TENSOR PRODUCT OF SEMISTABLE VECTOR BUNDLES OVER CURVES IN CHARACTERISTIC ZERO

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Abstract. In this expository paper, we give an essentially self-contained, algebraic proof of the fact that over a smooth projective curve in characteristic zero, the tensor product of two semistable vector bundles is semistable.

Contents

0. Main theorem

0.1. The goal of this exposition is to give an essentially self-contained, algebraic proof of

Theorem 0.1 (Main theorem)**.** *Let X be a smooth projective curve over an algebraically closed field k of characteristic zero. Let E_1 and E_2 be vector bundles over X. If E_1 and E_2 are semistable in the sense of Mumford-Takemoto, then so is $E_1 \otimes E_2$.*

Here, a vector bundle E over a nonsingular projective curve X is *semistable in the sense of Mumford-Takemoto* if for all subbundle F of E, there holds $\mu(F) \leq \mu(E)$. Here, the *slope* $\mu(E)$ of a vector bundle E is defined as

$$\mu(E) := \frac{\deg(E)}{\operatorname{rank}(E)}$$

(and similarly for F). We will simply say E is *semistable* in what follows. We will see later (§2.1) that E is semistable if and only if for all quotient bundle Q of E, there holds $\mu(E) \leq \mu(Q)$.

0.2. The proof of Thm. 0.1 is very modularized. Its essential ingredients are the following results:

Proposition 0.2. *Let $f : Y \to X$ be a degree-d finite morphism of smooth projective curves over an algebraically closed field k. Let E be a vector bundle over X, then $\deg(f^*E) = d \cdot \deg(E)$, and*

- *(i) f^*E semistable $\implies$ E semistable;*
- *(ii) If the field extension $K(X) \hookrightarrow K(Y)$ induced by f is separable, then the converse is true, i.e. E semistable $\implies$ f^*E semistable.*

Proposition 0.3. *Let X be a smooth projective curve over an algebraically closed field k of characteristic zero, and E_1, E_2 be ample vector bundles over X. Then $E_1 \otimes E_2$ is also ample.*

Here, amplitude for a vector bundle is defined as in Hartshorne [6]. We will survey its properties in §3. In particular, we will see that any quotient bundle of an ample vector bundle is ample.

Proposition 0.4. *Let X be a smooth projective curve over an algebraically closed field k, and E be a vector bundle over X. Then*

(i) E ample $\implies \deg(E) > 0$;
(ii) If E is semistable and k has characteristic zero, then the converse is true, i.e. $\deg(E) > 0 \implies E$ ample.

Assuming Prop. 0.2, 0.3, and 0.4, the proof of Thm. 0.1 is the following simple argument:

Proof of Thm. 0.1. Suppose $E_1 \otimes E_2$ is not semistable. Then there exists a vector bundle Q with a surjection $E_1 \otimes E_2 \to Q$, such that $\mu(E_1 \otimes E_2) = \mu(E_1) + \mu(E_2) > \mu(Q)$. It follows from Prop. 0.2 that for any degree-d finite morphism $f : Y \to X$, where Y is another smooth projective curve, there holds

$$\mu(f^*E_1) + \mu(f^*E_2) - \mu(f^*Q) = d \cdot (\mu(E_1) + \mu(E_2) - \mu(Q))$$

Hence we may consider such a morphism with d sufficiently large, and assume

$$\mu(f^*E_1) + \mu(f^*E_2) - \mu(f^*Q) \geq 2$$

Fix a degree-1 line bundle L over Y. Then there exist integers n_1, n_2 such that

$$n_1 > -\mu(f^*E_1), \quad n_2 > -\mu(f^*E_2), \quad n_1 + n_2 \leq -\mu(Q)$$

Using the identity $\deg(f^*E_1 \otimes L^{\otimes n_1}) = \deg(E_1) + n_1 \mathrm{rank}(E_1)$, and similar identities for $\deg(f^*E_2 \otimes L^{\otimes n_2})$ and $\deg(f^*Q \otimes L^{\otimes (n_1+n_2)})$, we see that

$$\deg(f^*E_1 \otimes L^{\otimes n_1}) > 0, \quad \deg(f^*E_2 \otimes L^{\otimes n_2}) > 0, \quad \deg(f^*Q \otimes L^{\otimes (n_1+n_2)}) \leq 0$$

Since k has characteristic zero, the field extension $K(X) \hookrightarrow K(Y)$ induced by f is separable. So Prop. 0.2(ii) implies that f^*E_1 and f^*E_2 are both semistable. Since tensoring with a line bundle does not change semistability, $f^*E_1 \otimes L^{\otimes n_1}$ and $f^*E_2 \otimes L^{\otimes n_2}$ are both semistable. Prop. 0.4(ii) now shows that they are both ample vector bundles. Consider the surjection

$$f^*E_1 \otimes f^*E_2 \to f^*Q \to 0$$

and its result after tensoring with $L^{\otimes (n_1+n_2)}$:

$$(f^*E_1 \otimes L^{\otimes n_1}) \otimes (f^*E_2 \otimes L^{\otimes n_2}) \to f^*Q \otimes L^{\otimes (n_1+n_2)} \to 0$$

The vector bundle on the left is ample by Prop. 0.3; so is its quotient $f^*Q \otimes L^{\otimes (n_1+n_2)}$. However, we have showed that $\deg(f^*Q \otimes L^{\otimes (n_1+n_2)}) \leq 0$, so we get a contradiction to Prop. 0.4(i). □

The rest of this exposition is devoted to the proofs of Prop. 0.2, 0.3, and 0.4. These results hold in more general context, and we will prove stronger versions of them in what follows. In §1, we introduce the notion of Mumford-Takemoto stability for coherent sheaves over a higher dimensional schemes, in relation to other stability conditions such as Gieseker stability. In §2, we study the behavior of semistability under pullback by a finite morphism. We first identify the degree of a coherent sheaf under such pullback, and then use techniques of Galois descent to deduce a higher-dimensional version of Prop. 0.2. This section follows the exposition of Huybrechts and Lehn [9]. We introduce amplitude in §3, and prove Prop. 0.3 in characteristic zero using representation theory of $\mathrm{GL}(r)$. This section follows the original papers by Hartshorne [6] and Atiyah [1]. Finally, we prove Prop. 0.4 in §4, using Kleiman and Seshadri criteria, for pseudo-ample respectively ample line bundles. A good reference for the proofs of these criteria is Hartshorne [7].

All of the important results presented in this paper can be found in literature. The author claims no originality beyond the level of exposition.

Acknowledgement. This paper partially fulfills the completion of a reading course at Columbia University. The author thanks his instructor and advisor Aise Johan de Jong for all his kind advice, mathematical insight, and incredible patience in teaching.

1. Mumford-Takemoto stability

1.1. **Associated points.** Let X be a Noetherian scheme, and E be a coherent sheaf over X. A point $x \in X$ is an *associated point* of E if $\mathfrak{m}_{X,x} \subset \mathcal{O}_{x,X}$ is an associated prime of the stalk E_x. Since the formation of associated primes commutes with localization ([3, Thm. 3.1(c)]), this condition is equivalent to the following: for any affine neighborhood $U = \mathrm{Spec}(A)$ of x, where $E|_U = \widetilde{M}$ for some finitely generated A-module M, x corresponds to an associated prime $\mathfrak{p}$ of M.

The closed set $\mathrm{Supp}(E)$ is equipped with a subscheme structure whose sheaf of ideals $\mathcal{I}$ fits into the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{E}nd(E) \tag{1.1}$$

The *dimension of* E, denoted by $\dim(E)$, is defined to be the Krull dimension of the scheme $\mathrm{Supp}(E)$. On an affine open $U = \mathrm{Spec}(A)$, such that $E|_U = \widetilde{M}$ for some finitely generated A-module M, the sheaf $\mathcal{I}$ corresponds to the ideal $\mathrm{Ann}(M)$. In particular, we have

Lemma 1.1. *Let M be a module over a ring A. The generic points of* $\mathrm{Supp}(M)$ *are precisely the minimal primes over* $\mathrm{Ann}(M)$. □

Lemma 1.2. *Let M be a module over a ring A. Then a prime $\mathfrak{p} \subset A$ is an associated prime of M if and only if $A/\mathfrak{p}$ embeds into M.*

Proof. Suppose $\mathfrak{p} = \mathrm{Ann}(f)$ for some $f \in M$. Then the A-module map $A \xrightarrow{\cdot f} M$ has kernel $\mathfrak{p}$, so $A/\mathfrak{p} \hookrightarrow M$. Conversely, if $A/\mathfrak{p} \hookrightarrow M$, take the element of M corresponding to $1 + \mathfrak{p} \in A/\mathfrak{p}$. Then its annihilator is precisely $\mathfrak{p}$. □

Lemma 1.3. *Let M be a finitely generated module over a Noetherian ring A. Then any prime minimal over* $\mathrm{Ann}(M)$ *is an associated prime of M. In particular, any generic point of* $\mathrm{Supp}(E)$*, where E is a coherent sheaf on a Noetherian scheme X, is an associated point of E.*

Proof. We first observe the following basic

Claim 1.4. *Assuming the hypothesis of the lemma, there exists an associated prime of M.*

Indeed, any ideal I satisfying

$$(*)\ I \text{ is maximal among all ideals of the form } \mathrm{Ann}(f) \text{ for some } f \in M$$

is necessarily prime: if $a, b \in I = \mathrm{Ann}(f)$ and $b \notin I$, then $I + (a) \subset \mathrm{Ann}(bf)$. Since A is Noetherian, such a maximal ideal must exist.

We now prove the lemma. Let $\mathfrak{p}$ be any prime minimal over $\mathrm{Ann}(M)$. Consider the finitely generated $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$, where $A_{\mathfrak{p}}$ is Noetherian. The claim guarantees the existence of an associated prime of $M_{\mathfrak{p}}$, but $\mathfrak{p}A_{\mathfrak{p}}$ is the only prime containing $\mathrm{Ann}(M_{\mathfrak{p}})$, so $\mathfrak{p}A_{\mathfrak{p}}$ is the unique prime in $\mathrm{Ass}(M_{\mathfrak{p}})$. Since the formation of associated primes commutes with localization, $\mathfrak{p} \in \mathrm{Ass}(M)$. □

Lemma 1.5. *Let M be a finitely generated module over a Noetherian ring A. Then*

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}(M)} \mathfrak{p} = 0 \cup \{\textit{zero divisors of } M\}$$

where $\mathrm{Ass}(M)$ *denotes the set of associated primes of M.*

Proof. The containment "$\subset$" is clear. For "$\supset$", note that any zero divisor of M is contained in some ideal satisfying $(*)$, and is therefore an element of some $\mathfrak{p} \in \mathrm{Ass}(M)$. □

To translate some of the commutative algebra of finitely generated modules to results about coherent sheaves, we need

Lemma 1.6 (extension of coherent sheaves)**.** *Let X be a Noetherian scheme, U an open subset, F a coherent sheaf on U, and G a quasi-coherent sheaf on X such that $F \hookrightarrow G|_U$. Then there exists a coherent subsheaf $F' \hookrightarrow G$ on X with $F'|_U = F$.*

Proof. This is [8, Ex. 5.15]. □

The following result classifies the associated points of a coherent sheaf in terms of its coherent subsheaves.

Proposition 1.7. *Let X be a Noetherian scheme, and E be a coherent sheaf over X, then*

(i) A point $x \in X$ is an associated point of E if and only if it is a generic point of $\mathrm{Supp}(F)$ *for some coherent subsheaf $F \hookrightarrow E$.*

(ii) Assume furthermore that X is integral. Then any $x \in X$, with the property that E_x is a torsion $\mathcal{O}_{X,x}$-module, is a specialization of some associated point of E that is not the generic point of X.

Recall that a point η in any topological space Y is a *generic point* of Y if η is the only point that specializes to η. If $Y = \mathrm{Spec}(B)$ is an affine scheme, then its generic points are precisely the minimal primes of B.

Proof of Prop. 1.7. (i) Let $x \in X$ be a generic point of $\mathrm{Supp}(F)$ for some coherent subsheaf $F \hookrightarrow E$. Consider an open affine neighborhood $U = \mathrm{Spec}(A)$ of x, with $E|_U = \widetilde{M}$, $F|_U = \widetilde{N}$, where $N \subset M$ are finitely generated A-modules. Then $x \cong \mathfrak{p}$ is an associated prime of N by Lem. 1.3, and consequently an associated prime of M. To prove the converse, assume $x \cong \mathfrak{p}$ now to be an associated point of E, then $N := A/\mathfrak{p} \hookrightarrow M$ by Lem. 1.2 and is a finitely generated submodule of M. Furthermore, $\mathfrak{p}$ is a (*the*) generic point of $\mathrm{Supp}(N)$, because of Lem. 1.1 and $\mathrm{Ann}(N) = \mathfrak{p}$. By Lem. 1.6, $\widetilde{N}$ extends to a coherent sheaf $F \hookrightarrow E$ on X, and x is a generic point of F.

(ii) Again, let $U = \mathrm{Spec}(A)$ be an affine neighborhood of $x \cong \mathfrak{p}$, and $E|_U = \widetilde{M}$ for some finitely generate A-module M. The hypothesis means that $M_{\mathfrak{p}}$ is a torsion $A_{\mathfrak{p}}$-module, i.e. some nonzero element of $A_{\mathfrak{p}}$ is a zero divisor of $M_{\mathfrak{p}}$. By Lem. 1.5 and the fact that formation of associated primes commutes with localization, there exists a nonzero prime $\mathfrak{q} \subset \mathfrak{p} \subset A$ with $\mathfrak{q} \in \mathrm{Ass}(M)$. Since X is integral, its generic point corresponds to the prime $(0) \subset A$, so the lemma follows. □

1.2. **Pure sheaves.** We introduce three equivalent ways of defining a pure sheaf, one of which involves a generalized Serre's criterion. Let E be a coherent sheaf over a Noetherian scheme X, and let $c := \dim(X) - \dim(\mathrm{Supp}(E))$. Then *Serre's criterion* $S_{k,c}$ for E is the following:

$$S_{k,c}: \ \mathrm{depth}(E_x) \geq \min\{k, \dim(\mathcal{O}_{X,x}) - c\} \text{ for all } x \in \mathrm{Supp}(E)$$

Proposition 1.8. *Let E be a coherent sheaf over a Noetherian scheme X. Then the following are equivalent:*

(i) Every associated point of E has the same dimension.

(ii) For every coherent subsheaf $F \hookrightarrow E$, there holds $\dim(\mathrm{Supp}\, F) = \dim(\mathrm{Supp}\, E)$.

If, furthermore, X is an integral scheme of finite type over a field k, then the above conditions are also equivalent to

(iii) E satisfies Serre's criterion $S_{1,c}$.

Any coherent sheaf E satisfying the above conditions is called a *pure sheaf.*

Proof. (i) $\iff$ (ii) is an immediate consequence of Prop. 1.7(i). Indeed, for (i) $\implies$ (ii), consider a generic point x of $\mathrm{Supp}(F)$. Then by Prop. 1.7, x is an associated point of E, hence of the same dimension as any other associated point of E. By Lem. 1.3, any generic point of E is an associated point of E. Thus $\dim(\mathrm{Supp}\, F) = \dim(\mathrm{Supp}\, E)$. The converse follows from a similar argument.

