible by $0!_S 1!_S \cdots n!_S$; moreover there exist such integers making the product precisely $0!_S 1!_S \cdots n!_S$;
4. The number of polynomial functions from $S$ to $\mathbb{Z}/n\mathbb{Z}$ is given by

\[
\prod_{k=0}^{n-1} \frac{n}{\gcd(n, k!_S)}.
\]

We will not prove these statements but refer to [1] for detailed demonstrations.

As one would expect, determining the generalized factorial function for a generic $S$ is rather difficult, and usually no explicit formula is available. But Example 1 showcases the best case possible, where there exists an ordering which is a $p$-ordering for every prime $p$. Such an ordering is called a simultaneous ordering. There are considerable discussions on the existence of simultaneous orderings for subsets of $\mathbb{Z}$, and we are mostly concerned about those of $[2]$, in which the authors considered $S$ to be the image set of a polynomial, either on the natural numbers or on the integers. We are going to recall some of their results in the next section.

As their results suggest, the subsets of $\mathbb{Z}$ that do exhibit such an ordering are quite scarce. It is natural to ask if we can relax the requirement somewhat to obtain “almost simultaneous orderings” that are still helpful in terms of determining the generalized factorial functions. One way, of course, is to require that the ordering is a $p$-ordering for all but finitely many primes. It turns out that we could generate much more subsets that exhibit such an ordering. We will talk more about this in the following sections.

We end this introductory section with the following obvious lemma:

**Lemma 2.** Fix $S \subset \mathbb{Z}$, and let $\alpha$ be a nonzero integer, $\beta$ be any integer. If $S$ exhibits a simultaneous ordering (rep. almost simultaneous ordering) \{\alpha_n\}, then $\alpha S + \beta$ exhibits a simultaneous ordering (rep. almost simultaneous ordering) \{\alpha \alpha_n + \beta\}. In the case of simultaneous ordering, we have moreover

\[
k!_{\alpha S + \beta} = \alpha^k k!_S
\]

2. Simultaneous orderings: Known results

In this section we recall some results obtained in [2]. Throughout this section $f$ denotes a non-constant polynomial with coefficients in $\mathbb{Z}$ such that $f(\mathbb{N})$ (or $f(\mathbb{Z})$) is a subset of $\mathbb{Z}$. The following lemma is an easy consequence of Lemma 2:
Lemma 3. Suppose $f(\mathbb{Z})$ admits a simultaneous ordering (or almost simultaneous ordering) \{f(a_n)\}. Let $g(x) = \alpha f(\pm (x - \lambda)) + \mu$, where $\alpha, \lambda, \mu \in \mathbb{Z}$. Then $g(\mathbb{Z})$ admits a simultaneous ordering (or almost simultaneous ordering) \{g(\pm a_n + \lambda)\}.

Remark. If we consider $f(\mathbb{N})$ instead, with other conditions unchanged, then $g(\pm N + \lambda)$ has a simultaneous (or almost simultaneous ordering) \{g(\pm a_n + \lambda)\}.

This lemma enables us to slightly reduce our set of candidates when searching for those do admit a simultaneous ordering: It is enough to consider those with relatively prime coefficients and zero constant coefficient.

The next lemma imposes a rather strong restriction on any possible simultaneous orderings of the image of polynomials:

Lemma 4. Suppose $f(x)$ is a non-constant polynomial with integer coefficients such that $f(\mathbb{N})$ admits a simultaneous ordering \{f(a_n)\}, where $a_n \in \mathbb{N}$. Then there exists an integer $m$ such that for all $n \geq m$ we have $a_{n+1} = a_n + 1$.

Proof. We may assume the leading coefficient of $f$ to be positive, by virtue of the previous lemma. Thus there exists a number $M$ such that $f$ is strictly increasing on $[M, \infty)$, and $f$ reaches maximum at $x = M$ on $[0, M]$.

Let $a_l$ be the first integer such that $a_l > M$. Now $\prod_{k=0}^{L-1}(f(a_l) - f(a_k))$ divides $\prod_{k=0}^{L-1}(f(x) - f(a_k))$ for all natural numbers $x > M$. Since both products are positive, and $f$ is increasing on $[M, \infty)$, necessarily we have $a_l$ is the least possible choice, $M + 1$.

Now consider $a_{l+1}$. If $a_{l+1} > M$, then similarly we have $a_{l+1} = a_l + 1$. Otherwise $a_{l+1} \leq M$. Now consider $a_{l+2}$ and so on. We have if $a_{l+s} > M$, necessarily $a_{l+s} = 1 + \max\{a_l, \ldots, a_{l+s-1}\}$. Since there are only finitely many integers in $[0, M]$, we have after a while all elements in the ordering $> M$. From that point, by our previous argument, we can see that $a_{n+1} = a_n + 1$, as desired.

This simple proof relies much on an elementary fact: if $a, b > 0$ are two integers such that $a$ divides $b$, we necessarily have $a \leq b$. We will constantly use this fact in later parts of this report. On another note, the proof does not depend on the polynomials per se: any function from \mathbb{N} to \mathbb{Z} that is ultimately increasing will satisfy such a property.

Before we proceed to more specific results, we try to list certain examples of polynomials whose image admits simultaneous orderings:

Example 2. The set \{n^2 : n \geq 0\} $\subset \mathbb{Z}$ admits a simultaneous ordering $0, 1, 2^2, 3^2, \ldots$. Indeed, for any $n > 0$, we have $\prod_{k=0}^{n-1}(n^2 - k^2) = \prod_{k=0}^{n-1}(n - k)(n + k) = \frac{1}{2} (2n)!$, and $\prod_{k=0}^{n-1}(X^2 - k^2) = \prod_{k=0}^{n-1}(X - k)(X + k) = X \sum_{k=n+1}^{n-1} (X + k) = \frac{1}{2} (X + n) \sum_{k=n+1}^{n-1} (X + k) + \frac{1}{2} (X - n) \sum_{k=n+1}^{n-1} (X + k) + \frac{1}{2} \sum_{k=n+1}^{n-1} (X + k)$. We have $(2n)!$ divides either of the product in the last expression, since they are both the product of $2n$ consecutive integers.

Example 3. The set \{n(n + 1)/2 : n \geq 0\} admits a simultaneous ordering $0, 1, 3, 6, \ldots$. Indeed, for any $n > 0$, we have $\prod_{k=0}^{n-1}\frac{1}{2}(n+1) - k(k+1)) = \frac{1}{2n} \prod_{k=0}^{n-1}(n - k)(n + k + 1) = \frac{1}{2n} (2n)!$, and $\prod_{k=0}^{n-1}\frac{1}{2}(X+1) - k(k+1)) = \frac{1}{2n} \prod_{k=0}^{n-1}(X - k)(X + k + 1) = \frac{1}{2n} \prod_{k=-n+1}^{n-1} (X + k)$. By a similar reasoning as in the previous example, we have our claim.

Example 4. The set \{n(2n + 1) : n \in \mathbb{Z}\} admits a simultaneous ordering. In fact, \{n(2n + 1) : n \in \mathbb{Z}\} = \{n(n + 1)/2 : n \geq 0\}.
In some ways these examples are essentially what we have. We have the following result:

**Proposition 5.** The set \( \{ n^r : n \in \mathbb{N} \} \) admits a simultaneous ordering if and only if \( r = 1 \) or \( r = 2 \). The set \( \{ n^{2s+1} : n \in \mathbb{Z} \} \) does not admit a simultaneous ordering for all \( s \geq 1 \).

The key to the proof of the proposition is to realize that the ordering has to be the natural one \( 0, 1, 2^r, 3^r, \ldots \) or \( 0, 1, -1, 2^r, -2^r, \ldots \) if it exists at all. We refer to [2] for details.

The following result characterizes all degree 2 polynomials whose image on \( \mathbb{N} \) or \( \mathbb{Z} \) admits a simultaneous ordering:

**Theorem 6.** Let \( f \in \text{Int}(\mathbb{Z}) := \{ g \in \mathbb{Q}[X] : g(\mathbb{Z}) \subset \mathbb{Z} \} \) be a polynomial of degree 2. Then the subset \( f(\mathbb{N}) \) admits a simultaneous ordering if and only if \( f \) is of one of the forms:

\[
\alpha X(X - 2\lambda) + \beta \quad \text{or} \quad \alpha \frac{X(X - 1 - 2\lambda)}{2} + \beta \quad \text{where} \quad \alpha, \beta \in \mathbb{Z}, \lambda \in \mathbb{N};
\]

The set \( f(\mathbb{Z}) \) admits a simultaneous ordering if and only if \( f \) is of one of the forms:

\[
\alpha X(X - 2\lambda) + \beta, \quad \alpha \frac{X(X - 1 - 2\lambda)}{2} + \beta \quad \text{or} \quad \alpha X(2X - 1 - 2\lambda) + \beta \quad \text{where} \quad \alpha, \beta, \lambda \in \mathbb{Z}.
\]

Again we refer to [2] for details. These results echo the expectation that subsets of \( \mathbb{Z} \) admitting simultaneous orderings are scarce.

