Construction of acylindrical hyperbolic 3-manifolds with quasifuchsian boundary

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1 The basic example and its deformations

Consider the following Coxeter diagram:

![Coxeter diagram](image)

Figure 1: The Coxeter diagram

One immediately notices that there are isometric substructures. The full subgraph with vertices 1, 2, 3, 4, 5 and the full subgraph with vertices 6, 7, 8, 9, 10 are isometric as graphs, and any points between them are either not connected (meaning they are orthogonal), or connected by a dotted line (meaning they are ultraparallel). A compact hyperbolic polyhedron $P$ realizing this diagram exists, by Andreev’s theorem (see [RHD]). Shown below in Euclidean space, it looks like a dodecahedron, except that a pair of opposite faces (those corresponding to 1 and 6 in the diagram) are quadrilaterals instead of pentagons. All the angles of the two quadrilateral faces are $\pi/3$. The pentagons are orthogonal to the quadrilateral they intersect, and every pair of adjacent pentagons are orthogonal.

The subgroup $\widetilde{\Gamma}$ of the group of isometries of $\mathbb{H}^3$ generated by reflections in all faces of $P$ except Face 1 is discrete, by Poincaré’s polyhedron theorem (see for example [Mas]). There is an index 2 subgroup $\Gamma$ consisting of all orientation preserving elements in $\widetilde{\Gamma}$. By Selberg’s lemma, $\Gamma$ in turn contains a finite index torsion free subgroup, which is the fundamental group of a convex

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*This is an excerpt from a much more detailed paper, slightly modified to make the explanations relatively self-contained. For details, background knowledge and other discussions, see the longer version.
**cocompact** acylindrical hyperbolic three manifold whose *convex core* has totally geodesic boundary by construction. A fundamental domain $\tilde{P}$ for the action of $\Gamma$ can be obtained by funneling out through Face 1, extending Faces 2, 3, 4, 5 to the sphere at infinity across Face 1.

The polyhedron $P$ provides a hands-on way to deform the group. Let $P(\alpha)$ be a hyperbolic polyhedron with the same faces and dihedral angles as $P$, except that the dihedral angle between Faces 1 and 2 is $\alpha$ instead of $\pi/2$. According to Andreev’s theorem, when $\pi/6 < \alpha \leq \pi/2$, there exists a unique hyperbolic polyhedron (up to hyperbolic motions) realizing these dihedral angles. Also note in particular in this notation, we have $P(\pi/2) = P$.

Let us fix $\alpha \in (\pi/6, \pi/2)$. Let $\tilde{\Gamma}(\alpha)$ be the group generated by reflections in all faces of $P(\alpha)$ except Face 1, and $\Gamma(\alpha)$ its orientation preserving subgroup. Using Poincaré’s polyhedron theorem, it is still true that $\Gamma(\alpha)$ is a discrete subgroup of $PSL(2, \mathbb{C})$. Moreover, reflections in 7, 8, 9, 10 still generate a non-elementary Fuchsian group, and hence $\Gamma(\alpha)$ is still non-elementary. Now the group generated by 2, 3, 4, 5 is no longer Fuchsian; otherwise, there exists a hyperbolic plane orthogonal to all of 2, 3, 4, 5 and hence $P(\alpha)$ is isomorphic to $P$, a contradiction.

Thus the subgroup generated by 2, 3, 4, 5 is a quasi-Fuchsian group, and $\Gamma(\alpha)$ is nonrigid.

When $\alpha = \pi/6$, there does not exist a compact hyperbolic polyhedron realizing the dihedral angles. However, there exists one with two points at infinity. As $\alpha \to \pi/6$, the vertices of the edge between Faces 1 and 2 are pushed towards the sphere at infinity. At $\alpha = \pi/6$, there exists a polyhedron with finite volume and two points at infinity realizing all the dihedral angles. However, reflections in Faces 2, 3, 4, 5 still generate a quasi-Fuchsian group.

It is possible to further deform the polyhedron, so that $\alpha$ decreases to 0, barring some modifications we have to make on the polyhedron. Because of the angles, should such a polyhedron exists, Faces 1, 2, 3 do not meet at a vertex in $\mathbb{H}^3$. But given a configuration of three hyperbolic planes with dihedral angles $\pi/2, \pi/3, \alpha$, there exists a unique hyperbolic plane orthogonal to all of them. With this in mind, we consider a combinatorial polyhedron as in Figure 3a and assign dihedral angles between faces mostly the same as those with correspondingly numbered faces in $P$, except that the assigned dihedral angle between Faces 1 and 2 is $\alpha$, and the newly added Faces 11 and 12 are perpendicular to all their adjacent ones.