We now prove (i) $\iff$ (iii). The condition $S_{1,c}$ says that for every $x \in \mathrm{Supp}(E)$ with $\dim(\mathcal{O}_{X,x}) > c$, there holds $\mathrm{depth}(E_x) \geq 1$. Note that

(a) $\dim(\mathcal{O}_{X,x}) > c$ if and only if x is not a generic point of $\mathrm{Supp}(E)$. (This is because $\dim(A/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}) = \dim(A)$ when A is an integral finite type k-algebra.)

(b) $\mathrm{depth}(E_x) \geq 1$ if and only if x is an associated point of E.

Statement (b) deserves some explanation: $\operatorname{depth}(E_x) = 0$ if and only if every element in $\mathfrak{m}_{X,x}$ is a zero divisor of E_x, if and only if

$$\mathfrak{m}_{X,x} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(E_x)} \mathfrak{p}$$

by Lem. 1.5, if and only if $\mathfrak{m}_{X,x}$ is itself an associated prime of E_x by prime avoidance. Combining (a)(b), the implications (i) $\iff$ (iii) are clear. □

The concept of pure sheaves generalizes that of torsion-free sheaves, as showed by

Lemma 1.9. *Let E be a coherent sheaf over an integral Noetherian scheme X. Then E is torsion-free if and only if E is pure and* $\dim(\operatorname{Supp} E) = \dim(X)$.

Proof. If E is torsion-free, then for every $x \in \operatorname{Supp}(E)$, the map $\mathcal{O}_{X,x} \to (\mathcal{E}nd\ E)_x$ is injective. Hence $\mathcal{I}_x = 0$ in the sequence (1.1). This shows that $\operatorname{Supp}(E)$ contains an open neighborhood of x. Since X is integral, $\operatorname{Supp}(E) = X$. To see that E is pure, note that $\mathfrak{m}_{X,x}$ is not an associated prime of E_x for any $x \in \operatorname{Supp}(E)$.

Conversely, if X is not torsion-free, take any torsion point $x \in X$ of E, and Prop. 1.7(ii) provides some y which is an associated point of E, and not the generic point of X. In particular, the dimension of $\overline{\{y\}}$ is strictly less than $\dim(X)$. Hence either E is not pure, or $\dim(\operatorname{Supp}\ E) \neq \dim(X)$ (or both). □

1.3. **Hilbert polynomial.** Let X be a Noetherian scheme, and E be a coherent sheaf over X. Fix a line bundle L over X. Then a section $s \in H^0(X, L)$ is *E-regular* if the sheaf morphism $E \xrightarrow{\cdot s} E \otimes L$ is injective.

Lemma 1.10. *The section $s \in H^0(X, L)$ is E-regular if and only if its vanishing locus $|s|$ does not contain any associated point of E.*

Proof. The question is purely local, and we may assume $X = \operatorname{Spec}(A)$, where A is a Noetherian ring, $L \cong \mathcal{O}_X = \widetilde{A}$, and $E = \widetilde{M}$ where M is a finitely generated A-module. The morphism $E \xrightarrow{\cdot s} E \otimes L$ corresponds to the A-module morphism $M \xrightarrow{\cdot s} M$, where s is regarded as an element in A. It is now clear that this morphism is injective if and only if s does not belong to $\operatorname{Ann}(f)$ for any $f \in M$. □

A sequence $s_1, \cdots, s_m \in H^0(X, L)$ is an *E-regular sequence* if each s_i is regular for the quotient sheaf of the morphism

$$(s_1, \cdots, s_{i-1})E \otimes (L^\vee)^{i-1} \to E$$

Regular divisors and regular sequences of divisors are defined in terms of their corresponding sections, in an obvious way.

Lemma 1.11. *Suppose X is a projective scheme over an infinite field k, and L is globally generated. Then the set of E-regular sections in $H^0(X, L)$ is a dense open subset.*

Proof. Since X is quasi-compact, and any module over a Noetherian ring has finitely many associated primes (see, for example, [3, Thm. 3.1(a)]), the coherent sheaf E has only finitely many associated points $x_1, \cdots, x_m$. Let $X_i = \overline{\{x_i\}}$ be equipped with the induced reduced subscheme structure, and let I_{X_i} denote its ideal sheaf. Hence $I_{X_i} \otimes L$ is a subsheaf of L. Given a section $s \in H^0(X, L)$, there holds $s \in H^0(X, I_{X_i} \otimes L) \iff s \in \mathfrak{m}_{x_i} L_{x_i}$.

On the other hand, since L is globally generated, the map $H^0(X, L) \to L_{x_i}/\mathfrak{m}_{x_i} L_{x_i}$ is surjective. Thus there exists $s \in H^0(X, L)$ that does not lie in $H^0(X, I_{X_i} \otimes L)$. Now that each $H^0(X, I_{X_i} \otimes L)$ is a proper subspace of the finite dimensional vector space $H^0(X, L)$ over an infinite field k, their complement is open and dense. □

Let X be a projective scheme over an algebraically closed field, and L be a fixed ample line bundle over X. For each coherent sheaf E over X, we define the *Hilbert polynomial* $P_L(E)$, or $P(E)$ when there is no ambiguity about the line bundle L, by the following integer-valued function in m:

$$P_L(E) := \chi(E \otimes L^m), \quad \text{where } L^m = L^{\otimes m}$$

The following results justify the terminology:

Lemma 1.12. *Suppose X is a projective scheme over a field k, and L is any line bundle over X. Let E be a coherent sheaf of dimension d. If $H_1, \cdots, H_d \in |L|$ is an E-regular sequence, then*

$$\chi(E \otimes L^m) = \sum_{i=0}^{d} \chi(E|_{\bigcap_{j\leq i} H_j}) \binom{m+i-1}{i}, \quad \text{for all } m \in \mathbb{Z} \tag{1.2}$$

Here

$$\binom{\alpha}{q} = \frac{\alpha(\alpha-1)\cdots(\alpha-q+1)}{q(q-1)\cdots 1} \quad \alpha \in \mathbb{Z}, \quad 0 \leq q \in \mathbb{Z}$$

is the generalized binomial coefficient.

Proof. We induct on the dimension d. For $d = 0$, the statement is trivial since E is supported on points. For notational simplicity, we will use $E(m)$ to denote $E \otimes L^m$. Since H_1 is an E-regular section, the sequence

$$0 \to E(m) \to E(m+1) \to E(m+1)|_{H_1} \to 0$$

is exact, and $E(m)|_{H_1}$ is of dimension $d-1$. We then obtain by induction hypothesis that

$$\chi(E(m+1)) = \chi(E(m)) + \chi(E(m+1)|_{H_1}) = \chi(E(m)) + \sum_{i=1}^{d} \chi(E|_{\bigcap_{j\leq i} H_j}) \binom{m+i-1}{i-1}$$

Therefore, by property of binomial coefficients, and then subtracting $\chi(E)$ from both sides,

$$\chi(E(m+1)) - \sum_{i=0}^{d} \chi(E|_{\bigcap_{j\leq i} H_j}) \binom{m+i}{i} = \chi(E(m)) - \sum_{i=0}^{d} \chi(E|_{\bigcap_{j\leq i} H_j}) \binom{m+i-1}{i}$$

i.e. if we let $f(m)$ be the difference of left and right-hand-side of (1.2), then $f(m+1) = f(m)$ for all $m \in \mathbb{Z}$. The proof will be competed if we can check $f(0) = 0$. Indeed, when $m = 0$, the right-hand-side of (1.2) has only one nonzero term $\chi(E)$. □

In particular, the first two (normalized) coefficients are given by

$$\alpha_d(E) = \chi(E|_{\bigcap_{j\leq d} H_j}), \quad \alpha_{d-1}(E) = \frac{d-1}{2} \cdot \chi(E|_{\bigcap_{j\leq d} H_j}) + \chi(E|_{\bigcap_{j\leq d-1} H_j}) \tag{1.3}$$

Corollary 1.13. *Suppose X is a projective scheme over an infinite field k, and L be a line bundle over X. Then for every coherent sheaf E over X, the expression $\chi(E \otimes L^m)$ is a numerical polynomial in m. Furthermore, if L is globally generated, this expression is given by (1.2).*

Proof. Let $d = \dim(E)$. First suppose L is globally generated. Then by Lem. 1.11, there exists an E-regular sequence $H_1, \cdots, H_d \in |L|$. Consequently the previous lemma applies, and $\chi(E \otimes L^m)$ is given by the expression in (1.2). This expression is a numerical polynomial in m.

For a general line bundle L, because X is projective, we may take an ample line bundle N and an integer r such that $L \otimes N^{\otimes r}$ and $N^{\otimes r}$ are both globally generated. Thus L can be expressed as $L = L_1 \otimes L_2^\vee$, where L_1 and L_2 are both globally generated line bundles. The problem is then reduced to the previous special case. □

In §4, we will define the multivariate Hilbert polynomial, and use it to define intersection numbers.

1.4. **The category $\mathrm{Coh}_{d,d'}(X)$.** Let X be a Noetherian scheme, and $\mathrm{Coh}(X)$ be the abelian category of coherent sheaves on X. Let $\mathrm{Coh}_d(X)$ be the abelian category of coherent sheaves on X of dimension at most d. Then, for every $d' \leq d$, the category $\mathrm{Coh}_{d'}(X)$ is a full subcategory of $\mathrm{Coh}_d(X)$.

Recall that a nonempty, full subcategory $\mathcal{A}'$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if for any exact sequence

$$x' \to x \to x''$$

with $x', x'' \in \mathrm{Ob}(\mathcal{A}')$, we have $x \in \mathrm{Ob}(\mathcal{A}')$ as well. If $\mathcal{A}'$ is a Serre subcategory of $\mathcal{A}$, then the subset of morphisms

$$S = \{s \in \mathrm{Mor}(\mathcal{A}) : \ker(s),\ \mathrm{coker}(s) \in \mathrm{Ob}(\mathcal{A}')\} \tag{1.4}$$

is a multiplicative set of morphisms of $\mathcal{A}$ ([10, Tag 04VC]). Localizing at the set S, we obtain the *quotient category* $\mathcal{C} = \mathcal{A}/\mathcal{A}'$, which is an abelian category together with an exact functor $\mathcal{A} \to \mathcal{C}$. Every morphism $x \to y$ in $\mathcal{C}$ is an equivalence class of roofs (cf. [5]):

$$x \xleftarrow{s} z \to y, \quad \text{where} \quad s \in S$$

Lemma 1.14. *For each $d' = 0, \cdots, d$, the category $\mathrm{Coh}_{d'}(X)$ is a Serre subcategory of $\mathrm{Coh}_d(X)$. Furthermore, its corresponding multiplicative system S, defined by (1.4), is saturated.*

Proof. Given an exact sequence of coherent sheaves $E' \to E \to E''$, the first assertion follows from $\mathrm{Supp}(E) = \mathrm{Supp}(E') \cup \mathrm{Supp}(E'')$. For the second assertion, consider any three composable morphisms f, g and h:

$$E' \xrightarrow{f} E \xrightarrow{g} F \xrightarrow{h} F'$$

with $g \circ f, h \circ g \in S$. We want to show that $g \in S$ as well. Indeed, it follows from the exact sequences

$$\mathrm{coker}(g \circ f) \to \mathrm{coker}(g) \to 0, \quad \text{and} \quad 0 \to \ker(g) \to \ker(h \circ g)$$

that $\mathrm{coker}(g)$, $\ker(g)$ are both elements of $\mathrm{Coh}_{d'}(E)$. Thus S is saturated. □

We define the abelian category $\mathrm{Coh}_{d,d'} := \mathrm{Coh}_d(X)/\mathrm{Coh}_{d'-1}(X)$ for each $d' = 1, \cdots, d$, and set $\mathrm{Coh}_{d,0}(X) := \mathrm{Coh}_d(X)$.

Now, we let X be a projective scheme over an algebraically closed field k, and let L be a fixed ample line bundle on X. The Hilbert polynomial can be regarded as a map $P : \mathrm{Ob}(\mathrm{Coh}(X)) \to \mathbb{Q}[m]$. For every coherent sheaf E, $P(E)$ is a polynomial in m of degree $d = \dim(E)$. Let $\mathbb{Q}[m]_d$ be the subset of $\mathbb{Q}[m]$ that consists of polynomials of degree at most d, and let $\mathbb{Q}[m]_{d,d'}$ be $\mathbb{Q}[m]_d/\sim$, where the equivalence relation $\sim$ is given by $P \sim Q$ if and only if $P - Q \in \mathbb{Q}[m]_{d'-1}$.

Lemma 1.15. *P induces a well-defined map (still denoted by) $P : \mathrm{Ob}(\mathrm{Coh}_{d,d'}(X)) \to \mathbb{Q}[m]_{d,d'}$.*

Proof. The case for $d' = 0$ is trivial. We now assume $d' \geq 1$.

Let E and F be coherent sheaves isomorphic in $\mathrm{Coh}_{d,d'}(X)$, and let the roof

$$E \xleftarrow{s} G \xrightarrow{f} F \tag{1.5}$$

represent such an isomorphism, with $s \in S$. Consider the following exact sequences for each $m \in \mathbb{Z}$:

$$0 \to \mathrm{im}(s)(m) \to E(m) \to \mathrm{coker}(s)(m) \to 0, \quad \text{and} \quad 0 \to \ker(s)(m) \to G(m) \to \mathrm{im}(s)(m) \to 0$$

Since $\chi(\mathrm{coker}(s)(m))$ and $\chi(\ker(s)(m))$ are polynomials in m of degree at most $d' - 1$, we obtain

$$\chi(E(m)) \equiv \chi(\mathrm{im}(s)(m)) \equiv \chi(G(m)) \mod \mathbb{Q}[m]_{d'-1}$$

Hence $P(E) = P(G)$ in $\mathbb{Q}[m]_{d,d'}$. The diagram (1.5), passed to the quotient category $\mathrm{Coh}_{d,d'}(X)$, shows that f is an isomorphism in $\mathrm{Coh}_{d,d'}(X)$. Since S is saturated, we see that as a morphism in $\mathrm{Coh}_d(X)$, $f \in S$ (cf. [10, Tag 05Q9]). The above argument applied to f shows that $P(G) = P(F)$ in $\mathbb{Q}[m]_{d,d'}$. □

We defined the *reduced Hilbert polynomial* by $p(E) := P(E)/\alpha_d(E)$.

1.5. **(Semi)stability.** Let X be a projective scheme over an algebraically closed field k, with a fixed ample line bundle L. Note that there is a linear ordering on $\mathbb{Q}[m]_{d,d'}$ given by the lexicographic ordering of the coefficients $(\alpha_d, \cdots, \alpha_{d'})$. For $d' = 0$, this ordering amounts to

$$P(\leq) < Q \iff P(m)(\leq) < Q(m) \quad \text{for sufficiently large } m$$

where $P, Q \in \mathbb{Q}[m]_d$.

Definition 1.16. A coherent sheaf E of dimension d on X is *semistable* in the category $\mathrm{Coh}_{d,d'}(X)$, if for all proper coherent subsheaf $F \hookrightarrow E$, there holds

$$\alpha_d(E)P(F) \le \alpha_d(F)P(E) \quad \text{in} \quad \mathbb{Q}[m]_{d,d'} \tag{1.6}$$

and is *stable* in the category $\mathrm{Coh}_{d,d'}(X)$ if, in addition, strict inequality holds whenever the right-hand-side is nonzero.