There are two paths of generalizing or furthering these established discussions. On can consider polynomials of degree 3 and try to characterize all such polynomials whose image on \( \mathbb{N} \) or \( \mathbb{Z} \) admits a simultaneous ordering. This will be done in part in the next section. One would expect that such polynomials are even scarcer, as cubic polynomials grow faster, rendering higher possibility of introducing new prime factors. This is reflected in our result: such a polynomial with a simultaneous ordering is yet to be found. Alternatively, one can study if we would have more candidates if we consider only almost simultaneous orderings, the exact meaning of which will be clarified in later sections. Although almost simultaneous orderings cannot completely reduce the calculations of generalized factorial functions to a single sequence, it can still reduce the labor significantly. It turns out that we have quite a lot of candidates in this regard. We will explain this in greater detail later.

3. **Simultaneous orderings for the image of cubic polynomials**

As we have pointed out, if \( \{ n^3 : n \in \mathbb{N} \} \) should admit a simultaneous ordering, it would be the natural ordering \( 0, 1, 2^3, \ldots \). Yet problem arises rather early on: \( 2^3(2^3 - 1) = 56 \) does not divide \( 3^3(3^3 - 1) = 756 \). We will calculate more examples to see that this captures a general phenomenon.

**Example 5.** Consider the set \( \{ n^3 + n : n \in \mathbb{N} \} \). It is easy to see that should there be a simultaneous ordering, it would be the natural ordering \( 0, 2, 10, 30, \ldots \). But \( 8 \cdot 10 \) does not divide \( 28 \cdot 30 \).

**Example 6.** Consider the set \( \{ n^3 + n^2 : n \in \mathbb{N} \} \). Again should there be a simultaneous ordering, it would have to be the natural one \( 0, 2, 12, 36, \ldots \). But \( 10 \cdot 12 \) does not divide \( 34 \cdot 36 \).

As suggested by these two examples, if a cubic polynomial is increasing on \( \mathbb{N} \), we have the simultaneous ordering of its image on the natural numbers must be the natural one if it should exist. Yet analyzing the first few terms would give a contradiction. Thus we have the following result:

**Theorem 7.** Let \( f(X) = AX^3 + BX \) or \( f(X) = AX^3 + BX^2 \), where \( A > 0, A + B > 0, (A, B) = 1 \). Then the image of \( f \) on the natural numbers does not admit a simultaneous ordering.

**Proof.** We have noted that if there should exist such an ordering, the ordering is the natural ordering. Our strategy is to analyze the first few terms to get restrictions on \( A \) and \( B \).
**Case 1**  Let \( f(X) = AX^3 + BX \) with \( A,B > 0, (A,B) = 1 \). Then the first few terms are \( f(0) = 0, f(1) = A + B, f(2) = 8A + 2B, f(3) = 27A + 3B \). We remark already that if there is a simultaneous ordering, it must be \( f(0), f(1), \ldots, f(n), \ldots \), except possibly that the first two terms might be interchanged.

Under such assumptions, we have \( A + B \) divides \( Am^3 + Bm = (A + B)m^3 + B(m - m^3) \), and thus divides \( Bm(m + 1)(m - 1) \) for all \( m \geq 1 \). Since \( (A + B, B) = (A, B) = 1 \), we have \( A + B \) divides \( (m - 1)m(m + 1) \) for all \( m \geq 1 \), or equivalently, \( A + B \) divides 6.

Moreover, we have \((8A + 2B)(7A + B) \) divides \((27A + 3B)(26A + 2B) \), or equivalently, \((4A + B)(7A + B) \) divides \(3(9A + B)(13A + B) \). We have

\[(7A + B, 9A + B) = (2A, 7A + B) = (2A, A + B)\]

If \( A + B \) is not divisible by 2, then \((7A + B, 9A + B) = (A, A + B) = 1\), we thus have \( 7A + B \) divides \( 39A + 3B = 18A + 3(7A + B) \). Therefore, \( 18A = k(7A + B) \) for some positive integer \( k \), or equivalently, \((18 - 7k)A = kB \). Since \((18 - 7k, k) = (18, k)\), we have \( A = k/(18, k) \) and \( B = (18 - 7k)/(18, k) = 18/(18, k) - 7A = l - 7A \). Clearly \((A, l) = 1\).

If \( A + B \) are of different parity, we have \( l \) should be odd, thus \( l = 1, 3 \) or 9. As \( B > 0 \), we have \( l > 6A \) or \( A < l/6 \), thus we must have \( l = 9 \). Then \( A = 1, B = 2 \), and the first few terms are \(0, 3, 12, 33, \ldots\) But \( 9 \cdot 12 \) does not divide \( 33 \cdot 30 \).

If \( A + B \) is divisible by 2, then we must have both \( A \) and \( B \) are odd. Moreover, \((7A + B, 9A + B) = (2A, A + B) = 2\). Thus we have \((7A + B) \) divides \( 6(13A + B) = 36A + 6(7A + B) \). Thus we see the argument goes with the previous case with 18 replaced by 36. Define \( k \) analogously to the previous case, and we have \( A = k/(36, k) \) and \( B = l - 7A, \) where \( l = 36/(36, k) \). As \( A \) and \( B \) are both odd, we have \( l \) is even. Moreover \( B > 0 \) implies \( A < l/7 \). Hence \( l = 12, 18, 36 \).

If \( l = 12 \), we must have \( A = 1, B = 5 \). Now we have the polynomial \( f(X) = X^3 + 5X \), with first few terms \(0, 6, 18, 42, 84 \). But \( 18 \cdot 12 \) does not divide \( 84 \cdot 78 \).

If \( l = 18 \), we must have \( A = 1, B = 11 \). But \( A + B \) has to divide 6.

If \( l = 36 \), we have \( A = 1, B = 29 \) or \( A = 3, B = 15 \) or \( A = 5, B = 1 \). Since we have \( A + B \) divides 6 and \((A, B) = 1 \), we are left with the last case only. Consider the polynomial \( f(X) = 5X^3 + X \), with first few terms \(0, 6, 42, 138, 324 \). We have \( 42 \cdot 36 \) divides \( 324 \cdot 318 \), which is impossible.

Another round of careful discussion gives the other half of the theorem:

**Case 2**  Let \( f(X) = AX^3 + BX^2 \), where \( A > 0, A + B > 0, (A,B) = 1 \). As in the previous case, if there should be a simultaneous ordering for \( f(\mathbb{N}) \), it must be \( f(0), f(1), \ldots, f(n), \ldots \), except possibly the position of the first two. The first few terms are \(0, A + B, 8A + 4B, 27A + 9B \).

Just as the first case, we have \( A + B \) divides \( Am^3 + Bm^2 = (A + B)m^3 + B(m^2 - m^3) \), and thus \( A + B \) divides \( B(m - 1)m^2 \). Again as \( (A + B, B) = 1 \), we have \( A + B \) divides \( m^2(m - 1) \), implying \( (A + B) \) divides 2.

Also, we have \((8A + 4B)(7A + 3B) \) divides \((27A + 9B)(26A + 8B) \), or equivalently \((4A + 2B)(7A + 3B) \) divides \(3(9A + 3B)(13A + 4B) \). Now

\[(3) \quad (7A + 3B, 9A + 3B) = (2A, 7A + 3B) = (2A, 3A + 3B) = (2A, 3(A + B))\]

If \( A + B = 1 \) and \((A, 3) = 1 \), we have \( 7A + 3B \) divides \( 39A + 12B = 4(7A + 3B) + 11A \), hence \( 7A + 3B = 4A + 3 \) divides 11A. Suppose \( 11A = (4A + 3)k \) for some positive integer \( k \). We have \((11 - 4k)A = 3k \). Since \((A, 3) = 1 \), we must have \( k = At, 11 - 4k = 3t \). As \( t = (11 - 4k, k) = (11, k) = 1 \) or 11. But the case \( t = 11 \) is impossible, as it would imply \( k = -5.5 \). Now we must have \( t = 1 \), implying \( k = 2, A = 2, B = -1 \). The first few terms of \( f(X) = 2X^3 - X^2 \) are \(0, 1, 12, 45, 112 \). But \( 12 \cdot 11 \) does not divide \( 111 \cdot 112 \).
Suppose \( A + B = 1 \) and \( A \) is divisible by 3. Let \( A = 3a \). Then \((7a + B, 9a + B) = 1\). Thus \(7a + B = 4a + 1\) divides \(3(13A + 4B) = 3(9A + 4) = 3(27a + 4) = 3(11a + 16a + 4)\), thus divides \(33a\). Therefore \(33a = k(4a + 1)\) or equivalently \((33 - 4k)a = k\) for some positive integer \(k\). Since \(33 - 4k = (33 - 4k, k) = (33, 4k)\), we have \(33 - 4k = 1\), or 3 (impossible), or 11 (impossible), or 33 (impossible). Therefore \(k = 8\), \(a = 8\), \(B = 1 - 24 = -23\). The first few terms of \(f(X) = 24X^3 - 23X^2\) are 0, 1, 100, 441. But 100 \(\cdot\) 99 does not divide 441 \(\cdot\) 440.

If \(A + B = 2\) then \(A, B\) must be both odd. We have \((2A + 4)(4A + 6)\) divides \((6A + 6)(9A + 8)\), or equivalently \((2A + 3)(2A + 3)\) divides \((9A + 1)(9A + 8)\). But \((2A + 3, A + 1) = 1\), thus \(2A + 3\) divides \((9A + 8) = 81A + 72 = 33A + 48A + 72\), implying \(2A + 3\) divides \(33A\). Thus \(33A = k(2A + 3)\) for some positive integer \(k\). Therefore \((33 - 2k)A = 3k\). We have \((33 - 2k, k) = (33, k)\).