Appealing to Andreev’s theorem again, we have there exists a unique compact hyperbolic polyhedron realizing these angles as long as $\alpha \in (0, \pi/6)$. Let us still denote these polyhedron by $P(\alpha)$.
(a) Two additional faces so that $\alpha$ can decrease to 0.

(b) We can further deform the polyhedron so that Faces 1 and 2 no longer intersect.

(c) We can change the angle between Faces 3 and 12 while pushing Face 11 to infinity.

(d) Finally, deform the dihedral angle between Faces 1 and 12.

Figure 3: The deformation process

and the corresponding group obtained from reflections in Faces $2 - 10$ by $\Gamma(\alpha)$. It is clear that despite the complications in construction, they are still representations of the same abstract group.

When $\alpha = 0$, Faces 1 and 2 no longer intersect in $\mathbb{H}^3$, but rather at infinity. The edge between 1 and 2 is pushed to a point at infinity, and the two new Faces 11 and 12 passes through this point at infinity as well.

It is possible to deform the polyhedron even further; see Figure 3b. Faces 1 and 2 no longer intersect, but Faces 11 and 12 intersect in an dihedral angle $\beta \in (0, \pi)$. Let us denote this polyhedron $Q(\beta)$, and the corresponding group generated by Faces $2 - 10$ by $\Delta(\beta)$. Note that $P(0) = Q(0)$. Still, $\Delta(\beta)$ gives a representation of the same abstract group, with the subgroup generated by reflections in Faces 2, 3, 4, 5 being quasi-Fuchsian.

When $\beta = \pi/2$ (and actually throughout the deformation process above), the dihedral angle between Face 1 and Face 11 is $\gamma = \pi/2$. To further deform, we can let $\gamma$ decrease with other dihedral angles fixed. By Andreev’s theorem again, we have a compact polyhedron until we reach $\gamma = 0$. Let us denote the polyhedron by $R(\gamma)$ in this process, and the corresponding group by $E(\gamma)$. Note that $R(\pi/2) = Q(\pi/2)$.

When $\gamma = 0$, the edge between Faces 11 and 1 is pushed to a vertex at infinity, a vertex which Face 12 also passes. We can then increase the angle $\delta$ between Faces 3 and 12. See Figure 3c. When $\delta = \pi/6$, Face 11 is pushed to infinity, but we can still continue our deformation to increase $\delta$, where Faces 2, 3, 12 intersect in a vertex, see Figure 3d. We can do this until $\delta = \pi/2$. 


Finally, the angle between Faces 12 and 1 is $\epsilon = \pi/2$. We can decrease this angle $\epsilon$ as close to 0 as possible. If $\epsilon = 0$, the edge between 1 and 12 is pushed to a point at infinity. Now, the group generated by reflections in Faces $2 - 10$ has parabolic elements and the corresponding orbifold has two isomorphic boundary components, both of them $(0; 3, 3, \infty)$-orbifolds. Also, as Faces 1 and 12 are orthogonal to other faces adjacent to them, the group we get is actually rigid.

Relating to the discussion in the next section, the whole deformation process can be simplified as constructing a compact hyperbolic polyhedron as Figure 3d, with the dihedral angle between 1 and 12 decreasing from $\pi$ to 0. However, such a polyhedron does not always have acute dihedral angles, and Andreev’s theorem does not guarantee its existence.

**Remark 1.1.** In the proof of Andreev’s theorem in [RHD], two operations are essential. The first is *Whitehead move*, where we change the combinatorial data of a polyhedron in the following way shown in Figure 4, where 1, 2, 3, 4 are faces and line segments are edges.

![Figure 4: A Whitehead move](image)

To realize this move, we can decrease the dihedral angle assigned to the edge between 1 and 2 to 0, and hence push that edge to a vertex at infinity, and then increase the angle between 3 and 4.

The second operation is to push a triangular face to infinity and beyond, so that the face becomes a vertex. Notice that in our deformation process above, we are essentially using these two operations (or their reverse).

**Remark 1.2.** In the example, instead of choosing $\pi/3$, we can of course let the angles of the quadrilateral Face 1 be $\pi/n$ for any integer $n \geq 3$, and similarly let the angles of the quadrilateral Face 6 be $\pi/m$ for any integer $m \geq 3$. The deformation process described above still works with minor modifications. In particular, for each fixed pair of integers $(n, m)$ with $n, m \geq 3$, we obtain a one-dimensional family of polyhedra and Kleinian groups.