Let $T_i(E)$ denote the maximal coherent subsheaf of E of dimension i. We say that E is *pure of dimension d in* $\mathrm{Coh}_{d,d'}(X)$, if E is a coherent sheaf of dimension d, and $T_{d-1}(E) \cong 0$ in $\mathrm{Coh}_{d,d'}(X)$. Note that the condition $T_{d-1}(E) \cong 0$ in $\mathrm{Coh}_{d,d'}(X)$ is equivalent to $T_{d-1}(E) = T_{d'-1}(E)$, and purity in the category $\mathrm{Coh}_{d,0}(X)$ agrees with the earlier definition in §1.2.

Lemma 1.17. *A coherent sheaf E of dimension d on X is (semi)stable in $\mathrm{Coh}_{d,d'}(X)$ if and only if E is pure in $\mathrm{Coh}_{d,d'}(X)$ and for all proper coherent subsheaf $F \hookrightarrow E$ of dimension d, there holds*

$$p(F)(\le) < p(E) \quad \text{in} \quad \mathbb{Q}[m]_{d,d'} \tag{1.7}$$

where p denotes the reduced Hilbert polynomial.

Proof. For the "if" part, given that E is pure, any coherent subsheaf F of dimension at most $d-1$ must have dimension at most $d'-1$. Hence $P(F) = 0$ in $\mathbb{Q}[m]_{d,d'}$, and (1.6) holds with both sides equal to zero.

For the "only if" part, we need to show that E is pure provided that it is semistable. Indeed, let $F = T_{d-1}(E)$, then $\alpha_d(F) = 0$ and (1.6) implies that $P(F) = 0$ in $\mathbb{Q}[m]_{d,d'}$, i.e. F is of dimension at most $d'-1$. Hence $T_{d-1}(E) = T_{d'-1}(E)$. □

The following lemma is straightforward consequence of the definition:

Lemma 1.18. *Let E be a coherent sheaf of dimension d on X and let $0 \le j < i \le d$. Then*

$$E \text{ is semistable in } \mathrm{Coh}_{d,j}(X) \implies E \text{ is semistable in } \mathrm{Coh}_{d,i}(X)$$

Furthermore, if E is pure in $\mathrm{Coh}_{d,j}(X)$, then

$$E \text{ is stable in } \mathrm{Coh}_{d,i}(X) \implies E \text{ is stable in } \mathrm{Coh}_{d,j}(X)$$

In particular, given a pure sheaf E of dimension d in $\mathrm{Coh}_d(X) = \mathrm{Coh}_{d,0}(X)$ and any $0 \le j < i \le d$, we have the following chain of implications:

$$\begin{aligned} &E \text{ is stable in } \mathrm{Coh}_{d,i}(X) \implies E \text{ is stable in } \mathrm{Coh}_{d,j}(X) \\ \implies &E \text{ is semistable in } \mathrm{Coh}_{d,j}(X) \implies E \text{ is semistable in } \mathrm{Coh}_{d,i}(X) \end{aligned}$$

Definition 1.19. A coherent sheaf E of dimension d on X is *Gieseker (semi)stable* if it is (semi)stable in the category $\mathrm{Coh}_{d,0}(X)$, and is *Mumford-Takemoto (semi)stable* (or *μ-(semi)stable*) if it is (semi)stable in the category $\mathrm{Coh}_{d,d-1}(X)$.

In other words, E is Mumford-Takemoto (semi)stable if and only if $T_{d-1}(E) = T_{d-2}(E)$, and for all proper coherent subsheaf $F \hookrightarrow E$ of dimension d, there holds

$$\frac{\alpha_{d-1}(F)}{\alpha_d(F)}(\le) < \frac{\alpha_{d-1}(E)}{\alpha_d(E)} \tag{1.8}$$

We define the *degree* and *rank* of a d-dimensional coherent sheaf E by

$$\deg(E) = \alpha_{d-1}(E) - \mathrm{rank}(E)\alpha_{d-1}(\mathcal{O}_X), \quad \mathrm{rank}(E) = \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)} \tag{1.9}$$

which will depend on the line bundle L. Thus (1.8) is equivalent to

$$\frac{\deg(F)}{\mathrm{rank}(F)}(\le) < \frac{\deg(E)}{\mathrm{rank}(E)} \tag{1.10}$$

The degree defined by (1.9) generalizes the usual definition using Chern classes:

Lemma 1.20. *Let E be a vector bundle of rank r on a nonsingular projective scheme X of dimension d, over a field k. Then*

$$\alpha_{d-1}(E) - r\alpha_{d-1}(\mathcal{O}_X) = \deg(c_1(E).H^{d-1}) \tag{1.11}$$

Proof. By the Hirzebruch-Riemann-Roch theorem,

$$\chi(E(m)) = \deg(\mathrm{ch}(E(m)).\mathrm{td}(\mathcal{T}))_0$$

where $\mathcal{T}$ is the tangent sheaf of X, and the Chern character

$$\mathrm{ch}(E(m)) = \mathrm{ch}(E).\mathrm{ch}(\mathcal{O}(m)) = (r + c_1(E) + \cdots).\left(\sum_{i=0}^{d} \frac{m^i H^i}{i!}\right)$$

and the Todd class

$$\mathrm{td}(\mathcal{T}) = 1 + \frac{1}{2}c_1(\mathcal{T}) + \cdots$$

Hence $\alpha_{d-1}(E)$, being $(d-1)!$ times the second leading coefficient of $\chi(E(m))$, equals

$$\begin{aligned}\alpha_{d-1}(E) =&(d-1)! \deg\left(\frac{1}{(d-1)!}\left(c_1(E) + \frac{r}{2}c_1(\mathcal{T})\right).H^{d-1}\right)\\ =&\deg(c_1(E).H^{d-1}) + \frac{r}{2}\deg(c_1(\mathcal{T}).H^{d-1})\end{aligned}$$

Letting $E = \mathcal{O}_X$ in this formula, we have $\alpha_{d-1}(\mathcal{O}_X) = \frac{1}{2}\deg(c_1(\mathcal{T}).H^{d-1})$. Hence $\alpha_{d-1}(E) = \deg(c_1(E).H^{d-1}) + r\alpha_{d-1}(\mathcal{O}_X)$ as desired. □

Note that if E is a vector bundle over a curve X, the degree and rank of E do not depend on the line bundle L. In the curve case, one also observes that the notions of Gieseker stability and Mumford-Takemoto stability agree.

1.6. **A criterion for (semi)stability.** Let X be a Noetherian, and E be a coherent sheaf over X of dimension d. A subsheaf F of E is *saturated* if the quotient sheaf E/F is zero or pure of dimension d. The *saturation* of a subsheaf $F \subset E$ is the kernel of the surjective morphism $E \to (E/F)/T_{d-1}(E/F)$.

Lemma 1.21. *Let $E \in \mathrm{Coh}_{d,d'}(X)$ be pure. The following are equivalent:*

(i) E is (semi)stable in the category $\mathrm{Coh}_{d,d'}(X)$;
(ii) For any proper saturated subsheaf F of E, there holds

$$p(F)(\leq) < p(E) \quad in \quad \mathbb{Q}[m]_{d,d'}$$

(iii) For any proper purely d-dimensional quotient sheaf Q of E, there holds

$$p(E)(\leq) < p(Q) \quad in \quad \mathbb{Q}[m]_{d,d'}$$

Proof. The implication (i) $\Longrightarrow$ (ii) is trivial from the definition. (ii) and (iii) are equivalent because by definition, F is a proper nonzero saturated subsheaf of E if and only if the quotient E/F is a proper purely d-dimensional quotient sheaf of E.

It now suffices to prove (ii) $\Longrightarrow$ (i). Since E is pure, it suffices by Lem. 1.17 to show that for any subsheaf F of E (necessarily of dimension d), whose saturation is $\tilde{F}$, we must have $p(F) \leq p(\tilde{F})$. Indeed, let $Q = E/F$ be the quotient sheaf, and consider the exact sequence

$$0 \to T_{d-1}(Q) \to Q \to \tilde{Q} \to 0$$

It follows that $\alpha_d(Q) = \alpha_d(\tilde{Q})$ (where both could be zero), and $P(\tilde{Q}) \leq P(Q)$ as the leading coefficient of $T_{d-1}(Q)$ is positive. Hence $\alpha_d(F) = \alpha_d(\tilde{F})$ and $P(\tilde{F}) \geq P(F)$. The result follows. □

2. Stability under finite pullback

2.1. **Normality.** In this section, we will only be concerned with Mumford-Takemoto (semi)stability, which will be hitherto referred to simply as "semi(stability)." Its behavior is under pullback by finite morphisms is most easily understood over *normal* projective k-schemes. This is essentially due to the following

Lemma 2.1. *Let E be a torsion-free coherent sheaf on a normal, locally Noetherian scheme X. Then E is locally free outside codimension 2, i.e. the closed subset $Z = \{x \in X \mid E_x \text{ is not free}\}$ is of codimension at least 2.*

Proof. Consider a point $x \in X$ of codimension 1, i.e. $\dim(\mathcal{O}_{X,x}) = 1$. By the assumption, $\mathcal{O}_{X,x}$ is also normal and Noetherian, and thus a discrete valuation ring. As such, $\mathcal{O}_{X,x}$ is a principal ideal domain. Since E_x is a torsion-free $\mathcal{O}_{X,x}$-module, it is free. Hence $x \notin Z$, and it follows that every generic point of Z is of codimension at least 2. □

In particular, over a nonsingular projective curve, any torsion-free coherent sheaf is a vector bundle. Hence a vector bundle E is (semi)stable if $p(F)(\leq) < p(E)$ holds for every subbundle F. Together with Lem. 1.21(iii) and Lem. 1.9, we also see that a vector bundle E is (semi)stable if $p(E)(\leq) < p(Q)$ for every proper quotient bundle Q.

2.2. **Degree under finite pullback.** As (semi)stability is characterized by degree/rank ratio, we have to first understand how degree behaves under pullback. Note the the following general result:

Lemma 2.2. *Let $f : Y \to X$ be a finite morphism of Noetherian schemes, and L is a line bundle on X. Then*
 (i) L ample $\implies$ f^*L ample;
 (ii) If f is surjective, then f^*L ample $\implies$ L ample.

Proof. (i) Let F be a coherent sheaf on Y. Then f_*F is a coherent sheaf on X since f is finite. Thus for some $d > 0$, there is a surjection $\mathcal{O}_X^n \to f_*F \otimes L^{\otimes d}$ for some n. Since f^* is a right-exact functor, we get a surjection $\mathcal{O}_Y^n \to f^*f_*F \otimes (f^*L)^{\otimes d}$. Since f is affine, the natural map $f^*f_*F \to F$ is surjective. We obtain a surjection by composition: $\mathcal{O}_Y^n \to f^*f_*F \otimes (f^*L)^{\otimes d} \to F \otimes (f^*L)^{\otimes d}$. For (ii), see for example [7, Prop. 4.4]. □

We will study the behavior of degree under finite pullback between normal projective schemes. Before doing so, a simple observation is in order:

Observation 2.3. *Assume X and Y are normal projective schemes over an algebraically closed field k, and $f : Y \to X$ is a finite morphism of degree d. Then*
 (i) $f_*\mathcal{O}_Y$ is a torsion-free sheaf on X of rank d.
 (ii) Given any open subset $U \subset X$, the pushforward functor $f_* : \mathrm{Coh}(f^{-1}U) \to \mathrm{Coh}(U)$ is exact.

Proof. (i) Since "torsion-free" is a local property, we can check the first statement of (i) on an affine open $U = \mathrm{Spec}(A) \subset X$. Since f is finite, hence affine, $f^{-1}U = \mathrm{Spec}(B)$, and $f_*\mathcal{O}_Y(U) = B$. Since A and B are both domains, B is a torsion-free A-module via the natural map $A \to B$ induced by f. To compute the rank of $f_*\mathcal{O}_Y$, take some $x \cong \mathfrak{p} \subset A$ where $(f_*\mathcal{O}_Y)_x = B_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. On the other hand,

$$\mathrm{frac}(B_\mathfrak{p}) = \mathrm{frac}(B) \cong \mathrm{frac}(A)^{\oplus d} = \mathrm{frac}(A_\mathfrak{p})^{\oplus d}$$

Hence $B_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus d}$ and $\mathrm{rank}(f_*\mathcal{O}_Y) = d$.
(ii) Since finite morphisms are stable under base change, $f : f^{-1}U \to U$ is still finite. It is a consequence of the theorem on formal functions ([10, Tag 02OE]) that for any morphism of schemes $f : Y \to X$ and $x \in X$, if
(a) X is locally Noetherian
(b) f is proper, and
(c) $f^{-1}(x)$ is finite

then for any coherent sheaf F on Y, we have $(R^p f_* F)_x = 0$ for all $p > 0$. The desired statement follows from this result. □

We will keep referring to the following set-up in the proof of the next few propositions:

Set-up 2.4. Assume X and Y are normal projective schemes over an algebraically closed field k, and $f : Y \to X$ is a finite morphism of degree d. Fix a very ample line bundle L over X such that f^*L is very ample over Y.

Take $U = \{x \in X \mid E_x \text{ and } (f_*\mathcal{O}_Y)_x \text{ are both free}\}$. Then $Z := X - U$ and $f^{-1}Z = Y - f^{-1}U$ are both of codimension at least 2, and f^*E is locally free on $f^{-1}U$. We may successively take divisors $H_1, \cdots, H_n \in |L|$ such that the following open conditions hold:

(i) The sequence $H_1, \cdots, H_n$ is $E \otimes f_*\mathcal{O}_Y$-regular;

Justification. Lem. 1.11 shows that the property of a divisor in $|L|$ being $E \otimes f_*\mathcal{O}_Y$ is an open condition. □

(ii) The sequence of scheme-theoretic inverse images $f^{-1}H_1, \cdots, f^{-1}H_n \in |f^*L|$ is f^*E-regular;

Justification. There is a k-linear map between the finite dimensional vector spaces $H^0(X, L) \to H^0(Y, f^*L)$. The pre-image of a dense open subset of $H^0(Y, f^*L)$ is dense open in $H^0(X, L)$. □

(iii) Each H_j intersects Z transversally; and

(iv) $\bigcap_{j \le d} H_j$ is a normal projective variety for all $1 \le d \le n$.

Justification. The openness of this condition follows from Bertini's theorem for normality. □

In particular, $C = \bigcap_{j \le n-1} H_j$ is a normal, thus nonsingular, projective curve properly contained in U.