Suppose \((A, 3) = 1\). Then \(A = k/(33, k)\) and \(3 = (33 - 2k)/(33, k) = 33/(33, k) - 2A := l - 2A\). Now \(l = 3, 11, 33\), the corresponding \(A = 1, 4, 15\). The last two are impossible. Now the first few terms of \(f(X) = X^3 + X^2\) are 0, 2, 12, 36, 80. But 12 \(\cdot\) 10 does not divide 36 \(\cdot\) 34.

Now suppose \(A = 3a\). Then \((33 - 2k)a = k\). We have \(33 - 2k = 1, 3, 11, 33\). The last one is impossible; for the other, the corresponding \(a = 16, 5, 1\). The first one is impossible. Thus we are left with \(f(X) = 15X^3 - 13X^2\) and \(3X^3 - X^2\). They both fail by a simple test.

Although slightly lengthy, the core idea of the proof is simple: \(a|b\) implies \(|a| \leq |b|\), and this implication together with monotonicity fixes what a simultaneous ordering looks like; then some analysis of the first few terms (as it turns out, the first three suffice) gives the result.

This result does not deal with the most general case; two conditions are imposed so that the problem can be solved. First of all, we require certain monotonicity on the cubic polynomial, and this condition forces the simultaneous ordering to be the natural one and comes in handy when we try to restrict the range of certain parameters. One would naturally ask what would happen if we remove such a requirement. According to Lemma 4, this will only influence the first few terms. In theory, we can analyze more terms to relax this condition, but it would be extremely tedious if possible at all. On the other hand, we consider only cubic polynomials with no first order term or with no second order term. A general cubic polynomial \(f(X) = AX^3 + BX^2 + CX\) with \((A, B, C) = 1\) and increasing on the natural numbers could be similarly analyzed, yet the condition \((A, B, C) = 1\) is far more difficult to use than \((A, B) = 1\). However we still have the following result concerning this case:

**Proposition 8.** Let \(f(X) = AX^3 + BX^2 + CX\) be a cubic polynomial with nonnegative integer coefficients. Suppose \((A, B, C) = 1\). Then \(A + B + C\) divides 6.

**Proof.** As in the previous case, if there is a simultaneous ordering for \(f(N)\), it must be \(f(0), f(1), \ldots\), except possibly the position of the first two.

Hence we have \(A + B + C\) divides \(An^3 + Bn^2 + Cn\) for all \(n \geq 1\). Hence it divides \(A(n^3 - n) + B(n^2 - n) = n(n - 1)(An + A + B)\). Clearly \(A + B + C\) divides 2t, and any factor in t comes from \((3A + B, 3(4A + B))\), which divides \((3A + B, 3)(A + B, 4A + B) = (2A + B, 3)(A, B)\) dividing \(3(A, B)\). Hence \(A + B + C\) divides 6\((A, B)\). But \((A + B + C, (A, B)) = (C, (A, B)) = 1\), we have that \(A + B + C\) divides 6 (and if \(3A + B\) is not a multiple of 3, or equivalently \(B\) is not a multiple of 3, then we have \(A + B + C = 1\) or 2).

Now if we impose the condition \(A, B, C > 0\), which is stronger than requiring mere monotonicity, we can actually list all the possible combinations and find out that they are all unqualified candidate for what we look for. Yet the slightly more general case as presented in the proposition remains unsolved.

4. **Almost simultaneous orderings for the image of quadratic polynomials**

As promised, we now move on to the problem of almost simultaneous orderings. We have casually used this noun in the previous sections and we now give a more formal definition:
Definition 2. Let \( S \subset \mathbb{Z} \) be a subset. A sequence \( \{a_n\} \) is called an almost simultaneous ordering if it is a \( p \)-ordering for all but finitely many primes. If we want to specify the exceptions, we can say it is an almost simultaneous ordering except for \( p_1, \ldots, p_k \).

4.1. Examples of almost simultaneous orderings. Recall that we have essentially only two quadratic polynomials whose image on the natural numbers admit a simultaneous ordering. Note that in Examples 2 and 3 where we check that \( \{n^2 : n \geq 0\} \) and \( \{n(n+1)/2 : n \geq 0\} \) admit simultaneous orderings we rely much on the property of the original factorial function, namely the property that \( n! \) divides every product of \( n \) consecutive integers. A variation of this property turns out to be very important in our discussion of almost simultaneous orderings. We start with an example.

Example 7. We consider the polynomial \( f(X) = X(X+2) \), and the sequence \( \{2k\}_{k \geq 0} \). For each \( n > 0 \), we have \( \prod_{k=0}^{n-1} (2n(2n+2) - 2k(2k+2)) = 4^n \prod_{k=0}^{n-1} (n-k)(n+k+1) = 4^n(2n)! \) and in general \( \prod_{k=0}^{n-1} (X(2n+2) - 2k(2k+2)) = \prod_{k=0}^{n-1} (X-2k)(X+k+2) = \prod_{k=-n+1}^{n} (X+2k) \). The latter is a product of \( 2n \) numbers, consecutive but with step 2. If \( X \) is an even number, then clearly \( 4^n(2n)! \) divides this product. If \( X \) is odd, then this product is odd as well. But one might expect that the power of other primes in this product is at least that of \( (2n)! \). If such an expectation is true, we have clearly the image of this polynomial on the natural numbers admits an almost simultaneous ordering except for 2.

The following generic lemma fills the gap in the above example, and would be extremely useful later when we deal with the general case:

Lemma 9. Let \( x \in \mathbb{Z} \) and \( m, n \in \mathbb{N} \). Consider the product \( S(m,n,x) = x(x+m)(x+2m) \cdots (x+(n-1)m) \). For any prime \( p \) that does not divide \( m \), we have the \( p \)-power in \( S(m,n,x) \) is no less than that in \( n! \).

Proof. We prove the statement by comparing the power of \( p \) in \( S(m,n,x) \) with that in another product of \( n \) consecutive integers. If any of the factor in the product is zero, then the statement trivially holds. Thus we assume that none of them is zero. Suppose \( x + im \) is the first one that is a multiple of \( p \), and suppose \( x + im = p^j s \), where \( s \) is not divisible by \( p \). Suppose \( s = sp + r_1 \) for \( 1 \leq r_1 \leq p - 1 \). So \( x + im = r_1 p^j + s_1 p^{j+1} \). As \((m,p) = 1\), we have in \( \mathbb{Z}/p\mathbb{Z} \) a generator. Suppose \( t_1 m \equiv r_1 (\text{mod } p) \), where \( 1 \leq t_1 \leq p - 1 \). Now \( x + im + (p - t_1) p^j m = (pm - t_1m + r_1) p^j + s_1 p^{j+1} = (s_1 + m - t_1m/p) p^{j+1} \). Suppose \( s_1 + m - t_1m/p = s_2 + s_3 \), where \( 0 \leq s_3 \leq p - 1 \). Then \( x + im + (p - t_1) p^j m = (r_2 + m/p) p^{j+1} + s_2 p^{j+2} \). Again, suppose \( t_2 m \equiv r_2 (\text{mod } p) \), where \( 0 \leq t_2 \leq p - 1 \). Now \( x + im + (p - t_1) p^j m + (p - t_2 - t_2) p^{j+1} m = (pm + r_2 - t_2m) p^{j+1} + s_2 p^{j+2} = (s_2 + m + r_2 - t_2m/p) p^{j+2} \). Continue this process until \( x + im + (p - t_1) p^j m + (p - t_2) p^{j+1} m + \cdots + (p - t_k) p^{j+k-1} m > x + nm \).

Let \( P = t_1 p^j + t_2 p^{j+1} + \cdots + t_k p^{j+k-1} \). Consider the product \( (P - i) \cdots P \cdots (P + n - i - 1) \) As \( x + im \) is the first one that is a multiple of \( p \), we must have \( i < p \). Thus \( P - i > 0 \). Note that when \( e \leq l \), \( x + im + am \) is a multiple of \( p^e \) if and only if \( \alpha \) is a power of \( p^e \), if and only if \( P + \alpha \) is a power of \( p^e \). When \( l < e < l + k - 1 \), \( x + im + am \) is a multiple of \( p^e \) if and only if \( \alpha = (p - t_1) p^j + (p - t_2 - t_1) p^{j+1} + \cdots + (p - t_{e-1}) p^{j+e-1} \) by our construction, and this is equivalent to say that \( P + \alpha \) is a power of \( p^e \). By our condition on \( k \), these are all the possible powers. Hence the power of \( p \) in \( S(m,n,x) \) is the same as that in \( (P - i) \cdots P \cdots (P + n - i - 1) \). But this product can be divided by \( n! \), implying our assertion in the lemma.

With this lemma in hand and Example 7 as guide, the following is no surprise:

Theorem 10. Let \( m \) be a natural number. Then the image of \( f(X) = X(X+m) \) on \( \{0,1,2,\ldots\} \) has an almost simultaneous ordering. In fact, let \( m = p_1^{r_1} \cdots p_k^{r_k} \) be the unique factorization of
m, where \( p_1, \ldots, p_k \) are distinct primes, \( r_1, \ldots, r_k \geq 1 \), then \( \{f(0), f(1), f(2), \ldots\} \) has an ordering \( f(0), f(m), f(2m), \ldots \) which is a \( p \)-ordering for all primes \( p \) except \( p_1, \ldots, p_k \).