## 2 Quasi-Fuchsian $(0; n, n, n, n)$-reflection groups

Fix an integer $n \geq 3$. Let $Q$ be a quadrilateral in the extended complex plane whose sides are either line segments or arcs, and whose interior angles are all $\pi/n$. Denote the four circles on which the sides of $Q$ lie by $C_i, 1 \leq i \leq 4$, where the $C_1$ and $C_3$ are opposite sides. Treating the extended complex plane as the boundary at infinity of $\mathbb{H}^3$, these circles determine hyperbolic planes embedded in $\mathbb{H}^3$, which we also denote by $C_i$. The planes $C_1$ and $C_3$ do not intersect, and the same is true for $C_2$ and $C_4$. Assume the hyperbolic distance between $C_1$ and $C_3$ is $\cosh^{-1}(s)$, and the distance between $C_2$ and $C_4$ is $\cosh^{-1}(t)$. Then

**Proposition 2.1.** 1. For any quadrilateral $Q$, we have $(s - 1)(t - 1) \leq 4 \cos^2 \frac{\pi}{n}$.
2. Any pair of positive numbers \((s, t)\) satisfying \((s - 1)(t - 1) \leq 4 \cos^2 \frac{\pi}{n}\) uniquely determine the quadrilateral \(Q\) up to Möbius transforms.

3. Let \(\Gamma_Q\) be the group generated by reflections in \(C_i\). Then \(\Gamma_Q\) is Fuchsian if and only if \((s - 1)(t - 1) = 4 \cos^2 \frac{\pi}{n}\).

Proof. It is convenient to use the hyperboloid model of \(\mathbb{H}^3\). Recall that given an elliptic vector \(e = (e_0, e_1, e_2, e_3)\), the corresponding circle is given by \((e_0 - e_3)x^2 + (e_0 - e_3)y^2 - 2e_1x - 2e_2y + e_0 + e_3 = 0\). If \(e_0 \neq e_3\), this circle is centered at \(\left(\frac{e_1}{e_0 - e_3}, \frac{e_2}{e_0 - e_3}\right)\) with radius \(1/|e_0 - e_3|\).

Given \(s, t\), let us first construct a quadrilateral \(Q\) having the desired property. Let \(\alpha = \left(\sqrt{\frac{s - 1}{2}}, 0, \sqrt{\frac{s + 1}{2}}, 0\right)\) and \(\gamma = \left(\sqrt{\frac{s - 1}{2}}, 0, -\sqrt{\frac{s + 1}{2}}, 0\right)\). Then \((\alpha, \gamma) = -s\) and hence the corresponding hyperbolic planes are of distance \(\cosh^{-1}(s)\) apart. In the extended complex plane, the corresponding circles have centers on the \(y\)-axis, and are reflections of each other in the \(x\)-axis. Let

\[
\beta = \left(\sqrt{\frac{2}{(s - 1)}} \cdot \cos \frac{\pi}{n}, \sqrt{\frac{t + 1}{2}}, 0, \sqrt{\frac{4 \cos^2 \frac{\pi}{n} - (t - 1)(s - 1)}{2(s - 1)}}\right)
\]

\[
\delta = \left(\sqrt{\frac{2}{(s - 1)}} \cdot \cos \frac{\pi}{n}, -\sqrt{\frac{t + 1}{2}}, 0, \sqrt{\frac{4 \cos^2 \frac{\pi}{n} - (t - 1)(s - 1)}{2(s - 1)}}\right)
\]

and then \((\beta, \delta) = -t, (\alpha, \beta) = -\cos(\pi/n)\) and so on. Hence the quadrilateral bounded by the circles corresponding to \(\alpha, \beta, \gamma, \delta\) is as desired.

For 1, it is easy to see that for any quadrilateral, we can find a conformal isomorphism of the extended plane so that \(C_1\) and \(C_3\) are circles corresponding to \(\alpha\) and \(\gamma\). Applying a rotation fixing \(C_1\) and \(C_3\) if necessary, we may assume the center of \(C_2\) and \(C_4\) are symmetric about the \(y\)-axis. Since they intersect both \(C_1\) and \(C_3\) in an \(\pi/n\) angle, they also have the same radius. It is then easy to show that these circles are determined by \(\beta\) and \(\gamma\) above. In particular, \((s - 1)(t - 1) \leq 4 \cos^2(\pi/n)\) and uniqueness is shown as well.

For 3, note that \(\Gamma_Q\) is Fuchsian if and only if there exists a circle intersecting \(C_i\) orthogonally; equivalently, if there is a unit vector \(c = (c_0, c_1, c_2, c_3)\) so that \(c\) is orthogonal to \(\alpha, \beta, \gamma, \delta\). In follows that \(c_0 = c_1 = c_2 = 0\) and \(c_3 = \pm 1\). Also, we must have \(\sqrt{\frac{4 \cos^2 \frac{\pi}{n} - (t - 1)(s - 1)}{2(s - 1)}} = 0\) and hence \((s - 1)(t - 1) = 4 \cos^2(\pi/n)\).