Proposition 2.5. *Let $f : Y \to X$ be a finite morphism of degree d between normal projective varieties over an algebraically closed field k. Given a very ample invertible sheaf L on X, such that f^*L is very ample on Y, then for any torsion-free coherent sheaf E on X,*

$$\alpha_n(f^*E) = \alpha_n(E \otimes f_*\mathcal{O}_Y), \quad \textit{and} \quad \alpha_{n-1}(f^*E) = \alpha_{n-1}(E \otimes f_*\mathcal{O}_Y) \tag{2.1}$$

Proof. We proceed in two steps:

Step 1. Suppose $C \hookrightarrow X$ is a closed subscheme and E and $f_*\mathcal{O}_Y$ are locally free on an open subscheme of X containing C. We claim that

$$\chi(f^*E|_{f^{-1}C}) = \chi(E \otimes f_*\mathcal{O}_Y|_C)$$

Indeed, consider the base change diagram

$$\begin{array}{ccc} f^{-1}C & \xrightarrow{\tilde{j}} & Y \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{j} & X \end{array}$$

The Euler characteristic may be computed on the closed subschemes C and $f^{-1}C$, so it suffices to prove

$$\chi_{f^{-1}C}(\tilde{j}^* f^* E) = \chi_C(j^*E \otimes j^* f_*\mathcal{O}_Y)$$

Note that $\tilde{j}^* f^* E = \tilde{f}^* j^* E$ since the above diagram commutes. Hence

$$\chi_{f^{-1}C}(\tilde{j}^* f^* E) = \chi_{f^{-1}C}(\tilde{f}^* j^* E) = \chi_C(\tilde{f}_* \tilde{f}^* j^* E) = \chi_C(j^* E \otimes \tilde{f}_* \mathcal{O}_{f^{-1}C})$$

using the projection formula in the last equality. Since $\mathcal{O}_{f^{-1}C} = \tilde{j}^*\mathcal{O}_Y$, it now suffices to prove that $j^* f_* \mathcal{O}_Y = \tilde{f}_* \tilde{j}^* \mathcal{O}_Y$. This is clear locally: given a tensor product diagram of rings

$$\begin{array}{ccc} B \otimes_A A/I & \longleftarrow & B \\ \uparrow & & \uparrow \\ A/I & \longleftarrow & A \end{array}$$

it follows that

$$(B \text{ as an } A\text{-module}) \otimes_A A/I = (B \otimes_A A/I \text{ as an } A/I\text{-module})$$

Step 2. Consider Set-up 2.4.

Since $H_1, \cdots, H_n$ is an $E \otimes f_*\mathcal{O}_Y$-regular sequence, and $f^{-1}H_1, \cdots, f^{-1}H_n$ is an f^*E-regular sequence, (1.3) applies and we obtain

$$\alpha_n(f^*E) = \chi(f^*E|_{\bigcap_{j\leq n} f^{-1}H_j}), \quad \alpha_n(E \otimes f_*\mathcal{O}_Y) = \chi(E \otimes f_*\mathcal{O}_Y|_{\bigcap_{j\leq n} H_j})$$
$$\alpha_{n-1}(f^*E) = \frac{n-1}{2} \cdot \chi(f^*E|_{\bigcap_{j\leq n} f^{-1}H_j}) + \chi\left(f^*E|_{\bigcap_{j\leq n-1} f^{-1}H_j}\right)$$

and similarly for $\alpha_{n-1}(E \otimes f_*\mathcal{O}_Y)$. Since $\bigcap_{j\leq n-1} H_j$ and $\bigcap_{j\leq n} f^{-1}H_j$ are closed subschemes of U, over which E is finite locally free, Step 1 applies and the proof is complete. □

Proposition 2.6. *Let $f : Y \to X$ be a finite morphism of degree d between normal projective varieties over an algebraically closed k. Fix an ample line bundle L on X. Then for any torsion-free coherent sheaf E on X,*

(i) $\operatorname{rank}(f^*E) = \operatorname{rank}(E)$, *and*
(ii) $\deg(f^*E) = d \cdot \deg(E)$.

where the degree and rank of f^*E are calculated with respect to the ample line bundle f^*L.

Proof. We first reduce the problem to the case where both L and f^*L are very ample. Indeed, suppose X is of dimension n. Then $\deg_{L^q}(E) = q^{n-1}\deg_L(E)$, and $\deg_{f^*L^q}(f^*E) = q^{n-1}\deg_{f^*L}(f^*E)$. Thus the equality $\deg(f^*E) = d \cdot \deg(E)$ is invariant when L is replaced by L^q, but by such replacement with sufficiently large q, we may assume both L and f^*L are ample. Apply the same argument to the equality $\operatorname{rank}(f^*E) = \operatorname{rank}(E)$.

We can now consider Set-up 2.4. Since $\bigcap_{j\leq n} H_j$ is a disjoint union of points contained in U, where both E and $f_*\mathcal{O}_Y$ are locally free,

$$\alpha_n(f^*E) = H^0(\bigcap_{j\leq n} H_j, E \otimes f_*\mathcal{O}_Y) = d \cdot H^0(\bigcap_{j\leq n} H_j, E) = d \cdot \alpha_n(E) \tag{2.2}$$

Applying (2.2) to $E = \mathcal{O}_X$, then dividing (2.2) by the resulting equation, we find $\operatorname{rank}(f^*E) = \operatorname{rank}(E)$. This proves statement (i). Furthermore,

$$\alpha_{n-1}(f^*E) = \frac{n-1}{2} \cdot \alpha_n(f^*E) + \chi_C(i^*E \otimes i^* f_*\mathcal{O}_Y) \tag{2.3}$$

where $i : C \hookrightarrow X$ is the closed immersion. It follows from the Riemann-Roch theorem that

$$\chi_C(i^*E \otimes i^*f_*\mathcal{O}_Y) = d\left(c_1(i^*E) + \frac{\operatorname{rank}(E)}{2} \cdot c_1(\mathcal{T}_C)\right) + \operatorname{rank}(E)c_1(i^*f_*\mathcal{O}_Y) \tag{2.4}$$

where $\mathcal{T}_C$ is the tangent sheaf on C. The same computation for E yields

$$\alpha_{n-1}(E) = \frac{n-1}{2} \cdot \alpha_n(E) + \chi_C(i^*E), \quad \text{where} \tag{2.5}$$

$$\chi_C(i^*E) = c_1(i^*E) + \frac{\operatorname{rank}(E)}{2} \cdot c_1(\mathcal{T}_C) \tag{2.6}$$

Altogether, we obtain

$$\alpha_{n-1}(f^*E) = d \cdot \alpha_{n-1}(E) + \operatorname{rank}(E)c_1(i^*f_*\mathcal{O}_Y) \tag{2.7}$$

Apply (2.7) to the special case $E = \mathcal{O}_X$ and $f^*E = \mathcal{O}_Y$, so we get $\alpha_{n-1}(\mathcal{O}_Y) = d \cdot \alpha_{n-1}(\mathcal{O}_X) + c_1(i^* f_* \mathcal{O}_Y)$. Substituting this formula into (2.7) and using $\operatorname{rank}(E) = \operatorname{rank}(f^*E)$, we find

$$\alpha_{n-1}(f^*E) - \operatorname{rank}(f^*E)\alpha_{n-1}(\mathcal{O}_Y) = d \cdot (\alpha_{n-1}(E) - \operatorname{rank}(E)\alpha_{n-1}(\mathcal{O}_X))$$

Statement (ii) now follows from the definition of degree. □

2.3. **Galois descent.** Since (semi)stability involves consideration of all coherent subsheaves, it is natural to ask whether coherent subsheaves of the pullback sheaf f^*E correspond to coherent subsheaves of E. This is false in general, but for certain morphisms f we can obtain a positive result using Galois descent. In this subsection, we will survey key points of Galois descent over vector spaces.

Let k be a field, and $K \supset k$ be a finite Galois extension, with $G = \operatorname{Gal}(K/k)$. A fundamental result of Galois theory is the following

Lemma 2.7. *Let $G = \operatorname{Gal}(K/k)$ be as above. Consider the K-vector spaces*

(i) $K \otimes_k K$, where G acts on the left factor, and scalar multiplication by K acts on the right factor;
(ii) ${}^G K$, consisting of all functions $\varphi : G \to K$, equipped with G-action $(\tau\varphi)(\sigma) = \varphi(\sigma\tau)$, for all $\tau, \sigma \in G$.

Then the map $K \otimes_k K \to {}^G K$, defined by sending $\alpha \otimes \beta$ to the function $\sigma \mapsto \beta\sigma(\alpha)$, is a K-linear, G-equivariant isomorphism.

Proof. It is straightforward to check that the map $K \otimes_k K \to {}^G K$ is K-linear and G-equivariant. To show that it is an isomorphism, we first resort to the normal basis theorem: there exists $w \in K$ such that the collection $\{\sigma(w) : \sigma \in G\}$ forms a k-basis for K. Let

$$F(x) := \prod_{\sigma \in G} (x - \sigma(w)) \in k[x]$$

be the minimal polynomial of w. Then there is an isomorphism $k[x]/F(x) \xrightarrow{\sim} K$ given by sending x to w. Applying $\otimes_k K$ to the short exact sequence of k-modules

$$0 \to (F(x)) \to k[x] \to K \to 0$$

we see that as K-vector spaces,

$$K \otimes_k K \cong K[x]/F(x) \cong \prod_{\sigma \in G} K[x]/(x - \sigma(w)) \cong {}^G K$$

where the last isomorphism is given by sending a function $f(x) \in K[x]$ to the function on G defined by $\sigma \mapsto f(\sigma(w))$. To show that this entire composition $K \otimes_k K \xrightarrow{\sim} {}^G K$ sends $\alpha \otimes \beta$ to the function $\sigma \mapsto \beta\sigma(\alpha)$, note that in the first isomorphism, we are picking the unique polynomial $f(x) \in k[x]/F(x)$ such that $f(w) = \alpha$, and the element in $K[x]/F(x)$ corresponding to $\alpha \otimes \beta$ is $\beta f(x)$. Since $f(\sigma(w)) = \sigma(f(w)) = \sigma(\alpha)$, we see that passing through the last two isomorphisms, we indeed get $\sigma \mapsto \beta\sigma(\alpha)$. □

We define the *category of twisted G-representations* $G\mathrm{-}\mathbf{Rep}_k^K$ by the following:

(i) An object of this category is a K-vector space W, equipped with a k-linear action of G, such that for all $w \in W$, $\alpha \in K$, and $g \in G$, there holds $g(\alpha w) = g(\alpha)g(w)$;
(ii) A morphism $F : W_1 \to W_2$ is a K-linear morphism of vector spaces that commutes with the action of G.

In particular, the category of twisted G-representations is a subcategory of $K\mathrm{-}\mathbf{Mod}$. In this language, we may generalize Lem. 2.7 to

Proposition 2.8. *Let W be a twisted G-representation. Consider the K-vector spaces*

(i) $W \otimes_k K$, where G acts on the left factor W, and scalar multiplication by K acts on the right factor;
(ii) ${}^G W$, consisting of all functions $\varphi : G \to W$, equipped with G-action $(\tau\varphi)(\sigma) = \varphi(\sigma\tau)$, for all $\tau, \sigma \in G$.

Then the map $W \otimes_k K \to {}^G W$, defined by sending $w \otimes \alpha$ to the function $\sigma \mapsto \alpha\sigma(w)$, is a K-linear, G-equivariant isomorphism.

Proof. The map $W \otimes_k K \to {}^G W$ is clearly K-linear and G-equivariant, so we only need to show its bijectivity. It suffices to show that the map (which does not respect K-linearity or G-equivariance) $W \otimes_k K \to {}^G W$ defined by sending $w \otimes \alpha$ to the function $\sigma \mapsto \sigma^{-1}(\alpha)w$ is bijective. This new map, however, is compatible with direct sum in the following sense: fix a basis $W \cong \bigoplus_i K$, then the following diagram commutes:

$$\begin{array}{ccc} W \otimes_k K & \longrightarrow & {}^G W \\ \downarrow & & \downarrow \\ (\bigoplus_i K) \otimes_k K & \longrightarrow & {}^G(\bigoplus_i K) \end{array}$$

where the lower arrow is formed componentwise by the map $K \otimes_k K \to {}^G K$ sending $\alpha \otimes \beta$ to $\sigma \mapsto \sigma^{-1}(\beta)\alpha$. Hence it suffices to show that the map $K \otimes_k K \to {}^G K$ is bijective, and it is equivalent that the map $K \otimes_k K \to {}^G K$ sending $\alpha \otimes \beta$ to $\sigma \mapsto \beta\sigma(\alpha)$ is bijective. This latter statement is given precisely by Lem. 2.7. □

We define a functor $\mathcal{F} : k\text{−}\mathbf{Mod} \to G\text{−}\mathbf{Rep}_k^K$ by

(i) For any k-vector space V, $\mathcal{F}(V) = V \otimes_k K$, equipped with the G-action given by $g(v \otimes \alpha) = g(v) \otimes g(\alpha)$, for all $v \in V$, $\alpha \in K$, and $g \in G$;
(ii) For any k-linear morphism $f : V_1 \to V_2$, $\mathcal{F}(f) = f \otimes \mathbb{I} : V_1 \otimes_k K \to V_2 \otimes_k K$.

We define another functor $\mathcal{G} : G\text{−}\mathbf{Rep}_k^K \to k\text{−}\mathbf{Mod}$ by

(i) For any twisted G-representation W, $\mathcal{G}(W) = W^G$ with the induced k-vector space structure;
(ii) For any morphism $F : W_1 \to W_2$ of twisted G-representations, $\mathcal{G}(F)$ is the restriction of F to W_1^G; its image is necessarily contained in W_2^G.

Theorem 2.9. *There are natural isomorphisms $\mathcal{G} \circ \mathcal{F} \cong \mathbb{I}_{k-\mathbf{Mod}}$ and $\mathcal{F} \circ \mathcal{G} \cong \mathbb{I}_{G-\mathbf{Rep}_k^K}$. In particular, the categories $k\text{−}\mathbf{Mod}$ and $G\text{−}\mathbf{Rep}_k^K$ are equivalent.*

Proof. We first prove $\mathcal{G} \circ \mathcal{F} \cong \mathbb{I}_{k-\mathbf{Mod}}$. Indeed, there is a natural k-linear map $V \to (V \otimes_k K)^G$ for every k-vector space V. We want to prove that it is an isomorphism. The action of G on $V \otimes_k K$ is compatible with direct sum, i.e. fix a basis $V \cong \bigoplus_i k$, then the following diagram commutes:

$$\begin{array}{ccc} V & \longrightarrow & (V \otimes_k K)^G \\ \downarrow & & \downarrow \\ \bigoplus_i k & \longrightarrow & (\bigoplus_i k \otimes_k K)^G \end{array}$$

where the lower arrow is formed componentwise by the obvious map $k \to (k \otimes_k K)^G$. Hence we have reduced the problem to showing that $k \to (k \otimes_k K)^G = K^G$ is an isomorphism, which is clear by Galois theory.

With regard to morphisms, it suffices to show that for any two k-vector spaces V_1 and V_2, $\mathcal{F}$ induces a bijection $\mathrm{Hom}_{k-\mathbf{Mod}}(V_1, V_2) \to \mathrm{Hom}_{G-\mathbf{Rep}_k^K}(V_1 \otimes_k K, V_2 \otimes_k K)$. Indeed, since $(V_1 \otimes_k K)^G \cong V_1$ and $(V_2 \otimes_k K)^G \cong V_2$ as we have just proved, the functor $\mathcal{G}$ induces precisely the inverse of $\mathcal{F}$ on morphisms.