**Proof.** Now let \( a_t = f(tm) = tm(tm + m) = m^2t(t + 1), t = 0, 1, \ldots \). The assertion is equivalent to say that for any natural number \( n \), we have the \( p \)-powers in \( (a_n - a_0)(a_n - a_1) \cdots (a_n - a_{n-1}) \) minimized among \( (f(x) - a_0)(f(x) - a_1) \cdots (f(x) - a_{n-1}) \), where \( p \) is a prime that does not divide \( m \). Now \( a_s - a_t = m^2(s - t)(s + t + 1), \) so

\[
(a_n - a_0)(a_n - a_1) \cdots (a_n - a_{n-1}) = m^{2n}n(n+1)(n+2) \cdots 1 = m^{2n}(2n)!
\]

On the other hand, we have

\[
(f(x) - a_0)(f(x) - a_1) \cdots (f(x) - a_{n-1}) = x(x + m)(x + m + m) \cdots (x - (n - 1)m)(x + nm)
\]

which is a product of consecutive \( 2n \) numbers with step \( m \). So what we have to show is that the \( p \)-power in the factorization of the above expression is no less than that in \( (2n)! \), where \( p \) is a prime that does not divide \( m \). For this we refer to the generic lemma above.

4.2. **An attempt at characterization.** At this point one would be tempted to conclude that we have found all quadratic polynomials whose image on the natural numbers admits an almost simultaneous ordering. The result seems neat, and embraces the discussion of simultaneous ordering as a special case. The following result seems to affirm such an impression:

**Proposition 11.** Choose any prime number \( p \) other than 2. Then the image of the polynomial \( f(X) = X(X + 2) \) on natural numbers does not have a simultaneous ordering excluding \( p \). That is, if we only allow one exclusion, we have to exclude the prime 2.

**Proof.** Before we prove the statement, we want to briefly display the prime powers in the generalized factorial function on the image set of this polynomial. By Theorem 10, we know that the image set has a simultaneous ordering for all primes except 2. A easy calculation tells us that the power of \( k! \), where \( S \) is the image set, is the power of \( p \) in \( (2k)! \), i.e., \( w_p(k!_S) = w_p((2k)!) \) (recall that \( w_p(n) \) extract the highest \( p \)-power in \( n \)). We also consider a 2-ordering. Choose the first term to be \( f(0) = 0 \). The second one should be an odd number, we can choose \( f(1) = 3 \). Suppose the third one is \( f(m) \). Then \( (f(m) - 0)(f(m) - 3) = m(m+2)(m-1)(m+3) \), which has at least 4 as a factor. Consider \( m = 3 \), it has a factor of precisely 4. Thus we know \( w_2(1!_S) = 1 \) and \( w_2(2!_S) = 4 \).

Now we return to our proof. Suppose otherwise. Let \( f(a_0), f(a_1), \ldots \) be such an ordering. First consider \( p = 3 \). As \( w_p((1!_S)) = 1 \) for all primes \( q \), we know that \( f(a_1) - f(a_0) \) is a power of 3. Since \( f(a_1) - f(a_0) = (a_1-a_0)(a_1+a_0+2) \), we assume \( a_1-a_0 = 3^s \) and \( a_1+a_0+2 = 3^t \) for some nonnegative integers \( s \) and \( t \). Therefore we have \( a_1 = \frac{3^t+3^s}{2} - 1 \) and \( a_0 = \frac{3^t-3^s}{2} - 1 \).

Now as \( w_2(2!_S) = 4 \) and \( w_2(4!_S) = w_2(24) = 1 \) for all primes other than 2 and 3, we have \( (f(a_2) - f(a_1))(f(a_2) - f(a_0)) = 4 \) times a power of 3. Now \( (f(a_2) - f(a_1))(f(a_2) - f(a_0)) = (a_2-a_0)(a_2+a_0+2)(a_2-a_1)(a_2+a_1+2) \). By the first part, we know that \( a_0 \) and \( a_1 \) are of different parity. Thus one of \( a_2 - a_0 \) and \( a_2 - a_1 \) is odd, the other is even. We first assume \( a_2 - a_1 = \pm 3^u \) is odd. The other case is dealt with similarly. Thus \( a_2 = \frac{3^s+3^t}{2} \pm 3^u - 1 \). Thus \( a_2 + a_0 + 2 = 3^t \pm 3^u \) is even, \( a_2 - a_0 = 3^s \pm 3^u \) is even, and \( a_2 + a_1 + 2 = 3^t + 3^u \) is odd. But the only cases when \( 3^s \pm 1 = 2 \) are \( 3^s + 1 = 2 \) or \( 3^s - 1 = 2 \). The first case implies that \( s = u = t \), which cannot happen as then \( a_0 = -1 \). The second case implies either \( s = t \) (again cannot happen) or \( s = u - 1, t = u + 1 \). We have \( 3^{s+1} + 3^{s-1} - 3^s = 3^{s-1}(9 + 1 - 3) = 7 \cdot 3^{s-1} \), contradicting our requirement that the only power that appears is the power of 3.

Now consider when \( p \) is a prime other than 2 or 3. By a similar argument we have \( a_1 = \frac{3^t+p^r}{2} - 1 \) and \( a_0 = \frac{3^t-p^r}{2} - 1 \), where \( t, s \) are nonnegative integers. A similar argument also suggests that \( (f(a_2) - f(a_1))(f(a_2) - f(a_0)) \) is 12 times a power of \( p \). Again, we must have one of \( a_2 - a_0 \) and
\[ a_2 - a_1 \text{ is odd, the other is even. Assume first } a_2 - a_0 = \pm 3^\kappa p^u \text{ is odd, where } \kappa = 0, 1 \text{ and } u \text{ is a nonnegative integer. The other case can be dealt with similarly. Then } a_2 = \frac{p^r + p^s}{2} \pm 3^\kappa p^u - 1. \] Thus \[ a_2 + a_0 + 2 = p^t \pm 3^\kappa p^u \text{ is even, } a_2 - a_0 = p^s \pm 3^\kappa p^u \text{ is even, and } a_2 + a_1 + 2 = p^t + p^s \pm 3^\kappa p^u \text{ is odd.} \]

First consider \( \kappa = 1 \). The only case when \( p^r \pm 3 = 2 \) is \( 5 - 3 = 2 \), and \( 3 p^v \pm 1 = 2 \) can never happen. But then we must have \( s = t \), which is impossible.

Then consider \( \kappa = 0 \). The only case when \( p^v \pm 1 = 2 \) is when \( v = 0 \), and hence we must have at least one of \( p^t \pm p^s \) and \( p^s \pm p^u \) divisible by 3. Then other cannot be divisible by 3. Thus we must have minus sign, and either \( t = u, s = u - 1 \) or \( t = u + 1, s = u \), and in both cases \( p = 7 \). But we now have \( a_1 = \frac{7^t + 7^s}{2} - 1 \) and \( a_0 = \frac{7^{t+1}-7^s}{2} - 1 \), and \( a_2 = a_1 - 7^s = a_0 \), or \( a_2 = a_1 - 7^s + 1 = \frac{7^t - 7^{s+1}}{2} - 1 < 0 \). Both cases are impossible. \( \blacksquare \)

This slightly long demonstration is just a meticulous analysis of the first few terms, which we have applied for many times. As we have remarked before stating the theorem, this theorem seems consistent with the guess that Theorem 10 characterizes all quadratic polynomials whose image on the natural numbers admits a simultaneous ordering. In an attempt to prove so, however, we realize that this is not the case.

4.3. More examples of almost simultaneous orderings. We discover the following example while experimenting with specific cases in an attempt to show that Theorem 10 characterizes all quadratic polynomials whose image on the natural numbers admits an almost simultaneous ordering:

**Example 8.** We consider the set \( \{n(n+6) : n \geq 0\} \). Consider the sequence \( \{8k+1\} \). Then for any \( n > 0 \), we have \[ \prod_{k=0}^{n-1}((8n+1)(8n+7) - (8k+1)(8k+7)) = \prod_{k=0}^{n-1}64(n-k)(n+k+1) = 64^n(2n)!, \]
and in general \[ \prod_{k=0}^{n-1}(X(X+6) - (8k+1)(8k+7)) = \prod_{k=0}^{n-1}(X - 8k - 1)(X + 8k + 7) = \prod_{k=-n+1}^{n}(X + 8k - 1). \]
By Lemma 9 we can see that this sequence is an almost simultaneous ordering except for 2. On the other hand, by Theorem 10, this set has an almost simultaneous ordering except for 2, 3. \( \square \)

The construction used in this example can be generalized to give the following result:

**Theorem 12.** The image of \( f(X) = X(X+2k) \) on the natural numbers has a simultaneous ordering (except for the prime 2) for any integer \( k \).