Given a quadrilateral \(Q\) as above, one can make a \((0; n, n, n, n)\)-orbifold \(X\) by gluing two copies of \(Q\) as two sides of a pillow. If we normalize \(Q\) as in the proposition above, and draw it symmetric about the two axes, the segments on the axes glue together to give two geodesics on the orbifold. Let us denote the geodesic obtained from gluing two copies of the segment connecting \(C_1\) and \(C_3\) by \(\xi\) and the other by \(\eta\). The symmetries of the quadrilateral give two orientation reversing involutions \(\phi, \psi\) of the orbifolds. They interchange the cone points in pairs and fix the homotopy classes of \(\xi\) and \(\eta\). We have the following:
Lemma 2.2. The mapping classes represented by $\phi, \psi$ act on the projectivized space of measured laminations $\mathbb{P}ML(X)$ on $X$ nontrivially and the fixed points are the classes represented by simple closed geodesics $\xi$ and $\eta$.

Proof. Let $S$ be the space of free homotopy classes of non-peripheral unoriented simple closed curves on $X$. Then one can identify $(\mathbb{P}ML(X), S) \cong (\mathbb{RP}^1, \mathbb{QP}^1)$, and the elements of the mapping class group acts by $PGL(2, \mathbb{Z})$. Thus for any nontrivial mapping class, there are at most two fixed points in $\mathbb{P}ML(X)$.

For $\phi$ and $\psi$, it is clear that their actions on $S$ are nontrivial. Moreover, they both fix classes represented by $\xi$ and $\eta$. Therefore those are the only fix points.

With regards to Fenchel-Nielsen coordinates, let $\xi$ be the pants curve $\eta$ the transversal. By the classical work of Fenchel and Nielsen, the Teichmüller space $\mathcal{T}(\Sigma)$ of a $(0; n, n, n, n)$-orbifold $\Sigma$ is diffeomorphic to $\mathbb{R}^+ \times \mathbb{R}$ where the first coordinate gives the hyperbolic length $l$ of the pants curve and the second coordinate gives the twisting angle $\theta$. In our particular case, it is evident that $\theta = 0$.

Next we discuss a little about the geometry of the group $\Gamma_Q$ when $(s - 1)(t - 1) < 4 \cos^2(\pi/n)$. Again, $\Gamma_Q$ has a index 2 subgroup $\Gamma'_Q$ contained in $\text{Isom}^+(\mathbb{H}^3)$. The boundaries of the convex core of the quasi-Fuchsian manifold $\Gamma'_Q \backslash \mathbb{H}^3$ are $(0; n, n, n, n)$-orbifolds, bended along geodesic laminations $\lambda_1$ and $\lambda_2$ on the two sides respectively. We have the following:

Lemma 2.3. Both sides of the convex core are examples of the $(0; n, n, n, n)$-reflection groups we have discussed above, and they bend along $\xi$ on one side and $\eta$ on the other side.

Proof. The reflections along the real and imaginary axes descend to both of the boundaries as mapping classes $\phi$ and $\psi$ discussed above. Hence these reflections fix the bending laminations on either side. By our previous lemma, the bending laminations can only be $\xi$ and $\eta$.

For our example outlined in the previous section, it is clear that the corresponding quasi-Fuchsian groups of the boundary are precisely $(0; n, n, n, n)$-reflection groups. In particular, the bending laminations on two sides of the quasi-Fuchsian orbifolds are given by $\xi$ and $\eta$ as above. It remains to distinguish the two sides to give the bending laminations and bending angles on the boundaries of the acylindrical orbifolds.

Indeed, in the deformation process described above, should the bending laminations change, the deformation has to pass through the unique point corresponding to the rigid case. Hence the bending laminations remain the same. The shape of the polyhedron suggests what it is exactly: Faces 1 and 12 intersect in a segment $l$ of hyperbolic length $\cosh^{-1}(t)$, which is the distance between the Faces 3 and 5. Two copies of the pair 1 and 12 glue together to make an $(0; n, n, n, n)$-orbifold, and the two copies of segment $l$ glue together to make a closed geodesic $\eta$. The surface is precisely bended along this geodesic. Therefore, in the corresponding quasi-Fuchsian orbifold, the other side is bended along $\xi$, which is obtained by gluing two copies of the segment perpendicular to both Faces 2 and 4.
References