Now, it only remains to prove $\mathcal{F} \circ \mathcal{G} \cong \mathbb{I}_{G-\mathbf{Rep}_k^K}$ on objects. We will show that given any twisted G-representation, the natural map $W^G \otimes_k K \to W$ is an isomorphism of G-representations. This map is clearly K-linear and G-contravariant. Hence it suffices to show that it is bijective. Prop. 2.8 shows that there is a G-contravariant isomorphism of K-vector spaces $W \otimes_k K \to {}^G W$ sending $w \otimes \alpha$ to $\sigma \mapsto \sigma(\alpha)w$, thus inducing a K-vector space isomorphism $(W \otimes_k K)^G \to ({}^G W)^G$. This isomorphism factors through

$$\begin{array}{ccc} (W \otimes_k K)^G & \longrightarrow & W \\ & \overset{\sim}{\searrow} & \downarrow \Theta \\ & & ({}^G W)^G \end{array}$$

where the vertical arrow Θ maps any $w \in W$ to the constant function $G \to W$ with value w. Thus Θ is injective. It is also surjective since the composed arrow is an isomorphism. Hence all arrows in this diagram are bijective. It only remains to show that the natural map $W^G \otimes_k K \to (W \otimes_k K)^G$ is an isomorphism (Recall that G acts on $W \otimes_k K$ on the first factor). Indeed, W^G fits into the exact sequence in $k-$**Mod**:

$$0 \to W^G \to W \to \prod_{\sigma \in G} W$$

where the last map is given by $w \mapsto (\sigma(w) - w)_{\sigma \in G}$. Apply $\otimes_k K$ and we obtain

$$0 \to W^G \otimes_k K \to W \otimes_k K \to \prod_{\sigma \in G} W \otimes_k K$$

where the last map is $m \mapsto (\sigma(m) - m)_{\sigma \in G}$ for all $m \in W \otimes_k K$. Thus the natural map $W^G \otimes_k K \to W \otimes_k K$ identifies $W^G \otimes_k K$ as the subspace $(W \otimes_k K)^G$, as desired. □

2.4. **Torsion-free modules.** Certain constructions in the previous subsection can be carried over to modules. Let R be a ring, and G be a finite group of automorphisms of R. Let M be an R-module equipped with a G-action such that $g(rm) = g(r)g(m)$ for all $g \in G$, $r \in R$, and $m \in M$. We may define the functor $(\cdot)^G$ in an obvious way, such that M^G will be an R^G-module.

It is worth noting that given a G-equivariant exact sequence of R-modules equipped with G-actions:

$$0 \to M' \to M \to M'' \to 0$$

the induced sequence of R^G-modules

$$0 \to (M')^G \to M^G \to (M'')^G$$

is exact. In other words, the functor $(\cdot)^G$ is left-exact. It also commutes with localization at G-stable multiplicative subgroups:

Lemma 2.10. *Let R be a ring, and G be a finite group of automorphisms of R. Let S be a G-stable multiplicative subgroup of R. Let M be an R-module equipped with a G-action such that $g(rm) = g(r)g(m)$ for all $g \in G$, $r \in R$, and $m \in M$. Then the natural R^G-module morphism*

$$(S^G)^{-1} M^G \to (S^{-1}M)^G$$

is an isomorphism.

Proof. The natural R^G-module morphism $(S^G)^{-1} M^G \to (S^{-1}M)^G$ sending $\frac{m}{r}$ to $\frac{m}{r}$ is obviously well-defined. To check that it is injective, suppose $m \in M^G$, $r \in S^G$ satisfies $\frac{m}{r} = 0$ in $(S^{-1}M)^G$, i.e. $tm = 0$ for some $t \in S$. Since m is invariant under the action of G, there holds $\sum_{g \in G} g(t)m = 0$. Since $\sum_{g \in G} g(t) \in S^G$, we see that $\frac{m}{r} = 0$ in $(S^G)^{-1} M^G$ as well.

To check that this morphism is surjective, let us be given $\frac{m}{r} \in (S^{-1}M)^G$. Then for any $g \in G$, $\frac{m}{r} = \frac{g(m)}{g(r)}$, i.e. there exists some $t_g \in S$ such that $t_g(mg(r) - rg(m)) = 0$. Let $t = \prod_{g \in G} t_g$. Then $t(mg(r) - rg(m)) = 0$ for all $g \in G$. Consequently,

$$t \left(m \sum_{g \in G} g(r) - r \sum_{g \in G} g(m) \right) = 0, \quad \text{and thus} \quad \frac{m}{r} = \frac{\sum_{g \in G} g(m)}{\sum_{g \in G} g(r)} \quad \text{in} \quad (S^{-1}M)^G$$

while the element on the right-hand-side is in the image of $(S^G)^{-1} M^G$. □

Galois descent on vector spaces allows us to use descent method after localizing a coherent sheaf at the generic point. To extend this over the entire scheme, whether certain modules are torsion-free becomes an important question. Before we apply the descent method to morphisms of schemes, let us first characterize torsion-free modules with the following

Lemma 2.11. *Let M be a finitely generated module over a domain R. Then M is torsion-free if and only if there is an injection $M \to R^{\oplus n}$ for some n.*

Proof. The "if" direction is trivial. To prove the "only if" direction, consider the natural map $M \to M \otimes_R K$ where $K = \mathrm{frac}(R)$. It is injective if and only if M is torsion-free, because $M \otimes_R K = (R - \{0\})^{-1}M$, and for any $m \in M$, $\frac{m}{1} = 0$ if and only if $rm = 0$ for some nonzero $r \in R$.

Since M is assumed to be finitely generated, $M \otimes_R K$ is a finite dimensional K-vector space. Let $e_1, \cdots, e_n$ be a K-basis for $M \otimes_R K$. Suppose M is generated by $m_1, \cdots, m_k$. Then

$$m_i \otimes 1 = \sum_{j=1}^{n} \alpha_{ij} e_j, \quad \text{for some } \alpha_{ij} \in K$$

Let $r \in R$ be a common denominator for the α_{ij}'s. Then $r\alpha_{ij} \in R$ for all i and j. Hence

$$m_i \otimes 1 \in \bigoplus_{j=1}^{n} R \cdot \left(\frac{e_j}{r}\right)$$

In other words, the image of M in $M \otimes_R K$ is contained in the finite free R-module generated by all the $r^{-1}e_j$'s. Now, the torsion-free assumption will give the injection $M \to R^{\oplus n}$. □

Corollary 2.12. *(i) Let E be a coherent sheaf on a Noetherian integral scheme X. Then E is torsion-free if and only if there is an injection $E|_U \to \mathcal{O}_U^{\oplus n}$ for every open affine $U \subset X$.*
(ii) Let $f : Y \to X$ be a flat morphism of Noetherian integral schemes, and E be a torsion-free coherent sheaf on X. Then f^*E is also torsion-free.

Proof. (i) Since torsion-free is a local property, the "if" direction is clear. For the "only if" direction, consider any open affine $U \subset X$ and apply Lem. 2.11.
(ii) For any $y \in Y$, take an affine neighborhood V of y such that $f(V) \subset U$, where $U \subset X$ is open affine. Since f is flat, the sheaf injection $E|_U \to \mathcal{O}_U^{\oplus n}$ pulls back to an injection $f^*E|_V \to \mathcal{O}_V^{\oplus n}$. The result then follows from the fact that torsion-free is a local property. □

2.5. **Galois morphisms.** A finite morphism $f : Y \to X$ of schemes is *Galois*, if there is a finite group G of automorphisms of Y such that $f \circ g = f$ for all $g \in G$, and the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ induces an isomorphism $\mathcal{O}_X \xrightarrow{\sim} (f_*\mathcal{O}_Y)^G$.

Lemma 2.13. *Given a finite, surjective, Galois morphism $f : Y \to X$ of integral schemes. Then the function field $K(Y)$ is a Galois extension of $K(X)$ and the natural morphism $G \to \mathrm{Gal}(K(Y)/K(X))$ is an isomorphism.*

Proof. On an open affine, f corresponds to a morphism of domains $A \to B$, and G is a subgroup of automorphisms of B with $B^G = A$. It follows from Lem. 2.10 that $K(Y)^G = K(X)$, as $K(X) = \mathrm{frac}(A)$ and $K(Y) = \mathrm{frac}(B)$. Hence the field extension $K(Y) \supset K(X)$ is Galois with the group $G = \mathrm{Gal}(K(Y)/K(X))$. □

We now wrap up the discussion on Galois descent with the following technical

Proposition 2.14. *Let $f : Y \to X$ be a finite, surjective, Galois morphism of integral, Noetherian schemes, with Galois group G. Let E be a coherent sheaf over X, and F' be a G-invariant coherent subsheaf of f^*E. Then*
(i) There exists a coherent sheaf $\tilde{E}$ over X with a morphism $E \to \tilde{E}$, which is an isomorphism where E is locally free;
(ii) There exists a coherent sheaf F over X with an injective morphism $F \to \tilde{E}$;

(iii) There exists a morphism $f^*F \to F'$ which is injective with torsion quotient, where f is flat and E is locally free; it is an isomorphism where, furthermore, F' is saturated.

Furthermore, these morphisms fit into the following commutative diagram:

$$\begin{array}{ccc} F' & \hookrightarrow & f^*E \\ \uparrow & & \downarrow \\ f^*F & \longrightarrow & f^*\tilde{E} \end{array} \tag{2.8}$$

Proof. Locally, the morphism f is given by a finite ring injection $A \to B$; B is equipped with a G-action such that $B^G = A$. The sheaf E corresponds to a finite A-module M, and F' corresponds to a B-submodule N' of $M \otimes_A B$, such that N' is G-stable under the induced action of G on $M \otimes_A B$.

Note first that for any coherent sheaf E' over Y, locally corresponding to a finite B-module M', the A-module $(M')^G$ glues together to be a coherent sheaf over X. Indeed, $(M')^G$ is a finite A-module, since it is an A-submodule of M' which is a finite A-module; we have used the hypotheses that $A \to B$ is finite and A is Noetherian. Furthermore, given a B-module morphism $M'_1 \to M'_2$, there is an induced A-module morphism $(M'_1)^G \to (M'_2)^G$ given by restriction. Therefore, the glueing condition of M's over Y induces a glueing condition for the $(M')^G$'s.

We let $\tilde{E}$ be the coherent sheaf that is locally $(M \otimes_A B)^G$, and let F be the coherent sheaf that is locally $(N')^G$.

(i) Note that there an A-module morphism $M \to (M \otimes_A B)^G$. If M is free, this morphism fits into the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & (M \otimes_A B)^G \\ \downarrow & & \downarrow \\ A^{\oplus r} & \longrightarrow & (B^{\oplus r})^G \end{array}$$

where all other maps are isomorphisms. Furthermore, it is immediate to see that the local ring maps $M \to (M \otimes_A B)^G$ are compatible among distinct open affines, hence glueing together to a morphism $E \to \tilde{E}$.

(ii) Taking $(\cdot)^G$ on the exact sequence $0 \to N' \to M \otimes_A B$ identifies N as an A-submodule of $(M \otimes_A B)^G$. We therefore obtain an injective morphism $F \to \tilde{E}$.

(iii) To construct the morphism $f^*F \to F'$, simply note that the B-module morphism $N \otimes_A B \to M \otimes_A B$ factors through N'. Assuming M is free, then it follows from (i) and (ii) that we have an injection $N \to M$. Assuming B is a flat A-module, the map $N \otimes_A B \to M \otimes_A B$ is also injective; thus, so is the map $N \otimes_A B \to N'$. Let T' be the quotient of this morphism and obtain an exact sequence

$$0 \to N \otimes_A B \to N' \to T' \to 0$$

Apply $\otimes_B K(Y)$, where $K(Y) = \text{frac}(B)$, and observe that $N \otimes_A B \otimes_B K(Y) = N \otimes_A K(X) \otimes_{K(X)} K(Y)$ where $K(X) = \text{frac}(A)$. Furthermore, by Lem. 2.10, $N \otimes_A K(X) = (N' \otimes_B K(Y))^G$, and Thm. 2.9 implies that $(N' \otimes_B K(Y))^G \otimes_{K(X)} K(Y) = N' \otimes_B K(Y)$. Altogether, we get

$$N' \otimes_B K(Y) \to N' \otimes_B K(Y) \to T' \otimes_B K(Y) \to 0$$

Thus $T' \otimes_B K(Y) = 0$ and T' is a torsion B-module.

Now, we add the assumption that F' is saturated. Hence the B-module Q' in the exact sequence

$$0 \to N' \to M \otimes_A B \to Q' \to 0$$

is torsion-free. Applying $(\cdot)^G$ and we obtain an exact sequence of A-modules:

$$0 \to N \to M \to (Q')^G$$

Hence the quotient sheaf $Q := M/N$ is again torsion-free. The flatness of B over A then implies that $Q \otimes_A B = (M \otimes_A B)/(N \otimes_A B)$, and it follows from Lem. 2.11 that $Q \otimes_A B$ is again torsion-free. Inspecting the following two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_A B & \longrightarrow & N' & \longrightarrow & T' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \text{inj.}} & & \vdots & & \\ 0 & \longrightarrow & N \otimes_A B & \longrightarrow & M \otimes_A B & \longrightarrow & Q \otimes_A B & \longrightarrow & 0 \end{array}$$

exhibits T' as a submodule of $Q \otimes_A B$, and hence also torsion-free. Combining this with the result from Step 2, we see that $T' = 0$. In conclusion, we have found an isomorphism $N \otimes_A B \xrightarrow{\sim} N'$.

The commutativity of (2.8) can be checked immediately for the corresponding B-modules. □

2.6. **(Semi)stability under finite pullback.** We are now ready to establish the main result of this section:

Proposition 2.15. *Let $f : Y \to X$ be a finite surjective morphism of normal projective varieties over an algebraically closed field k. Suppose E is a torsion-free coherent sheaf over X of dimension* $\dim(X)$. *Then*

(i) f^*E stable $\implies$ E stable;
(ii) f^*E semistable $\implies$ E semistable;
(iii) If the induced field extension $K(X) \subset K(Y)$ is separable, then E semistable $\implies$ f^*E semistable.

Proof. Let $n = \dim(X) = \dim(Y)$. Recall from §1.5 that the coherent sheaf f^*E is (semi)stable if and only if $T_{n-1}(f^*E) = T_{n-2}(f^*E)$, and the slope inequality holds for every proper coherent subsheaf.

Claim 2.16. *$T_{n-1}(f^*E) = T_{n-2}(f^*E)$.*

Proof. Because being torsion-free is equivalent to being pure of dimension n by Lem. 1.9, this equality holds if and only if f^*E is torsion-free outside codimension 2. Outside codimension 2, $f_*\mathcal{O}_Y$ is locally free (cf. Set-up 2.4), so f is a flat morphism outside codimension 2. Since E is assumed to be torsion-free, we can simply take an open subset U of X, whose complement has codimension at least 2, and over which $f : f^{-1}U \to U$ is flat. Then f^*E is torsion-free over $f^{-1}U$ by Cor. 2.12. □

Now, suppose f^*E is stable (resp. semistable), and let F be any proper coherent subsheaf of E, torsion-free outside codimension 2. Then Prop. 2.6 shows that $\mu(f^*F) = d \cdot \mu(F)$ and $\mu(f^*E) = d \cdot \mu(E)$, where d is the degree of f. The morphism $f^*F \to f^*E$ is injective outside codimension 2, i.e. there is a coherent sheaf $\mathcal{K}$, supported on a codimension 2 subscheme, such that the sequence

$$0 \to \mathcal{K} \to f^*F \to f^*E$$

is exact. Thus $F' := f^*F/\mathcal{K}$ is a coherent subsheaf of f^*E with $\mu(F') = \mu(f^*F)$, and $\mu(F) > \mu(E)$ (resp. $\mu(F) \geq \mu(E)$) implies $\mu(F') > \mu(f^*E)$ (resp. $\mu(F') \geq \mu(f^*E)$). This proves (i) and (ii).

For (iii), we first reduce to the case where $K(Y)$ is a finite Galois extension of $K(X)$ with Galois group G. Indeed, since $K(Y)$ is a separable extension field of $K(X)$ of degree d, we may take the Galois closure $K \supset K(Y) \supset K(X)$, and consider the normalization $\tilde{Y}$ of Y in K, with canonical morphism $r : \tilde{Y} \to Y$.