**Proof.** We construct a desired ordering. Choose \( s \geq 1 \) such that \( 2^s \geq 2k \), and let \( t = 2^{s-1} - k \geq 0 \). Consider the sequence \( a_m = 2^s m + t \), \( m \geq 0 \). Then we have \( f(a_m) - f(a_1) = (a_m - a_1)(a_m + a_1 + 2k) = 2^s(m - l)(2^s m + t + 2^s l + t + 2k) = 2^s(m - l)(2^s m + l + t + 2k) = 2^{2s}(m - l)(m + l + 1) \)
\[ (f(a_m) - f(a_0))(f(a_m) - f(a_1)) \cdots (f(a_m) - f(a_{m-1})) = 2^{2ms} m(m-1)(m+1)(m+2) \cdots 2m = 2^{2ms}(2m)! \]
On the other hand \( f(n) - f(a_m) = (n - 2^s m - t)(n + 2^s m + t + 2k) \), and hence \( f(n) - f(a_0)) = (f(n) - f(a_1)) \cdots (f(n) - f(a_{m-1})) = (n - t)(n + t + 2k)(n - 2^s - t)(n + 2^s + t + 2k) \)
\[ \cdots (n - 2^s(m - l) - t)(n + 2^s (m - l) + t + 2k) = (n - 2^s(m - l) - t)(n - 2^s - t)(n - t)(n + t + 2k)(n + 2^s + t + 2k) \cdots (n + 2^s(m - l) + t + 2k) \]
As \( t + 2k - (-t) = 2^s \), we have this is a product of \( 2m \) consecutive numbers with step \( 2^s \). By Lemma 9, this product has the same or greater power of a prime \( p \) than that of \( (2m)! \) for all \( p \neq 2 \). This is enough to conclude the proof. \( \blacksquare \)

**Remark.** The discovery of such a construction enables us to find a larger pool of candidates. We can do similar constructions for other primes as well. For an odd prime \( p \), we consider again \( f(n) = n(n+k) \).
What we want to find is $s \geq 1, t \geq 0$ such that $2t + k = p^s$. Clearly this can be realized if $k$ is odd. Under this condition, *mutatis mutandis*, we can show that the image of $f$ on the natural numbers has an simultaneous ordering for all primes except $p$.

We can push a little bit further by considering almost simultaneous orderings except for a finite number of primes. Again consider $f(n) = n(n + k)$. What we want is instead $m \geq 1, t \geq 0$ such that $2t + k = m$, where $m$ only has those prescribed primes as prime factors. If $2$ is one of those primes then this can be realized when $k$ is even, and if not, we require $k$ is odd.

4.4. Another attempt at characterization. Up to now, we have shown that for all quadratic polynomials of the form $f(X) = X(X + 2k)$, the image of this polynomial on the natural numbers admits an almost simultaneous ordering except for 2. It is natural to ask whether these are all polynomials having this property. Having analyzed toy examples like $f(X) = X(X + 3)$ and $f(X) = X(2X + 1)$, we realize that the techniques used dealing with these specific examples can be generalized to give a partial answer to this question: if the coefficients of the quadratic polynomial are positive, then Theorem 12 has described all desired polynomials. Our strategy is nothing new: we simply analyze the first three terms of the ordering, should it exist at all. Before coming to this lengthy yet elementary analysis, we need to obtain certain information about the first few generalized factorials, which reflect how the first few terms in an almost simultaneous ordering behave. Therefore we have the following lemma:

**Lemma 13.** Let $f(X) = X(aX + b)$, where $a$ and $b$ are relatively prime integers. Denote by $S$ its image on the natural numbers. Then $w_p(!S) = 1$ for all primes $p \neq 2$, $w_p(2!S) = 1$ for all primes $p \neq 2, 3$, and $w_3(2!S) = 1$ or 3 (recall that $w_p(n)$ denotes the highest $p$-power in $n$).

**Proof.** We have $f(n) - f(m) = (n - m)(a(n + m) + b)$. In particular, consider $n = m + 1$. Then $f(n) - f(m) = 2am + a + b$. Since $(a, a + b) = (a, b) = 1$, then if $a, b$ are of different parity then $(2a, a + b) = 1$. In this case by Dirichlet’s theorem, we have there are infinitely $m$ such that $f(m + 1) - f(m)$ is prime, hence we must have $w_p(!S) = 1$ for all primes $p$. Otherwise $a, b$ must be both odd, then $(a, (a + b)/2) = 1$. Again by Dirichlet, we have there are infinitely $m$ such that $(f(m + 1) - f(m))/2$ is prime. Hence we have $w_p(!S) = 1$ for all primes $p \neq 2$, and $w_p(2!S) = 1$ for $p = 2$.

Fix a prime $p$ different from 2. Choose $m$ such that $f(m + 1) - f(m) = 2am + a + b$ is not divisible by $p$. We have

$$(f(n) - f(m))(f(n) - f(m + 1)) = (n - m - 1)(n - m)(a(n + m) + b)(a(n + m) + a + b)$$

If $p$ divides $a$, then letting $n = m + 2$, we have $(f(m + 2) - f(m))(f(m + 2) - f(m + 1)) = 2(2am + a + b + a)(2ma + a + b + 2a)$ is not divisible by $p$. Now suppose $p$ does not divide $a$. Then if $p \geq 5$, there are at most one divisible by $p$ in $2am + 2a + b, 2am + 3a + b, 2am + 4a + b, 2am + 5a + b$, and hence there is one consecutive pair which is not divisible by $p$. By letting $n = m + 2, m + 3$ or $m + 4$ we will have $(f(n) - f(m))(f(n) - f(m + 1))$ is not divisible by $p$.

We are left with the case where $p = 3$ does not divide $a$. In this case, there are at most two divisible by 3 in $2am + 2a + b, 2am + 3a + b, 2am + 4a + b, 2am + 5a + b$ (and if there are two, they have to be the first and last ones), and we still have there is one consecutive pair which is not divisible by 3. By letting $n = m + 2, m + 3$ or $m + 4$, we will have $(f(n) - f(m))(f(n) - f(m + 1))$ has at most a single factor of 3.

Hence we have $w_p(2!S) = 1$ for all primes $p \neq 2, 3$, and $w_3(2!S) = 1$ or 3, as desired. □

With this lemma in hand, we can state and prove the following theorem:

**Theorem 14.** Let $f(X) = X(aX + b)$ where $a, b > 0$. If either $a > 1$ or $a = 1, b \geq 3$ is odd, then the image of $f$ on $\mathbb{N}$ does not have an almost simultaneous ordering except for the prime $2$.

We remark that Theorems 12 and 14 together say that the only degree 2 polynomials with positive coefficients whose image on the natural numbers have an simultaneous ordering for all odd primes are of the form $X(X + 2k)$ or $X(X + 1)$. 

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Proof. We first show the case where \( b \) is odd. Suppose otherwise. Let \( f(a_0), f(a_1), \ldots \) be such an ordering. Then \((a_1 - a_0)(a(a_0 + a_1) + b)\). When \( a > 1 \) or \( b \geq 3 \), we must have \((a(a_0 + a_1) + b) \geq 2\). If \( a_1 - a_0 \) is even then \((a(a_0 + a_1) + b)\) is odd, hence has an odd prime factor, which contradicts the lemma. Thus \( a_1 - a_0 = 1 \), and \( 2a_0 + a + b = 2^k \) for some \( k \). If \( a \) is even then \( k = 0 \). But \( a + b \geq 3 \), hence this is impossible. We assume from now on \( a \) is odd.

Now consider \((f(a_2) - f(a_0))(f(a_2) - f(a_1)) = (a_2 - a_0 - 1)(a_2 - a_0)(a(a_2 + a_0) + b)(a(a_2 + a_0) + a + b)\). One of \((a(a_2 + a_0) + b), (a(a_2 + a_0) + a + b)\) is odd, and it can only be \( 3 \), and thus we must have \( a = 2, b = 1, a_2 + a_0 = 1 \). So it is either \( a_0 = 0, a_1 = 1, a_2 = 1 \) (impossible), or \( a_0 = 1, a_1 = 2, a_2 = 0 \). Then \( 3a + b = 6 + 1 = 7 \), which is not a power of 2, a contradiction!

We are left with the case where \( b = 2k \) is even and \( a \geq 3 \) is odd. Now \((a_1 - a_0)(a(a_0 + a_1) + b)\) is a power of 2, and since \( a(a_0 + a_1) + b \geq 3 + 2 = 5 \), we must have it is even, and hence \( a_0 + a_1 \) is even, therefore \( a_1 - a_0 \) is even as well. Suppose \( a_1 - a_0 = 2^s \) and \((a(a_0 + a_1) + b) = 2^t \) for some \( s, t \geq 1 \). Then \( a_1 = 2^{s-1} + \left( 2^{t-1} - k \right)/a \) and \( a_0 = -2^{s-1} + \left( 2^{t-1} - k \right)/a \).

Now consider \((f(a_2) - f(a_0))(f(a_2) - f(a_1)) = (a_2 - a_1)(a(a_2 + a_0) + b)(a_2 + a_0) + a + b)\). All four factors in this product have the same parity, and hence can only be even, as this product has at most a single prime factor 3. One of \( a_2 - a_0 \) and \( a_2 - a_1 \) must be a power of 2. We first assume \( a_2 - a_1 = \pm 2^a \). Then \( a_2 = \pm 2^a + 2^{a-1} + \left( 2^{t-1} - k \right)/a \). Hence \( a_2 - a_0 = 0 \pm 2^a + 2^t, a(a_2 + a_0) + b = \pm a2^a + 2^k, a_2 + a_1 = b = \pm a2^a + a2^b + 2^f \). We have the following cases:

Case 1 \( a_2 - a_1 > 0 \). Then we have \( a(a_2 + a_0) + b = 2^{u_2} + 2^{f_1} \). Thus we have \( u = t \) and \( a + 1 \) is a power of 2, or \( a + 2^t = 3 \) (which is impossible as \( a \geq 3 \)). Since \( a_0 \geq 0 \), we must have \( 2^f > 2^{2^a} \geq 3 \cdot 2^2 \). Thus \( t \geq s + 2 \). Now \( u \geq s + 2 \), and \( a_2 - a_0 \) has an odd factor larger than 3, a contradiction!