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{r} & Y \\ {\scriptstyle \tilde{f}}\downarrow & \swarrow{\scriptstyle f} & \\ X & & \end{array}$$

Then, if E is semistable over X, and we can show that $\tilde{f}^*E$ is semistable, then the "only if" direction, together with $\tilde{f}^*E = r^*f^*E$ will show that f^*E is semistable. We may now assume that $G = \mathrm{Gal}(K(Y)/K(X))$. Since X and Y are normal, G induces an action on Y that commutes with f, and the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ induces an isomorphism $\mathcal{O}_X \xrightarrow{\sim} (f_*\mathcal{O}_Y)^G$. Hence f is a finite Galois morphism, and outside codimension 2, it is flat. Now, suppose f^*E is not semistable on Y. Let

$F' \hookrightarrow f^*E$ be its maximal destabilizing subsheaf; by definition, it is saturated, and $\mu(F') > \mu(f^*E)$. By its uniqueness, F' is stable under G.

We apply Prop. 2.14 to find morphisms of coherent sheaves $E \to \tilde{E}$, $F \hookrightarrow \tilde{E}$, and $f^*F \to F'$. Outside codimension 2, the first and third morphisms are isomorphisms. In particular, $\mu(F') = \mu(f^*F) = d{\cdot}\mu(F)$. Since $\mu(f^*E) = d{\cdot}\mu(E)$, it suffices to produce a subsheaf $\tilde{F}$ of E with $\mu(\tilde{F}) = \mu(F)$. Consider the quotient $Q := \tilde{E}/F$, and let F be the kernel of the induced map $E \to Q$. Therefore, we have the following two exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \tilde{E} & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \tilde{F} & \longrightarrow & E & \longrightarrow & \tilde{Q} & \longrightarrow & 0 \end{array}$$

where $\tilde{Q} = \mathrm{Im}(E \to Q)$. Since the two vertical arrows are isomorphisms away from codimension 2, there holds in $\mathbb{Q}[m]_{n,n-1}$:

$$\chi(\tilde{F}(m)) = \chi(E(m)) - \chi(\tilde{Q}(m)) = \chi(\tilde{E}(m)) - \chi(Q(m)) = \chi(F(m))$$

Therefore $\mu(\tilde{F}) > \mu(E)$ and we have a contradiction. □

The special case where X and Y are nonsingular projective curves proves Prop. 0.2, noting that a finite morphism of nonsingular projective curves is necessarily surjective.

3. Ample vector bundles

3.1. **Definition.** The concept of ample line bundles can be generalized to vector bundles of arbitrary rank. Given a vector bundle E over a scheme X. We will use $S^n(E)$ to denote the nth symmetric power of E.

Proposition 3.1. *Let E be a vector bundle over a Noetherian scheme X. The following are equivalent*

- *(i) For every coherent sheaf F over X, there exists an integer $n_0 > 0$ such that for all $n \geq n_0$, the sheaf $F \otimes S^n(E)$ is globally generated.*
- *(ii) The tautological line bundle $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$ is ample.*

When either (hence both) of the above condition holds, E is called an *ample vector bundle*.

Proof. This is [6, Prop. 3.2]. □

3.2. **Elementary properties.** The eventual goal of this section is to prove that the tensor product of ample vector bundles remains ample (in characteristic zero, or over curves). Hartshorne [6] used representation theory of $\mathrm{GL}(r)$ to prove this result. Before introducing his method, we discuss a number of results that can be obtained directly from the definition. Let X be a Noetherian scheme.

Lemma 3.2. *Let E be an ample vector bundle over X. Then any quotient vector bundle Q of E is ample.*

Proof. The any integer n, the vector bundle $S^n(Q)$ is a quotient of $S^n(E)$. Hence, for every coherent sheaf F over X, $F \otimes S^n(E)$ being globally generated will imply the same property of $F \otimes S^n(Q)$. □

Lemma 3.3. *Let E_1 and E_2 be vector bundles over X. Then $E_1 \oplus E_2$ is ample if and only if both E_1 and E_2 are.*

Proof. It follows from the previous lemma that $E_1 \oplus E_2$ ample $\implies$ both E_1 and E_2 are ample. We now assume E_1 and E_2 are ample and prove that $E_1 \oplus E_2$ is ample. Since $S^n(E_1 \oplus E_2) = \bigoplus_{p+q=n} S^p(E_1) \otimes S^q(E_2)$, it suffices to show that for every coherent sheaf F over X, there exists $n_0 \geq 0$ such that the sheaf $F \otimes S^p(E_1) \otimes S^q(E_2)$ is globally generated whenever $p + q \geq n_0$. Pick integers n_1, n_2, m_1, m_2 such that

- (i) $F \otimes S^n(E_1)$ is globally generated whenever $n \geq n_1$;

(ii) $S^n(E_2)$ is globally generated whenever $n \geq n_2$;
(iii) $F \otimes S^n(E_1) \otimes S^r(E_2)$ is globally generated for all $r = 0, \cdots, n_2 - 1$, whenever $n \geq m_1$;
(iv) $F \otimes S^l(E_1) \otimes S^n(E_2)$ is globally generated for all $l = 0, \cdots, n_1 - 1$, whenever $n \geq m_2$.

Now let $n_0 = \max\{n_1, m_1\} + \max\{n_2, m_2\}$, and suppose $p + q \geq n_0$. If $p \geq n_1$ and $q \geq n_2$, then $F \otimes S^p(E_1) \otimes S^q(E_2)$ is globally generated by (i)&(ii). Otherwise, if $0 \leq p < n_1$, then $q \geq m_2$, and (iv) shows that $F \otimes S^p(E_1) \otimes S^q(E_2)$ is globally generated. If $0 \leq q < n_2$, then $p \geq m_1$, and (iii) shows that $F \otimes S^p(E_1) \otimes S^q(E_2)$ is globally generated. □

Lemma 3.4. *Let E and F be vector bundles over X. If E is ample and F is globally generated, then $E \otimes F$ is ample.*

Proof. Since F is globally generated, there is a sheaf surjection $\mathcal{O}_X^{\oplus r} \to F$. Tensoring with E yields a sheaf surjection $E^{\oplus r} \to E \otimes F$. Thus $E \otimes F$ is a quotient bundle of a direct sum of ample vector bundles, and must itself be ample. □

Lemma 3.5. *Let E be a vector bundle over X. Then*

(i) E ample $\implies$ there exists an integer n_0 such that $S^n(E)$ is ample for all $n \geq n_0$.
(ii) $S^k(E)$ ample for some k $\implies$ E ample.

Proof. For (i), E ample $\implies$ there exists n_0 such that $S^n(E)$ is globally generated for all $n \geq n_0 - 1$. Hence $E \otimes S^n(E)$ is ample by Lem. 3.4, and so is its quotient $S^{n+1}(E)$. Thus $S^n(E)$ is ample for all $n \geq n_0$.

For (ii), assume $S^k(E)$ is ample, and consider any coherent sheaf F. There exists an integer m_0 such that $F \otimes S^r(E) \otimes S^m(S^k(E))$ is globally generated for all $r = 0, \cdots, k-1$, whenever $m \geq m_0$. By Lem. 3.2, the quotient sheaf $F \otimes S^{km+r}(E)$ is globally generated for all $r = 0, \cdots, k-1$, so long as $m \geq m_0$. Let $n_0 = m_0 k$. Then every $n \geq n_0$ can be expressed as $n = km + r$ with $m \geq m_0$ and $0 \leq r \leq k - 1$. Thus $F \otimes S^n(E)$ is globally generated. □

Lemma 3.6. *Let E be an ample vector bundle over X of rank r. Then its top exterior power $\Lambda^r E$ is an ample line bundle.*

Proof. Let n be sufficiently large, so that $S^n(E)$ is both ample and globally generated (possible by Lem. 3.5). It is a sheaf of rank $\binom{n+r-1}{r-1}$. We now use the linear-algebraic identity

$$\Lambda^{\binom{n+r-1}{r-1}}(S^n(E)) = \Lambda^r(E)^{\otimes \binom{n+r-1}{r}}$$

The left-hand-side is ample since it is a quotient of the ample vector bundle $S^n(E)^{\otimes \binom{n+r-1}{r-1}}$. Being a line bundle, $\Lambda^r(E)$ is ample if and only if some tensor power of it is ample. Hence $\Lambda^r(E)$ is also ample. □

A direct consequence of the above lemma is the following

Corollary 3.7. *Let E be an ample vector bundle over X. Then* $\deg(E) > 0$.

Proof. Say E has rank r. Since $\deg(E) = \deg(\Lambda^r E)$, and ample line bundles have positive degree, the result follows. □

We also need a functorial property of ample vector bundles with respect to finite morphisms:

Lemma 3.8. *Let $f : Y \to X$ be a finite morphism of Noetherian schemes, and E is a vector bundle on X. Then*

(i) E ample $\implies$ f^*E ample;
(ii) If X is normal and f is surjective, then f^*E ample $\implies$ E ample.

Proof. Note that we have a fiber product diagram:

$$\begin{array}{ccc} \mathbb{P}(f^*E) & \xrightarrow{\hat{f}} & \mathbb{P}(E) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

where $\mathcal{O}_{\mathbb{P}(f^*E)}(1) = \hat{f}^*\mathcal{O}_{\mathbb{P}(E)}(1)$. Thus the result follows from the corresponding Lem. 2.2 for line bundles. □

3.3. **Representations of** $\mathrm{GL}(r)$**.** In this subsection, we give a survey of some representation theoretic results about the linear algebraic group $\mathrm{GL}(r)$, over an algebraically closed field k. Our exposition (and notation) follows the appendix of [6]. Write $G = \mathrm{GL}(r)$, and denote some of its important subgroups by

(i) H is the abelian group of diagonal matrices $(h_1, \cdots, h_r)$;
(ii) P is the solvable group of upper triangular matrices; $X = [P, P]$ is its commutator subgroup consisting of upper triangular matrices with 1's on the diagonal;
(iii) Q is the solvable group of lower triangular matrices; $Y = [Q, Q]$ is its commutator subgroup consisting of lower triangular matrices with 1's on the diagonal;

Given a representation $T : G \to \mathrm{GL}(V)$, there is an induced G-action on the dual space $V^\vee = \mathrm{Hom}_k(V, k)$ given by $(g\varphi)(v) = \varphi(g^{-1}v)$ for all $g \in G$, $\varphi \in V^\vee$, and $v \in V$. This makes a G-representation $T^\vee : G \to \mathrm{GL}(V^\vee)$, known as the *contragredient representation.*

Furthermore, the globally sections $H^0(G, \mathcal{O}_G)$, identified as the space $A(G)$ of regular functions on k-points of G, is equipped with a *right translation* defined by $(g\theta)(g') = \theta(g'g)$ for all $g, g' \in G$ and $\theta \in A(G)$.

Lemma 3.9. *Given a representation T of a connected solvable linear algebraic group B over k, on a k-vector space V. Then there exists a nonzero $v \in V$, and a character $\chi : B \to k^*$ such that $b(v) = \chi(b)v$ for all $b \in B$.*

Given a G-representation T on a k-vector space V, the solvable groups P and Q inherit the representation T. Any character $\lambda : P \to k^*$ given by the lemma is called an *upper weight* of T, and any character $\mu : Q \to k^*$ given by the lemma is called a *lower weight* of T.

Proof of Lem. 3.9. Borel's fixed point theorem [2, Prop. 15.5] states that a connected solvable group acting regularly on a complete variety has a fixed point. Consider the projectivization $\mathbb{P}(V) = V/k^*$, on which B acts. It must have a fixed point $\bar{v}$. Take any $v \in V$ in the equivalence class of $\bar{v}$. Then $b(v) = \chi(b)v$ for a map $\chi : B \to k^*$. The map χ is a map of varieties, since it is a composition of maps of varieties:

$$\begin{array}{ccc} B & \xrightarrow{T} & \mathrm{GL}(V)^v \\ & \searrow^{\chi} & \downarrow \\ & & k^* \end{array}$$

where $\mathrm{GL}(V)^v$ denotes the linear automorphisms of V, with eigenvector v, and the vertical arrow $\mathrm{GL}(V)^v \to k^*$ evaluates the eigenvalue associated to v of a given linear automorphism. This arrow is a map of varieties, since we may pick a basis $e_1 = v, e_2, \cdots, e_n$ for V, and this map is the projection onto the g_{11}-coordinate for any $g = (g_{ij}) \in \mathrm{GL}(V)^v$. To see that χ is a homomorphism, note that

$$\chi(bb')v = (bb')(v) = b(b'(v)) = \chi(b)\chi(b')v$$

and therefore $\chi(bb') = \chi(b)\chi(b')$. □

Note that since $X = [P, P]$ and k^* is commutative, an upper weight $\lambda : P \to k^*$ is trivial on the subgroup X. Hence it is determined completely by its action on $P/X \cong H$. Similarly, a lower weight $\mu : Q \to k^*$ is trivial on Y, and is determined completely by its action on $Q/Y \cong H$.

Lemma 3.10. *Let $g = (g_{ij}) \in G$. Define $g_0 = 1$ and*

$$g_i = \det \begin{pmatrix} g_{11} & \cdots & g_{1i} \\ \vdots & & \vdots \\ g_{i1} & \cdots & g_{ii} \end{pmatrix} \quad \text{for} \quad i = 1, \cdots, r$$

If $g_i \neq 0$ for all i, then g can be written uniquely as $g = yhx$ with $y \in Y$, $h \in H$, and $x \in X$. Here $h = (h_1, \cdots, h_r)$ and $h_i = g_i/g_{i-1}$ for $i = 1, \cdots, r$.

Proof. We induct on the rank r. The case for $r = 1$ is trivial. Assume the result for $r - 1$. Then we have the unique decomposition for

$$g' = \begin{pmatrix} g_{11} & \cdots & g_{1(r-1)} \\ \vdots & & \vdots \\ g_{(r-1)1} & \cdots & g_{(r-1)(r-1)} \end{pmatrix} = y'h'x'$$

To construct the decomposition for g, let y, h, and x be defined by

$$y = \begin{pmatrix} y' & & 0 \\ & & \vdots \\ y_{r1} & \cdots & 1 \end{pmatrix}, \quad h = \begin{pmatrix} h' & & 0 \\ & & \vdots \\ 0 & \cdots & h_r \end{pmatrix}, \quad x = \begin{pmatrix} x' & & x_{1r} \\ & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

where $y_{r1}, \cdots, y_{r(r-1)}, h_r, x_{1r}, \cdots, x_{(r-1)r}$ are to be determined. The uniqueness of the decomposition $g' = y'h'x'$ shows that these are the only entries remaining to be determined. Indeed, by considering the identities

$$g_{ir} = \sum_{j \leq \min\{i,r\}} y_{ij} h_j x_{jr}, \quad \text{for} \quad i = 1, \cdots, r-1$$

we may successively determine $x_{1r}, \cdots, x_{(r-1)r}$. Similarly, by considering g_{rj} for $j = 1, \cdots, r-1$, we may successively determine $y_{r1}, \cdots, y_{r(r-1)}$. Altogether, the decomposition of g exists and is unique. Furthermore, $g_{r-1} = \det(g') = \det(h')$, and $g_r = \det(g) = \det(h) = \det(h')h_r$. Hence $h_r = g_r/g_{r-1}$. □

Following [6], we let $G_0 \subset G$ be the Zariski open subset consisting of matrices $g \in \mathrm{GL}(r)$ with $g_i \neq 0$ for $i = 1, \cdots, r$.