Case 2 \( a_2 - a_1 < 0 \). Then we have \( a(a_2 + a_0) + b = -2^{u_2} + 2^f \). Thus we have \( u = t \) and \( a - 1 \) is a power of 2, or \( t = u + v, a - 2^t = \pm 3 \), or \( 2^u - 1 = \pm 3, \pm 1 \) (impossible as we would have \( a = \pm 1, 0 \)). For the first possibility, again we must have \( t \geq s + 2 \). Therefore we have \( u = s + 2 \) precisely. But then \( a_2 < 0 \), which is impossible. For the second possibility, we have \( s - u = \pm 1, \pm 2 \). First consider \( s - u = 1, 2 \). In this case, we have \( 2a \pm 1 = \pm 1, \pm 3 \) or \( 4a \pm 1 = \pm 1, \pm 3 \), or \( 2a \pm 3 = \pm 1, 4a \pm 3 = \pm 1 \), all of them impossible. Thus we have \( u - s = 1, 2 \). Then either

(i) \( 2^u \pm 2 \pm 1 = 2^v \pm 3 = \pm 3, \pm 1 \), where we extract the only possible case \( v = 2, u - s = 1, a = 3 \); or
(ii) \( 2^u \pm 4 \pm 1 = 2^v \pm 5 = \pm 3, \pm 1 \), where we extract possible cases \( v = 2, u - s = 2, a = 3 \) and \( v = 3, u - s = 2, a = 7 \); or
(iii) \( 2^u \pm 9 = \pm 1 \), where we have \( v = 3, u - s = 1, a = 7 \); or
(iv) \( 2^u \pm 15 = \pm 1 \), where we have \( v = 4, u - s = 1, a = 15 \).

All of these cases are impossible as we will have \( a_2 < 0 \).

Now we assume \( a_2 - a_1 = \pm 3 \cdot 2^a \). Hence \( a_2 - a_0 = \pm 3 \cdot 2^a + 2^s, a(a_2 + a_0) + b = \pm 3a \cdot 2^a + 2^f, a_2 + a_1 + b = \pm 3a \cdot 2^a + a2^s + 2^f \), and they are all powers of 2. Again, we have the following cases:
1) \( a_2 - a_1 > 0 \). In this case we must have \( u = s = t \). But then \( a_0 < 0 \), which is impossible.
2) \( a_2 - a_1 < 0 \). In this case, we must have \( 2^u - 3a = \pm 1, t = u + v \). Now we have \( s = u, u + 1 \) or \( u + 2 \), and \( \pm 2^u + a2^a \) is a power of 2, i.e., \( a \pm 1, 2a \pm 1 \) or \( 4a \pm 1 \) is a power of 2, and the only possible case is \( a = 2^u \mp 1 \). Now \( 2^v \pm 3 \cdot 2^w = \pm 1 \). We must have \( v = 1 \), which is impossible as we will have \( a = 1 \).

By all of the above, we have the desired result. 

Note that the condition that \( a, b \) are positive is needed to obtain certain estimates for the first few terms of the ordering and eliminate certain cases. We would expect by analyzing more terms that we can relax this condition, but just as in the previous section, it would be very tedious if helpful at all. Hence we leave this problem as it is and hopefully there would be an easier way to finish it.
5. Almost simultaneous orderings for the image of cubic polynomials

It is natural to explore the same question for cubic polynomials. Because of the complexity of the problem, we will only discuss the polynomial \( f(X) = X^3 \). We know from [2] that the image of this polynomial on \( \mathbb{N} \) does not admit a simultaneous ordering. As for almost simultaneous ordering, it seems that there is lack of such as well. As always, we start with calculating the first few generalized factorials of the image of \( X^3 \) on the natural numbers:

**Lemma 15.** Let \( S = \{n^3 : n \geq 0\} \subset \mathbb{Z} \). Then

\[
w_p(1!_S) = 1, \quad w_p(2!_S) = \begin{cases} 1 & p \neq 2, 3, 7 \\ 2 & p = 2 \end{cases} \quad \text{and} \quad w_p(3!_S) = \begin{cases} 1 & p \neq 2, 3, 7 \\ 2^3 & p = 2 \\ 2^2 & p = 3 \\ 7 & p = 7 \end{cases}
\]

**Proof.** Since \( 1^3 - 0^3 = 1 \) divides all differences we have \( 1!_S = 1 \). Also, there is a \( p \)-ordering starting with 0, 1 for any prime \( p \). Now \( n^3(n^3 - 1) = n^3(n-1)(n^2 + n + 1) \) is even for any natural number \( n \), hence \( w_2(2!_S) \geq 2 \). Taking \( n = 3 \) we can conclude that \( w_2(2!_S) = 2 \). If \( n \equiv 2 \pmod{3} \), where \( p \) is a prime other than 2 and 7, we have \( n^3(n-1)(n^2 + n + 1) \) is not divisible by \( p \), hence \( w_p(2!_S) = 1 \). If \( p = 7 \), considering \( n \equiv 3 \pmod{7} \) gives \( w_7(2!_S) = 1 \) as well.

For \( p \) other than 2, 7, there is a \( p \)-ordering starting with 0, 1, 2 by our above discussion. Now \( n^3(n^3 - 1)(n^3 - 2^3) = n^3(n-1)(n-2)(n^2 + n + 1)(n^2 + 2n + 4) \). By considering \( n \equiv 0, 1, 2 \pmod{3} \), we have \( w_3(3!_S) \geq 3^2 \). Letting \( n = 4 \) gives \( w_3(3!_S) = 3^2 \). For \( p \neq 2, 3, 7 \), by considering \( n \equiv 4 \pmod{3} \) we have \( w_p(2!_S) = 1 \). Finally if \( p = 2, 7 \), there is a \( p \)-ordering starting with 0, 1, 3 by our discussion above. Now \( n^3(n^3 - 1)(n^3 - 3^3) = n^3(n-1)(n-3)(n^2 + n + 1)(n^2 + 3n + 9) \). This product must be a multiple of 8, and letting \( n = 5 \) we have exactly \( w_7(3!_S) = 2^4 \). This product must also be a multiple of 7. Letting \( n = 4 \) gives precisely \( w_7(3!_S) = 7 \).

This lemma affirms the expectation that cubic polynomials grow too fast and bring in many prime factors early on. This is why we would expect there is no almost simultaneous ordering allowing a few exceptions. This is indeed the case if we only allow one exception:

**Proposition 16.** Let \( S \) denote the same set as in the previous lemma. Then \( S \) does not admit an almost simultaneous ordering with exception \( p \) for any prime \( p \).

**Proof.** We deal with the case \( p = 2 \) and \( p \neq 2 \) separately. First assume \( p = 2 \). Suppose we have such an ordering \( a_0^3, a_1^3, a_2^3, \ldots \). Then \( a_1^3 - a_0^3 = (a_1 - a_0)(a_1^2 + a_0a_1 + a_0^2) \) is a power of 2. By taking out common powers of 2 we may assume one of \( a_0 \) and \( a_1 \) is odd. By exchanging them if necessary, we may assume \( a_1 \) is odd. Then \( a_1^2 + a_0a_1 + a_0^2 \) must be odd, hence 1. But \( a_1, a_0 \) are nonnegative, forcing \( a_1 = 1, a_0 = 0 \). Hence in general we have \( a_1 = 2^t, a_0 = 0 \).

Now \( a_2^3(a_3^3 - 2^{3t}) \) is also a power of 2. If \( a_2^3 = 1 \) then \( a_1 = 1 \) which is impossible. Thus \( a_2^3 \) is a power of 2. We must have \( a_3^3 = 2^{3t+1} \) or \( 2^{3t-1} \); yet both cases render an irrational \( a_2 \), impossible!

Now let \( p \neq 2 \). Suppose we have such an ordering. Then \( a_1^3 - a_0^3 = (a_1 - a_0)(a_1^2 + a_0a_1 + a_0^2) \) is a power of \( p \), and \( (a_2^3 - a_0^3)(a_3^3 - a_1^3) = (a_2 - a_0)(a_2 - a_1)(a_2^2 + a_2a_0 + a_0^2)(a_3^2 + a_2a_1 + a_1^2) \) is 2 times a power of \( p \). In particular, \( a_1 - a_0 \) is a power of \( p \), one of \( a_2 - a_0 \) and \( a_2 - a_1 \) is a power of \( p \), the other 2 times a power of \( p \). Note that this forces the first three terms \( a_0, a_1, a_2 \) to be \( a, a + p^t, a + 2p^t \) up to a reordering if \( p \neq 3 \), and if \( p = 3 \) we have additional cases \( a, a + 3^t, a + 3^{t+1} \) or \( a, a + 2 \cdot 3^t, a + 3^{t+1} \).

We consider \( a, a + p^t, a + 2p^t \) first. Now \( 3a^2 + 3ap^t + p^{2t}, 3a^2 + 6ap^t + 4p^{2t}, 3a^2 + 9ap^t + 7p^{2t} \) are all powers of \( p \). Suppose they are \( p^k \leq p^{l} \leq p^{r} \) respectively. Note that \( p^{k} - p^{l} = 3ap^t + 3p^{2t} = p^{l} - p^{k} \). Thus \( p^{k}(p^{s-k} + 1) = 2p^{l} \). As \( p \) is odd, we must have \( k = l, p^{s-k} + 1 = 2 \), and hence \( k = l = s \). But then \( 3ap^t + 3p^{2t} = 0 \), which is impossible.