A *coefficient* of a G-representation $T : G \to \mathrm{GL}(V)$ is any function $\theta \in A(G)$ of the form $\theta(g) = \varphi(g(v))$ for some fixed $v \in V$ and $\varphi \in V^\vee$.

Lemma 3.11. *Given a G-representation $T : G \to \mathrm{GL}(V)$, let λ be an upper weight of T corresponding to some $v \in V$, and μ be a lower weight of $T^\vee$ corresponding to some $\varphi \in V^\vee$. Let θ be the coefficient $\theta(g) = \varphi(g(v))$. Then*

$$\theta(g) = \lambda(h)\theta(e) = \mu(h^{-1})\theta(e) \quad \text{for all} \quad g = yhx \in G_0 \tag{3.1}$$

decomposed as in Lem. 3.10. Furthermore, if T is an irreducible representation, then $\theta(e) \neq 0$.

Proof. By direct calculation,

$$\theta(g) = \varphi(yhx(v)) = (y^{-1}\varphi)(hx(v)) = (\mu(y^{-1})\varphi)(\lambda(hx)(v)) = \varphi(\lambda(h)v) = \lambda(h)\theta(e)$$

using $\mu(y^{-1}) = 1$ and $\lambda(x) = 1$, and

$$\theta(g) = \varphi(yhx(v)) = (h^{-1}y^{-1}\varphi)(x(v)) = (\mu(h^{-1}y^{-1})\varphi)(\lambda(x)v) = \mu(h^{-1})\varphi(v) = \mu(h^{-1})\theta(e)$$

using the exact same identities. If $\theta(e) = 0$, then θ vanishes on G_0, a dense open subset of G, and hence is identically zero. However, since T is irreducible, there holds $\{g(v) : g \in G\} = V$. Thus $\varphi = 0$, which is impossible. □

Proposition 3.12. *Any irreducible G-representation T has a unique upper weight λ. Furthermore, $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$ for any diagonal matrix $h = (h_1, \cdots, h_r) \in H$, and $n_1 \geq \cdots \geq n_r$ are integers.*

Proof. We proceed in several steps.

Step 1. We first prove that any irreducible G-representation T has a unique upper weight. Indeed, fix a lower weight μ corresponding to some $\varphi \in V^\vee$, and let us be given any upper weight λ corresponding to $v \in V$. It follows from Lem. 3.11 that the coefficient $\theta \in A(G)$ defined by $\theta(g) = \varphi(g(v))$ satisfies $\theta(g) = \lambda(h)\theta(e) = \mu(h^{-1})\theta(e)$, for all $g = yhx \in G_0$. Since $\theta(e) \neq 0$, it follows that $\lambda(h) = \mu(h^{-1})$. So λ is uniquely determined.

Step 2. $\lambda : P \to k^*$ restricts to a character $\lambda : H \cong (k^*)^r \to k^*$. We claim that $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$ for any diagonal matrix $h = (h_1, \cdots, h_r) \in H$, where $n_i \in \mathbb{Z}$. Indeed, any character χ of $(k^*)^r$ defines an element in $A((k^*)^r) = k[x_1, \cdots, x_r, \frac{1}{x_1}, \cdots, \frac{1}{x_r}]$, a k-vector space with a basis given by $\{x_1^{n_1} \cdots x_r^{n_r} : n_i \in \mathbb{Z}\}$. Since each such monomial $x_1^{n_1} \cdots x_r^{n_r}$ is a character of $(k^*)^r$, it follows from linear independence of characters that every character of $(k^*)^r$ is of this form.

Step 3. We now prove that $n_1 \geq \cdots \geq n_r$. Since for every $g = yhx \in G_0$, there holds $\theta(g) = \lambda(h)\theta(e)$, by applying a scalar to v if necessary, we may assume

$$\theta(g) = \lambda(h) = h_1^{n_1} \cdots h_r^{n_r} = g_1^{n_1 - n_2} \cdots g_{r-1}^{n_{r-1} - n_r} g_r^{n_r}$$

where g_i is defined as in Lem. 3.10. Now, we have a regular function $\theta \in A(G)$, such that for all $g \in G_0$,

$$\theta(g) = g_1^{s_1} \cdots g_r^{s_r}$$

and we just need to show that all $s_1, \cdots, s_{r-1} \geq 0$. We proceed by induction. The result is vacuous for $r = 1$ (since there is no condition on s_r).

For $r = 2$, consider a family of matrices

$$g(x) = \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$$

Then the composition $\theta(g(x))$, as a function in x, is a regular function from $\mathbb{A}_k^1$ to $\mathbb{A}_k^1$. On the other hand, when $x \neq 0$, $g(x) \in G_0$, so $\varphi(g(x)) = x^{s_1}$ for $x \neq 0$. For x^{s_1} to define a regular function on $\mathbb{A}_k^1$, it is necessary and sufficient that $s_1 \geq 0$.

In the general case $r \geq 2$, we assume the result for $r - 1$, and consider the following two subgroups of G:

$$\begin{pmatrix} * & * & 0 \\ * & * & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \cong \mathrm{GL}(r-1), \quad \text{and} \quad \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \cong \mathrm{GL}(2)$$

The function θ restricts to a regular function on both subgroups, as $\theta_1(g) = g_r^{s_1} \cdots g_{r-2}^{s_{r-2}} g_{r-1}^{s_{r-1}+s_r}$ on the first subgroup, and as $\theta_2(g) = g_{r-1}^{s_{r-1}} g_r^{s_r}$. So induction hypothesis and base case $r = 2$ imply that $s_1, \cdots, s_{r-2} \geq 0$, and also $s_{r-1} \geq 0$. The proof is complete. □

We now establish the converse to Prop. 3.12:

Proposition 3.13. *Given r integers $n_1 \geq \cdots \geq n_r$, define a function $\theta \in A(G)$ by*

$$\theta(g) = g_1^{s_1} \cdots g_r^{s_r}$$

where $s_i = n_i - n_{i+1}$ for $i = 0, \cdots, r-1$, and $s_r = n_r$. Let V be the subspace of $A(G)$ generated by the right translates of θ. Then V defines an irreducible G-representation with upper weight $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$. Furthermore, any other irreducible representation of G with upper weight λ is isomorphic to V.

Proof. We proceed in two steps:

Step 1. We first show that V is a G-representation with upper weight $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$. Indeed, let $p \in P$ and $g \in G$ be arbitrary. We want to show that $(p\theta)(g) = \lambda(p)\theta(g)$. Note that

$$\lambda(p) = p_{11}^{n_1} \cdots p_{rr}^{n_r} = p_1^{s_1} \cdots p_r^{s_r} = \theta(p)$$

since $p_i = p_{11} \cdots p_{ii}$ is the determinant of the upper-left $(i \times i)$-minor of p. Furthermore, $(gp)_i = g_i p_i$ by straightforward calculation. We may now compute

$$(p\theta)(g) = \theta(gp) = (gp)_1^{s_1} \cdots (gp)_r^{s_r} = p_1^{s_1} \cdots p_r^{s_r} g_1^{s_1} \cdots g_r^{s_r} = \lambda(p)\theta(g)$$

as desired.

Step 2. We prove that given any irreducible G-representation on W with upper weight λ, there is an injective G-equivariant map $f : W \to A(G)$ whose image contains θ. This will show that $f(W)$ is an irreducible G-representation containing V, and thus V is irreducible and $f : W \xrightarrow{\sim} V$.

Indeed, let $\varphi \in W^\vee$ be an eigenvector associated to the lower weight of the contragredient representation $W^\vee$. For every $w \in W$, define $f(w) \in A(G)$ to be the coefficient $f(w)(g) = \varphi(g(w))$. It is clear that $f : W \to A(G)$ is G-equivariant. To show that f is injective, note that since W is irreducible, $\{g(w) : g \in G\} = W$. Hence there exists g such that $\varphi(g(w)) \neq 0$, i.e. for any $w \in W$, $f(w)$ is not the zero function on G. To show that the image of f contains θ, let $w \in W$ be an eigenvector corresponding to the upper weight of W. Then $\varphi(w) \neq 0$ by Lem. 3.11, so we may impose the normalization $\varphi(w) = 1$. It then follows that

$$f(w)(g) = \varphi(g(w)) = \lambda(h) = \theta(g) \quad \text{for all} \quad g = yhx \in G_0$$

where in the second equality, we used (3.1). Consequently $f(w) = \theta \in A(G)$, and the proposition is proved. □

In summary, we have obtained the following classification of irreducible representations of $\mathrm{GL}(r)$:

Theorem 3.14. *There is a bijective correspondence*

$$\mathcal{F} : \begin{pmatrix} \text{isomorphism classes of} \\ \text{irreducible } G\text{-representations} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{integer tuples } (n_1, \cdots, n_r) \\ \text{with } n_1 \geq \cdots \geq n_r \end{pmatrix}$$

given by associating, to an irreducible G-representation T, the tuple $(n_1, \cdots, n_r)$ occurring in $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$ where λ is the upper weight of T, and $h = (h_1, \cdots, h_r) \in H$ is a diagonal matrix.

Proof. The association $\mathcal{F}$ is well defined by Prop. 3.12. Prop. 3.13 gives a well defined association $\mathcal{G}$ in the other direction, and proves $\mathcal{F} \circ \mathcal{G} = \mathbb{I}$. The last part of Prop. 3.13 then proves $\mathcal{G} \circ \mathcal{F} = \mathbb{I}$. □

We now prove that every weight occurs in a tensor product of sufficiently high symmetric powers of the standard representation:

Proposition 3.15. *Given an integer tuple $c = (n_1, \cdots, n_r)$ with $n_1 \geq \cdots \geq n_r$, let λ denote the upper weight of its corresponding irreducible G-representation. Let V be the standard G-representation. Then there exist natural numbers $q_1, \cdots, q_t$ where $t = r!$ such that $S^{q_1}(V) \otimes \cdots \otimes S^{q_t}(V)$ has an upper weight λ. Furthermore, each $q_i \to \infty$ as $\sum n_i \to \infty$.*

Proof. In this proof, we will use $S(q_1, \cdots, q_t)$ to denote $S^{q_1}(V) \otimes \cdots \otimes S^{q_t}(V)$. For the first claim, it suffices give an eigenvector of $S(q_1, \cdots, q_t)$ with eigenvalue $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$. We define two operations on vectors in $S(q_1, \cdots, q_t)$. Given $v \in S(a_1, \cdots, a_j)$ and $w \in S(b_1, \cdots, b_k)$, we may form:

(i) $v \otimes w \in S(a_1, \cdots, a_j, b_1, \cdots, b_k)$; if v and w are eigenvectors with eigenvalues λ_v and λ_w, $v \otimes w$ is an eigenvector with eigenvalue $\lambda_v \lambda_w$.

(ii) $vw \in S(a_1 + b_1, \cdots, a_j + b_j, b_{j+1}, \cdots, b_k)$, assuming $j \leq k$; if v and w are eigenvectors with eigenvalues λ_v and λ_w, vw is again an eigenvector with eigenvalue $\lambda_v \lambda_w$.

Define $f_i \in V^{\otimes i}$ by

$$f_i = \sum_{\sigma \in S_i} \mathrm{sgn}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(i)}$$

where $e_1, \cdots, e_r$ is the basis of V with respect to which G acts. Then f_i is an eigenvector for P with eigenvalue $\lambda(h) = h_1 \cdots h_i$. We will use these f_i's to build the desired eigenvector in the following way:

Let $s_i = n_i - n_{i+1}$ for $i = 1, \cdots, r-1$, and $s_r = n_r$. Dividing s_i by $\frac{t}{i}$ with remainder, we have

$$s_i = \frac{t}{i} \cdot Q_i + R_i, \quad \text{where} \quad 0 \leq R_i < \frac{t}{i} \tag{3.2}$$

Then set

$$v = \prod_{i=1}^{r} \left(f_i^{\otimes \frac{t}{i}} \right)^{Q_i} (f_i)^{R_i}$$

Since f_i is an eigenvector for P, so is v, and its eigenvalue is given by

$$\lambda(h) = \prod_{i=1}^{r} (h_1 \cdots h_i)^{\frac{t}{i} \cdot Q_i + R_i} = \prod_{i=1}^{r} (h_1 \cdots h_i)^{s_i} = h_1^{n_1} \cdots h_r^{n_r}$$

Furthermore, $v \in S(q_1, \cdots, q_t)$ where

$$q_i = \begin{cases} Q_1 + \cdots + Q_r + R_i + \cdots + R_r & : 1 \leq i \leq r \\ Q_1 + \cdots + Q_r & : r < i \leq t \end{cases}$$

For the second claim, we need to show that the minimum of q_i's, namely $Q_1 + \cdots + Q_r$, tends to infinity as $\sum n_i \to \infty$. It follows from (3.2) that $is_i < tQ_i + t$. Hence

$$\sum_{i=1}^{r} n_i = \sum_{i=1}^{r} is_i < t \sum_{i=1}^{r} Q_i + tr$$

Since $t = r!$ and r is fixed, the result follows from this inequality. □

3.4. **Characteristic zero case.** Over a field k of characteristic zero, every representation of $\mathrm{GL}(r)$ is a direct sum of irreducible representations. Therefore, if W is a $\mathrm{GL}(r)$-representation admitting an upper weight $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$, then the irreducible representation V_c corresponding to the tuple $c = (n_1, \cdots, n_r)$ is necessarily a direct summand of W. This fact, combined with Prop. 3.15, will allow us to prove ampleness for any symmetric power bundle via reduction to a sufficiently high symmetric power.

Let $\mathrm{char}(k) = 0$, and X be a scheme of finite type over k. Let E be a vector bundle over X of rank r. For every representation $T : \mathrm{GL}(r) \to \mathrm{GL}(n)$, we may associate a vector bundle $T(E)$ of rank n in a natural way.

Lemma 3.16. *Let* $\mathrm{char}(k) = 0$*, and* X *be a scheme of finite type over* k*. Let* E *be an ample vector bundle over* X *of rank* r*, and* $T_c : \mathrm{GL}(r) \to \mathrm{GL}(V_c)$ *be the irreducible representation corresponding to the tuple* $c = (n_1, \cdots, n_r)$ *with* $n_1 \geq \cdots \geq n_r$*. Then if* $\sum n_i$ *is sufficiently large,* $T(E)$ *is again ample.*

Proof. Lem. 3.5(i) guarantees the existence of some integer n_0 such that $S^n(E)$ is ample and globally generated for all $n \geq n_0$. By Prop. 3.15, there exists an m_0 such that if $\sum n_i \geq m_0$, then $T(E)$ is a direct summand of some $S^{q_1}(E) \otimes \cdots \otimes S^{q_t}(E)$ where each $q_i \geq n_0$. Hence $S^{q_1}(E) \otimes \cdots \otimes S^{q_t}(E)$ is ample by Lem. 3.4, and being a quotient of an ample vector bundle, $T(E)$ is again ample by Lem. 3.2. □

Proposition 3.17. *Let* $\mathrm{char}(k) = 0$*, and* X *be a scheme of finite type over* k*. Let* E *be an ample vector bundle over* X*. Then any tensor power* $E^{\otimes n}$*, where* $n \geq 1$*, is ample.*

Proof. According to Lem. 3.5(ii), we only need to show that for some natural number m, the vector bundle $S^m(E^{\otimes n})$ is ample. Let V be the standard representation of $\mathrm{GL}(r)$, where r is the rank of E. By Lem. 3.16, it suffices to prove that for some m, all upper weights $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$ of the representation $S^m(V^{\otimes n})$ has sufficiently large $\sum n_i$. Let $e_1, \cdots, e_r$ be the basis of V on which $\mathrm{GL}(r)$ acts. Then $S^m(V^{\otimes n})$ has a basis $e_{\sigma(1)}^{\alpha_1} \cdots e_{\sigma(t)}^{\alpha_t}$, where $\sigma(1), \cdots, \sigma(t)$ take values from the set $\{1, \cdots, r\}$, possibly with repetitions, and $\sum \alpha_i = mn$. Hence any upper weight $\lambda(h) = h_1^{n_1} \cdots h_r^{n_r}$ of the representation $S^m(V^{\otimes n})$ satisfies $\sum n_i = mn$. This number can certainly be made sufficiently large by the choice of m. □

Being quotients of tensor powers, arbitrary symmetric powers and alternating powers of an ample vector bundle are also ample.