We are left with the two additional cases for \( p = 3 \). For \( a, a + 3^t, a + 3^{t+1} \), we have again \( 3a^2 + 3a \cdot 3^t + 3^{2t}, 3a^2 + 9a \cdot 3^t + 9 \cdot 3^{2t}, 3a^2 + 12a \cdot 3^t + 13 \cdot 3^{2t} \) are powers of 3. Suppose they are \( 3^k \leq 3^t \leq 3^r \).
respectively. Note that $3^s - 3^l = 3a \cdot 3^t + 4 \cdot 2^t = (3^t - 3^k)/2$. Thus $3^{l+1} = 3^k + 2 \cdot 3^s = 3^k (1+2 \cdot 3^s^{-k})$. We must have $l = k, 1 + 2 \cdot 3^s^{-k} = 3$, and hence $k = l = s$. But then $3a \cdot 3^l + 4 \cdot 2^t = 0$, which is impossible. The other case is completely analogous. ■

Note that we only use the prime factorization of $2!s$. As expected, if we want to prove a similar result allowing more than one exception, we will need the factorization of larger factorials. We first do a few experiments:

**Example 9.** We consider almost simultaneous orderings for $S = \{ n^3 : n \geq 0 \}$ allowing two exceptions $2, p$, where $p$ is an odd prime. If we consider the first three terms of such an ordering, should it exist, they must be cubes of $a, a + 2k^2, a + 2k^2 + 2l^2$ such that $2k^2p^2 + 2l^2p^2 = 2^3p^2$. By canceling out common factors and considering divisibility, we have the following cases: $2^\alpha + p^\beta = 1$ (impossible), $1 + 2^\alpha p^\beta = 1$ (impossible), $1 + 2^\alpha = p^\beta$, $1 + p^\beta = 2^\alpha$, or $1 + 1 = 2$.

An example of the first case is $p = 5$. It is expected that an ordering starts with cubes of 0, 1, 5 or 0, 4, 5 (there are other subtleties, but for purpose of illustration we just leave those out), but both of them introduce other primes in $2!s$, which contradicts our calculation in Lemma 15.

An example of the second case is $p = 7$. Again the orderings suggested by the equality $1 + 7 = 2^3$ do not work. However if we consider the ordering 0, 1, 2, 3, 4, it works up until this point. This suggests that we will have to deal with $p = 7$ separately.

All the other primes fall into the last case. It is expected that an ordering starts with something like cubes of 0, 1, 2. But that sequence introduces the prime 3, which does not belong to this case. ■

We now deal with each case in general. The method we use is nothing new, and is in line with what we have used in the proof of Proposition 16. They are more or less analogous, each with its own subtleties. We write them all down for completeness, but reading one of them should be enough to have an idea of how things behave.

**Proposition 17.** Let $p$ be a prime such that it does not satisfy either one of the following: $1 + p^\beta = 2^\alpha, 1 + 2^\alpha = p^\beta$. The $S$ does not have an almost simultaneous ordering with exceptions $2, p$.

**Proof.** Following Example 9, we have the first three terms have to be cubes of $a, a + 2^\alpha p^\beta, a + 2^\alpha p^\beta + 2^\alpha p^\beta$ where $a_0$ is not divisible by 2 or $p$. We have the following cases:

**Case 1** $k \geq s, l \geq t$. By taking out common factors we are essentially dealing with $a, a + 1, a + 2$. Now $3a^2 + 3a + 1, 3a^3 + 9a + 7, 3a^3 + 6a + 4$ are all numbers with only 2 and $p$ as possible prime factors. Note that the first two are odd, hence must be powers of $p$. The third one is either odd or a multiple of 4 (but not a multiple of 8). Thus they are either $p^\alpha, p^\beta, p^\alpha$, or $p^\alpha, p^\alpha, 4p^\beta$ respectively. In the first case $p^\alpha - p^\beta = 3a + 3 = p^\alpha - p^\beta$, thus we have $u = v = w$. But then $3a + 3 = 0$ which is impossible. In the latter case we have $p^\alpha - 4p^\beta = 4p^\alpha - p^\beta$, leading to $p = 7, w = u = v = 1$. But $p = 7$ does not satisfy the condition required in the proposition.

**Case 2** $k \geq s, l < t$. By taking out common factors we are essentially dealing with $a, a + p^\alpha, a + 2p^\alpha$ where $a$ is not a multiple of $p$ and $t > 0$. Now $3a^2 + 3ap^\alpha + p^2t$ is odd, hence must be a power of $p$. Since it is larger than 1, we have it must be a multiple of 3. But then $3a^2$ is a multiple of $p$, thus $p = 3$. But again, $p = 3$ does not satisfy the condition required in the proposition.

**Case 3** $k < s, l \geq t$. By taking out common factors we are essentially dealing with $a, a + 2^s, a + 2^{s+1}$ where $a$ is odd and $s > 0$. Now $3a^2 + 3a \cdot 2^s + 2^2a^2 + 9a \cdot 2^s + 7 \cdot 2^3 + 3a^3 + 6a \cdot 2^s + 4 \cdot 2^s$ are all odd, hence all powers of $p$. Yet this implies they are all equal and thus $3a \cdot 2^s + 3 \cdot 2^s = 0$, which is impossible.

**Case 4** $k < s, l < t$. Again, we are essentially dealing with $a, a + 2^sp^\alpha, a + 2^{s+1}p^\alpha$ where $a$ is odd and not a multiple of $p$, and $s, t > 0$. This time $3a^2 + 3a \cdot 2^sp^\alpha + 2^{2s}p^2t$ is odd and not a multiple of $p$, hence must be 1. This is impossible. ■
We note that in this proof, two primes stand out. They are 3 and 7. We have discussed about 7 in Example 9. For 3, note that we have in Case 2 above \(a, a = 3, a = 2 \cdot 3\), where \(a\) is not divisible by 3 and \(t > 0\). But then \(a^2 + a \cdot 3^t + 3^{2t-1}\) is odd and must be a power of 3. This cannot be, as we would have \(a\) is divisible by 3.

We consider our next case, primes satisfying \(1 + 2^\alpha = p^\beta\):

**Proposition 18.** Let \(p\) be a prime such that \(1 + 2^\alpha = p^\beta\) for some positive integers \(\alpha, \beta\). Then \(S\) does not have an almost simultaneous ordering with exceptions 2, \(p\).

**Proof.** Following Example 9, we know that for such a prime, such an ordering would start with cubes of \(a, a + 2^\alpha p^t, a + 2^{\alpha+1} p^t\) (which is impossible as the proof of Proposition 17 goes here as well), or \(a, a + 2^\alpha p^t, a + 2^{\alpha+1} p^t\), or \(a, a + 2^{\alpha+1} p^t, a + 2^\alpha p^{t+\beta}\). We focus on the ordering \(a, a + 2^\alpha p^t, a + 2^\alpha p^{t+\beta}\). The proof goes similarly for the other one. Now assume \(a = a_0 2^k p^t\), where \(a_0\) is not divisible by 2 or \(p\). Again we have following cases:

**Case 1** \(k \geq s, l \geq t\). By taking out common factors we are essentially dealing with \(a, a + 1, a + p^\beta\). Now consider \(3a^2 + 3a + 1, 3a^2 + 3ap^t + p^{3\beta}, 3a^2 + 3a(p^t + 1) + (p^{3\beta} + p^t + 1).\) The first two are all odd, hence must be powers of \(p\). The second one suggests that \(a\) is a multiple of \(p\), but the first one would then forces \(a = 0\). But then the third one is now \(p^{2\beta} + p^t + 1\), which is odd, larger than 1 and a power of \(p\). But this is impossible.

**Case 2** \(k \geq s, l < t\). By taking out common factors we are essentially dealing with \(a, a + p^t, a + p^{t+\beta}\), where \(a\) is not divisible by \(p\) and \(t > 0\). Now \(3a^2 + 3ap^t + p^{2t}\) is odd, hence a power of \(p\). But as it is larger than 1, it is a multiple of \(p\). Thus \(3a^2\) is a multiple of \(p\). Since \(p\) does not divide \(p\), we have \(p = 3\). But 3 does not satisfy the condition in the proposition.

**Case 3** \(k < s, l \geq t\). Again, we are essentially dealing with \(a, a + 2^\alpha, a + 2^\alpha p^t\), where \(a\) is odd and \(s > 0\). Now \(3a^2 + 3a \cdot 2^\alpha + 2^\alpha, 3a^2 + 3a \cdot 2^\alpha p^t + 2^\alpha p^{2\beta}, 3a^2 + 3a \cdot 2^\alpha(p^t + 1) + 2^\alpha(p^{2\beta} + p^t + 1)\) are all odd, hence all powers of \(p\). The second one implies \(a\) is a multiple of \(p\), but the first one forces \(a = 0, s = 0\), already a contradiction!

**Case 4** \(k < s, l < t\). We are essentially dealing with \(a, a + 2^\alpha p^t, a + 2^\alpha p^{t+\beta}\), where \(a\) is odd and not divisible by \(p\), and \(s, t > 0\). This time \(3a^2 + 3a \cdot 2^\alpha p^t + 2^\alpha p^{2t}\) is odd and not a multiple of \(p\), hence must be 1. This is impossible. \(\blacksquare\)

Note that there are considerable similarities in this proof and that of Proposition 17. With very much the same method we are able to handle the last case (except the notoriously behaved 7 of course!):

**Proposition 19.** Let \(p \neq 7\) be a prime satisfying \(1 + 2^\alpha = p^\beta\) for some positive integers \(\alpha, \beta\). Then \(S\) does not have an almost simultaneous ordering with exceptions 2, \(p\).