Proposition 3.18. *Let* $\mathrm{char}(k) = 0$*, and* X *be a scheme of finite type over* k*. Let* E_1 *and* E_2 *be ample vector bundles over* X*. Then* $E_1 \otimes E_2$ *is again ample.*

Proof. Let $E = E_1 \oplus E_2$, which is ample by Lem. 3.3. The above proposition shows that $S^2(E) = S^2(E_1) \oplus (E_1 \otimes E_2) \oplus S^2(E_2)$ is also ample; hence so is its direct summand $E_1 \otimes E_2$. □

4. Numerical criteria for amplitude

4.1. **Multiplicity and intersection number.** Let X be a projective scheme over an algebraically closed field k. Given an integral closed subscheme $Y \subset X$ of dimension d, and line bundles $L_1, \cdots, L_d$, we define the *intersection number* of $L_1, \cdots, L_d$ with Y by

$$(L_1. \cdots .L_d.Y) = \mathrm{Coeff}_{m_1 \cdots m_d} \chi(L_1^{m_1} \otimes \cdots \otimes L_d^{m_d} \otimes \mathcal{O}_Y)$$

where the right-hand-side denotes the coefficient in front of $m_1 \cdots m_d$ in the multivariate polynomial $\chi(L_1^{m_1} \otimes \cdots \otimes L_d^{m_d} \otimes \mathcal{O}_Y)$. Indeed, this definition is justified by the following:

Lemma 4.1. *For every coherent sheaf* E *over* X*, the expression* $\chi(L_1^{m_1} \otimes \cdots \otimes L_d^{m_d} \otimes E)$ *is a numerical polynomial in* $m_1, \cdots, m_d$*.*

Proof. We induct on d. The base case $d = 1$ is precisely Cor. 1.13. For the general case, observe that

$$P(m_1, \cdots, m_d) = \chi(L_1^{m_1} \otimes \cdots \otimes L_d^{m_d} \otimes E)$$

is a numerical polynomial in $m_2, \cdots, m_d$ for fixed m_1 by induction hypothesis, i.e.

$$P(m_1, \cdots, m_d) = \sum_{I=(I_1, \cdots, I_{d-1})} a_I(m_1) m_2^{I_1} \cdots m_d^{I_{d-1}} \tag{4.1}$$

where $I = (I_1, \cdots, I_{d-1})$ ranges through a finite number of multiindices, and $a_I(m_1)$ is a function of m_1. Furthermore, the base case shows that for fixed $m_2, \cdots, m_d$, $P(m_1, \cdots, m_d)$ is a numerical polynomial in m_1. Hence we may use special values of $(m_2, \cdots, m_d)$ in (4.1), to solve a system of linear equations in terms of $a_I(m_1)$ with integer coefficients. Thus each $a_I(m_1)$ is a polynomial in m_1 with rational coefficients. The fact that $P(m_1, \cdots, m_d)$ is numerical follows, because the Euler characteristic only takes integer values. □

If $D_1, \cdots, D_d$ are Cartier divisors on X, their *intersection number* with a d-dimensional integral closed subscheme $Y \subset X$ is defined in terms their associated line bundles $\mathcal{O}_X(D_1), \cdots, \mathcal{O}_X(D_d)$ in an obvious manner:

$$(D_1. \cdots .D_d.Y) := (\mathcal{O}_X(D_1). \cdots .\mathcal{O}_X(D_d).Y)$$

Remark 4.2. The expression $(L_1. \cdots .L_d.Y)$ is multilinear in L_1 through L_d. More precisely, if $L_1 = N_1 \otimes N_2$, then

$$(L_1. \cdots .L_d.Y) = (N_1.L_2. \cdots .L_d.Y) + (N_2.L_2. \cdots .L_d.Y) \tag{4.2}$$

Indeed, if we set $P(m_1, m_1', m_2, \cdots, m_d) = \chi(N_1^{m_1} \otimes N_2^{m_1'} \otimes L_2^{m_2} \otimes \cdots \otimes L_d^{m_d} \otimes \mathcal{O}_Y)$, then we may express the left-hand-side of (4.2) as

$$\begin{aligned} &\mathrm{Coeff}_{m_1 m_2 \cdots m_d} P(m_1, m_1', m_2, \cdots, m_d) + \mathrm{Coeff}_{m_1' m_2 \cdots m_d} P(m_1, m_1', m_2, \cdots, m_d) \\ =&\mathrm{Coeff}_{m_1 m_2 \cdots m_d} P(m_1, 0, m_2, \cdots, m_d) + \mathrm{Coeff}_{m_1' m_2 \cdots m_d} P(0, m_1', m_2, \cdots, m_d) \end{aligned}$$

which is precisely the right-hand-side.

Remark 4.3. Let Y be a d-dimensional integral closed subscheme of X, and L be a line bundle. Then

$$(L^d.Y) = d! \cdot (\text{leading coefficient of } \chi(L^m \otimes \mathcal{O}_Y))$$

Indeed, this follows from

$$(L^d.Y) = \text{Coeff}_{m_1 \cdots m_d} \chi(L^{m_1 + \cdots + m_d} \otimes \mathcal{O}_Y) = d! \cdot \text{Coeff}_{m^d} \chi(L^m \otimes \mathcal{O}_Y)$$

Given an integral curve C over an algebraically closed field k, and a closed point $P \in C$. We define the *multiplicity* of C at P by

$$m_P(C) = \min_{f \in \mathfrak{m}_P - \{0\}} \dim_k(\mathcal{O}_{C,P}/(f))$$

We relate this definition to intersection numbers:

Lemma 4.4. *Let C be an integral, projective curve over an algebraically closed field k. Let L be a line bundle over C, with a nonzero global section $s \in H^0(C, L)$. Then*

$$(L.C) = \sum_{\substack{P \in C \\ s \in \mathfrak{m}_P L_P}} \dim_k(\mathcal{O}_{C,P}/(s)) \tag{4.3}$$

Proof. The right-hand-side makes sense because s defines an element in $\mathfrak{m}_P$ up to multiplication by a unit. There is an injective sheaf morphism $\mathcal{O}_C \to L$ given by multiplication by s. Let Q be the cokernel of this morphism:

$$0 \to \mathcal{O}_C \xrightarrow{\cdot s} L \to Q \to 0 \tag{4.4}$$

Then Q is supported at those points $P \in C$ with $s \in \mathfrak{m}_P L_P$. Altogether, the right-hand-side of (4.3) admits a natural interpretation as an Euler characteristic:

$$\sum_{\substack{P \in C \\ s \in \mathfrak{m}_P L_P}} \dim_k(\mathcal{O}_{C,P}/(s)) = \sum_{\substack{P \in C \\ s \in \mathfrak{m}_P L_P}} \dim_k(L_P/(s)) = \chi(Q)$$

On the other hand, after tensoring (4.4) with L^m we obtain the following exact sequence

$$0 \to L^m \to L^{m+1} \to Q \to 0$$

using the fact that L is locally free and Q is supported at finitely many closed points. Thus, $\chi(L^{m+1}) - \chi(L^m) = \chi(Q)$. The expression $\chi(L^{m+1}) - \chi(L^m)$ is precisely the leading coefficient of $\chi(L^m)$, which by definition agrees with the intersection number $(L.C)$. □

In particular, if some section $s \in H^0(C, L)$ vanishes at $P \in C$, i.e. $s \in \mathfrak{m}_P L_P$, then

$$(L.C) \geq m_P(C) \tag{4.5}$$

4.2. **Numerical criteria for amplitude.** Let X be a projective scheme over an algebraically closed field k. A Cartier divisor D (or equivalently, its associated line bundle) is *pseudo-ample* if for all integral closed subscheme Y of dimension d, there holds $(D^d.Y) \geq 0$. We have the following two numerical criteria for pseudo-amplitude and amplitude:

Theorem 4.5 (Kleiman's criterion)**.** *Let D be a Cartier divisor on X. Then D is pseudo-ample if and only if $(D.C) \geq 0$ for all integral closed curve C in X.*

Proof. See, for example, [7, Thm. 6.1]. □

For an integral, projective curve C over an algebraically closed field k, we define

$$m(C) = \sup_{P \in C} m_P(C)$$

where $m_P(C)$ is the multiplicity of C at P.

Theorem 4.6 (Seshadri's criterion)**.** *Let D be a Cartier divisor on X. Then D is ample if and only if there exists some $\varepsilon > 0$ such that $(D.C) \geq \varepsilon \cdot m(C)$ for every integral closed curve C in X.*

Proof. See, for example, [7, Thm. 7.1]. □

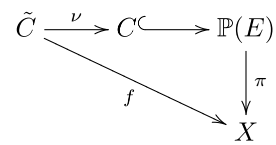
Proposition 4.7. *Let X be a smooth projective curve over an algebraically closed field k of characteristic zero, and E be a semistable vector bundle over X. Then $\deg(E) \geq 0 \implies \mathcal{O}_{\mathbb{P}(E)}(1)$ pseudo-ample.*

Proof. Let C be an integral closed curve in $\mathbb{P}(E)$, and let $\pi : \mathbb{P}(E) \to X$ be the canonical morphism. Then $\pi(C)$ is either a closed point, or the entire curve X. If $\pi(C)$ is a closed point, then C is contained in some fiber F. It follows that

$$(\mathcal{O}_{\mathbb{P}(E)}(1).C) = (\mathcal{O}_{\mathbb{P}(E)}(1)|_F.C) \geq 0$$

since the restriction $\mathcal{O}_{\mathbb{P}(E)}(1)|_F$ is an ample line bundle over F.

Suppose now that $\pi(C) = X$. Let $\tilde{C}$ be the normalization of C, with normalization map $\nu : \tilde{C} \to C$. Let $f : \tilde{C} \to X$ be the composed morphism, fitting into the commutative diagram



and the induced field extension $K(X) \subset K(\tilde{C})$ is separable. The natural surjection $\pi^*E \to \mathcal{O}_{\mathbb{P}(E)}(1)$ (see, for example, [8, Prop. II.7.11(b)]) pulls back to a surjection $f^*E \to \nu^*\mathcal{O}_{\mathbb{P}(E)}(1)$. Hence

$$\deg(\nu^*\mathcal{O}_{\mathbb{P}(E)}(1)) \geq \frac{\deg(f^*E)}{\operatorname{rank}(f^*E)} \geq 0$$

using Prop. 2.6. The following claim will establish the desired inequality $(\mathcal{O}_{\mathbb{P}(E)}(1).C) \geq 0$:

Claim 4.8. *Let L be a line bundle over C. Then $(L.C) = \deg(\nu^*L)$.*

Indeed, there is a sheaf exact sequence

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\tilde{C}} \to Q \to 0$$

where Q is supported on finitely many closed points. Tensoring with line bundle L^m gives

$$0 \to L^m \to L^m \otimes \nu_*\mathcal{O}_{\tilde{C}} \to Q \to 0$$

and it follows that $\chi_C(L^m \otimes \nu_*\mathcal{O}_{\tilde{C}}) = \chi_C(L^m) + H^0(C, Q)$. Since $L^m \otimes \nu_*\mathcal{O}_{\tilde{C}} = \nu_*\nu^*L^m$ by the projection formula, and ν_* is an exact functor on coherent sheaves,

$$H^i(C, L^m \otimes \nu_*\mathcal{O}_{\tilde{C}}) = H^i(C, \nu_*\nu^*L^m) = H^i(\tilde{C}, v^*L^m)$$

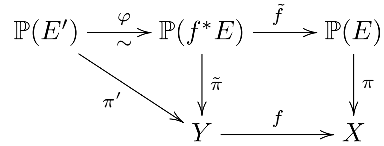
Thus $\chi_{\tilde{C}}(v^*L^m) = \chi_C(L^m) + H^0(C, Q)$, and it follows that the leading coefficients of $\chi_{\tilde{C}}(v^*L^m)$ and $\chi_C(L^m)$ agree. On the other hand, since $\tilde{C}$ is smooth, the Riemann-Roch theorem shows that $\chi_{\tilde{C}}(v^*L^m) = m \cdot \deg(v^*L) + 1 - g_{\tilde{C}}$, and hence the leading coefficient of $\chi_{\tilde{C}}(v^*L^m)$ is precisely $\deg(v^*L)$. This finishes the proof. □

Theorem 4.9. *Let X be a projective curve over an algebraically closed field k of characteristic zero, and E be a semistable vector bundle over X. Then $\deg(E) > 0 \implies E$ ample.*

Proof. Let $f : Y \to X$ be a morphism of smooth projective curves with $\deg(f) \geq \operatorname{rank}(E)/\deg(E)$, and let C be any integral closed curve in $\mathbb{P}(f^*E)$. Suppose $P \in C$ is a closed point with $m(C) = m_P(C)$. In other words, P is a "worst" singular point of C. Let Q be the image of P in Y under the canonical map $\tilde{\pi} : \mathbb{P}(f^*E) \to Y$, and let $E' = f^*E \otimes \mathcal{O}_Y(-Q)$. By the choice of f, there holds

$$\deg(E') = \deg(f)\deg(E) - \operatorname{rank}(E) \geq 0$$

The schemes $\mathbb{P}(E')$, $\mathbb{P}(f^*E)$, and $\mathbb{P}(E)$ fit into the following commutative diagram:



where φ is an isomorphism. Let $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ and $L' = \mathcal{O}_{\mathbb{P}(E')}(1)$; note that $\tilde{f}^*L = \tilde{f}^*\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}(f^*E)}(1)$. Furthermore, suppose $N = \tilde{\pi}^*\mathcal{O}_Y(Q)$, and $N' = (\pi')^*\mathcal{O}_Y(Q)$. Then N admits a global section s which vanishes at P. In particular, it follows from (4.5) that

$$(N.C) \geq m_P(C) = m(C)$$

There is also an isomorphism

$$\varphi^*\tilde{f}^*L \cong L' \otimes N' \tag{4.6}$$

which follows, for example, from [8, Lem. II.7.9].

By Prop. 2.15, f^*E is again semistable. Since tensoring with the invertible sheaf $\mathcal{O}_Y(-Q)$ does not change semistability, E' is still semistable. It follows from Prop. 4.7 that L' is pseudo-ample. We now let C' be the integral, closed curve in $\mathbb{P}(E')$ corresponding to C under the isomorphism φ, and compute

$$(\tilde{f}^*L.C) = (\varphi^*\tilde{f}^*L.C') = (L'.C') + (N'.C') \geq (N'.C') = (N.C) \geq m(C)$$

It follows from Seshadri's criterion that $\tilde{f}^*L$ is ample. Note that $\tilde{f}$ is finite and surjective. Applying Lem. 2.2, we see that L is itself ample. □

Prop. 0.4 is a combination of Cor. 3.7 and the above Thm. 4.9.

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