**Proof.** Following Example 9, we know that for such a prime, such an ordering would start with cubes of \(a, a + 2^\alpha p^t, a + 2^{\alpha+1} p^t\) or \(a, a + 2^\alpha p^t, a + 2^{\alpha+1} p^t\), or \(a, a + 2^{\alpha+1} p^t, a + 2^\alpha p^{t+\beta}\). The first one is impossible as the proof of Proposition 17 goes here except for the primes 3 and 7; but we have remarked after the proof that it is impossible for 3 as well, and we do not consider \(p = 7\). We focus on the second case and remark that the proof goes similarly for the third case. Suppose \(a = a_0 2^k p^t\) where \(a_0\) is not divisible by 2 or \(p\). And we once again begin our 4-case journey:

**Case 1** \(k \geq s, l \geq t\). Now we are essentially dealing with \(a, a + 1, a + 2^\alpha\). Now consider \(3a^2 + 3a + 1, 3a^2 + 3a \cdot 2^\alpha + 2^\alpha a, 3a^2 + 3a(2^\alpha + 1) + (2^\alpha a + 2^\alpha + 1).\) The first and the last are odd, hence powers of \(p\). If \(a = 0\), then \(2^\alpha a + 2^\alpha + 1\) is a power of \(p\). Since it is larger than 1, it is a multiple of \(p\). But \(2^\alpha \equiv 1 \pmod{p}\). Hence we must have \(p = 3\). Then \(a = 2\) and \(2^\alpha 2^\alpha + 2^\alpha + 1 = 21\) which is not a power of 3. Thus \(a \neq 0\). Therefore \(3a^2 + 3a + 1\) is larger than 1 and hence a multiple of \(p\). This means \(3a^2 + 3a(2^\alpha + 1) + (2^\alpha a + 2^\alpha + 1) - 3a^2 - 3a - 1 = 2^\alpha(3a + 2^\alpha + 1) = 2^\alpha(3a + p^\beta + 2)\) is a multiple of \(p\). Hence \(3a + 2\) is a multiple of \(p\). Now \(3a^2 + 3a + 1 - a(3a + 2) = a + 1\) is a multiple of \(p\), and thus so is \(3a + 3\). But \(3a + 2\) and \(3a + 3\) are relatively prime, a contradiction!
Case 2 \( k \geq s, l < t \). We are essentially dealing with \( a, a + p', a + 2^ap' \), where \( a \) is not divisible by \( p \) and \( t > 0 \). Then \( 3a^2 + 3ap' + p^{2t} \) is odd, hence a power of \( p \). It is larger than 1, thus a multiple of \( p \). Therefore \( 3a^2 \) is a multiple of \( p \). Thus \( p = 3 \). Then \( a^2 + a \cdot 3^l + 3^{2t-1} \) is a power of 3 larger than 1, leading to \( a^2 \) is a multiple of 3, which is impossible.

Case 3 \( k < s, l \geq t \). We are essentially dealing with \( a, a + 2^s, a + 2^{s+a} \) where \( a \) is odd and \( s > 0 \). Now both \( 3a^2 + 3a \cdot 2^s + 2^{2s} \) and \( 3a^2 + 3a \cdot 2^{s+1} + 2^{2s}(2^a + 2^a + 1) \) are odd and larger than 1, hence powers of \( p \) and thus multiples of \( p \). Now \( 2^s \equiv 1 \pmod{p} \), we have \( 3a^2 + 3a \cdot 2^{s+1} + 3 \cdot 2^{2s} \) is a multiple of \( p \). Thus \( 3a \cdot 2^s + 2^{2s+1} = 2(3a + 2^{s+1}) \) is a multiple of \( p \), and therefore so is \( 3a + 2^{s+1} \). Therefore \( 3a^2 + 3a \cdot 2^s + 2^{2s} - (3a + 2^{s+1}) = a \cdot 2^s + 2^{2s} = 2^s(a + 2^s) \) is a multiple of \( p \), hence so is \( a + 2^s \). Now \( 3a + 3 \cdot 2^s \) is a multiple of \( p \). But \((3a + 3 \cdot 2^s, 3a + 2^{s+1}) = (2^s, 3a + 2^{s+1}) \) which must be a power of 2, a contradiction!

Case 4 \( k < s, l < t \). We are essentially dealing with \( a, a + 2^s p^l, a + 2^{s+o} p^l \), where \( a \) is odd and is not divisible by \( p \), and \( s, t > 0 \). Then \( 3a^2 + 3a \cdot 2^s p^l + 2^{2s} p^{2t} \) is odd, hence a power of \( p \). It is larger than 1, thus a multiple of \( p \). Therefore \( 3a^2 \) is a multiple of \( p \). Thus \( p = 3 \). Then \( a^2 + a \cdot 3^l + 3^t \cdot 3^{2t-1} \) is a power of 3 larger than 1, leading to \( a^2 \) is a multiple of 3, which is impossible.

We are left, of course, with the case \( p = 7 \). As we have remarked, \( 0, 1, 2^3, 4^3 \) works in this case, as we can check with the help of Lemma 15. As a matter of fact, with a bit more analysis we can show that the first four terms must be cubes of \( 0, 1, 2, 4 \) up to a reordering or multiplication by powers of \( 2 \) and \( 7 \). To further analyze this case, we have to calculate \( w_p(4^1) \) as well.

Choose \( a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 4 \). Their cubes form the start of a \( p \)-ordering for \( p \neq 2, 7 \). Now \( a_1^2(a_2^2 - 2^3)(a_3^2 - 4^3) = a_1^2(a_2 - 1)(a_2 - 2)(a_3 - 4)(2a_3 + a_2 + 1)(a_2^2 + 2a_3 + 4)(a_2^3 + 2a_3 + 16) \). Letting \( a_4 = 7 \) we will have \( w_3(4^1) = 3^2 \). Letting \( a_4 = 3 \) gives \( w_p(4^1) = 1 \) except for \( p = 2, 3, 7, 13, 19, 37 \). Letting \( a_4 = 7 \) gives \( w_{13}(4^1) = 1 \). Finally letting \( a_4 = 5 \) gives \( w_p(4^1) = 1 \) for \( p = 19, 37 \) as well.

Having done the calculation, we now proceed to analyze an almost simultaneous ordering for \( S \) except \( 2, 7 \). Now suppose such an ordering exist. Then \( a_4, a_4 + 2^7t, a_4 + 2^7t + 1, a_4 + 2^7t + 2 \) are all powers of \( 2 \) and \( 7 \) except that only one of them can have a prime factor \( 3 \). But \( a_4 + 2^7t, a_4 + 2^7t + 1, a_4 + 2^7t + 2 \) are the same modulo \( 3 \), and \( a_4 \) cannot be a multiple of \( 3 \). Thus we must have \( a_4 + 2^7t = \pm 2^k7t \). Thus \( a_4 = 2^k7t \). The only possibilities are:

1. \( s = k, t = l, a_4 = 2^k7t \). This is impossible.
2. \( s = k + 1, t = l, a_4 = 2^k7t \). In this case we are essentially dealing with \( 0, 2, 4, 8, 1 \). But \( 8^3 - 1 = 511 = 7 \cdot 73 \), which is not acceptable.
3. \( s = k + 3, t = l - 1, a_4 = 2^k7t \). But then \( a_4 - 2^{k+1}7t \) is a multiple of \( 5 \), which is not acceptable.
4. \( s = k + 3, t = l, a_4 = 2^k7t + 1 \). But then \( a_4 + 2^k7t \) is a multiple of \( 9 \), again not acceptable.
5. \( s = k, t = l + 1, a_4 = 2^k7t \). But then \( a_4 - 2^k7t \) is a multiple of \( 5 \), not acceptable.
6. \( s = k, t = l - 1, a_4 = 2^k7t \). Now we are essentially dealing with \( 0, 1, 2, 4, 8 \), which is not acceptable as in (2).

With all the analysis we finally have:

**Theorem 20.** For any odd prime \( p \), the set \( S = \{ n^3 : n \geq 0 \} \subset \mathbb{Z} \) does not admit an almost simultaneous ordering with exceptions \( 2, p \).

6. FURTHER DISCUSSIONS

Although we have obtained substantial amount of results, we have not yet arrived at any complete result. Discussions on simultaneous orderings for cubic polynomials in Section 3 are restricted to those increasing on the natural numbers. Discussions on almost simultaneous orderings for quadratic polynomials are interesting, yet the characterization is only obtained for those with positive coefficients and allowing one exception 2. Finally in Section 5, we are only able to discuss the polynomial \( X^3 \), and obtain results only up to allowing two exceptions (and one of them being 2). It is reasonable to expect the same holds for finitely many primes, as the cubic polynomial increases very fast and easily.
brings prime factors early on. It is possible to obtain more by analyzing more terms of an ordering, yet it would be tedious if possible at all.

It is already noted in [1] that the construction of generalized factorials is not restricted to the integers, but can be extended to any Dedekind domain. In particular, any ring of integers in a number field is a Dedekind domain. Of course, there is also the associated concept of simultaneous orderings in such domains. In [3] the author discussed such orderings, and proved that simultaneous orderings do not exist for imaginary quadratic number rings. It is natural then to ask whether almost simultaneous orderings exist. There is no known result in this regard and it would surely be an interesting topic to explore.

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8. References

