Stable categories and t-structures

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1 Introduction

1.1 The purpose of this note is to introduce the reader to the modern theory of homological algebra, formulated in the language of higher categories. This approach is actually the third in a historical succession.

The first phase in the development of homological algebra started in the 1940s, although the "prehistory" of the subject stretches back to the 19^{th} century, when people computed kernels and cokernels to prove theorems in algebra and topology without much conceptual framework. During this period chain complexes, various kinds of resolutions, and derived functors were explicitly defined. Grothendieck's 1957 paper "Sur quelques points d'algèbre homologique," referred to as the Tôhoku paper, introduced abelian categories and the Grothendieck spectral sequence associated with a composition of derived functors. By this point most techniques which are necessary for explicit computations, even today, had been introduced.

In the early 1960s Grothendieck and his student Verdier introduced derived categories, the natural domains and targets of derived functors. This marked a transition in the development of homological algebra, after which the subject became noticeably more abstract. In the 1970s Verdier formulated the notion of triangulated category, which simultaneously generalized derived categories of abelian categories and the category of spectra from homotopy theory. Despite the impressive generality of this concept, it is inconvenient for many purposes, owing to shortcomings in the category theory of the time. For example, being triangulated is a structure on a category, as opposed to a condition. Also, the mapping cones axiomatized in the definition of triangulated category are unique up to *non-unique* isomorphism, and in particular are not functorial. This is a major flaw, since in all practical situations mapping cones (a.k.a. cofibers) can be constructed in a functorial way.

The third wave, during which ideas of homotopy theory became dominant, probably started with Quillen's introduction of model categories in 1967. Although popular among algebraic topologists, this theory has never really caught on in pure algebra and algebraic geometry. In the late 1980s differential graded categories, which are more suited to applications in algebra, were introduced. Although (strict) DG categories fix one deficiency of triangulated categories by making cones functorial, they are still based on ordinary (as opposed to higher) categories and as such not suitable for "external" constructions like limits of categories. Drinfeld famously asked "what do DG categories form?"

These issues can be resolved using the theory of ∞-categories, which has been expounded by Lurie in the encyclopedic volumes *Higher Topos Theory* and *Higher Algebra*, referred to below as [HTT] and [HA]. He uses the quasicategory model for ∞-categories, based on simplicial sets. We will work in model-independent manner which does not make use of set-theoretical constructions. This is rigorous as long as we are careful to cite appropriate theorems in *loc. cit.*. Perhaps in the future foundations of mathematics other than set theory will be adopted which make the basic properties of ∞-categories more tautological.

We assume the reader's familiarity with basic definitions and results about ∞-categories insofar as they are straightforward generalizations of ordinary category theory. So we will not define adjoint functors, limits and colimits, etc. To avoid an uncontrollable proliferation of lemniscates, from now on "category" will mean "$(\infty, 1)$-category in the sense of Lurie," as opposed to "$(1, 1)$-category in the sense of Saunders and Mac Lane."

1.2 The replacement for the structure of a triangulated category in higher category theory is the condition of being stable. The latter is really an enhancement of the former, in the sense that the homotopy category of a stable category is naturally triangulated. Also, the fact that being stable is a condition rather than additional structure is a very nice feature of the theory; even the additive structure is automatic. Moreover, in a stable category shifts, mapping cones, exact triangles, etc. have intrinsic meaning.

1.3 In many cases, the derived categories of two non-equivalent abelian categories are equivalent. A related problem is that sometimes a stable or triangulated category is not the derived category of any abelian category, but nonetheless one needs to extract an abelian category from it. A t-structure on a category is an additional datum which makes this possible: the resulting abelian category is called the heart of the t-structure. The "t" stands for truncation, because t-structures axiomatize truncation of complexes and in particular taking their cohomology. Moreover, an exact triple gives rise to a long exact sequence in the abelian category.

2 Stable categories

2.1 An object in a category $\mathscr{C}$ is called a *zero object* if it is both initial and terminal. A zero object is unique up to contractible choice, and if one exists we call $\mathscr{C}$ *pointed*. For example, the category of pointed spaces is pointed. For any objects X and Y in a pointed category there is a *zero morphism*, defined as the composition $X \to 0 \to Y$.

Given a morphism $f : X \to Y$ in a pointed category $\mathscr{C}$, we define the *fiber* $\mathrm{fib}(f)$ and the *cofiber* $\mathrm{cofib}(f)$ to be the limit and colimit, respectively, of the diagram

$$X \underset{0}{\overset{f}{\rightrightarrows}} Y.$$

Of course, in general they may or may not exist. If $\mathscr{C}$ has cofibers there is an endofunctor Σ called *suspension* which is given on objects by

$$\Sigma X = \mathrm{cofib}(X \to 0).$$

Similarly, a category with fibers has a *loop functor* Ω satisfying

$$\Omega X = \mathrm{fib}(0 \to X).$$

Observe that if both Σ and Ω exist then they are mutually adjoint, since by definition

$$\mathrm{Hom}_{\mathscr{C}}(\Sigma X, Y) = \Omega\,\mathrm{Hom}_{\mathscr{C}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, \Omega Y).$$

Example 2.1.1. In the derived category of an abelian category (bounded or unbounded), the shift functor [1] is suspension, so its inverse $[-1]$ is the loop functor.

If we consider the connective subcategory consisting of complexes which vanish in positive cohomological degrees, then Σ is still [1], but Ω is $[-1]$ followed by connective truncation.

Example 2.1.2. The suspension and loop functors in the category of pointed spaces, which inspired the names and notations for the general notions we are discussing, are not equivalences. For example, the suspension of any space is connected, and the loop space only depends on the connected component containing the base point. By definition $\pi_1(X) = \pi_0(\Omega X)$, and more generally $\pi_n(X) = \pi_0(\Omega^n X)$.

The following strange-looking lemma will be used to establish a criterion for a category to be stable.

Lemma 2.1.3. *Let $\mathscr{C}$ be a pointed category which admits fibers and cofibers, and suppose we are given a morphism $f : X \to Y$ in $\mathscr{C}$. Then the unit morphism $X \to \Omega\Sigma X$ canonically factors through the morphism*

$$X \to \mathrm{fib}(Y \to \mathrm{cofib}(f)).$$

Proof. We construct the desired morphism $\mathrm{fib}(Y \to \mathrm{cofib}(f)) \to \Omega\Sigma X$ as follows. The pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

fits into the larger diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{cofib}(f) & \longrightarrow & \Sigma X, \end{array}$$

where the outer square and therefore the square on the right are also pushouts. After replacing X by $\mathrm{fib}(Y \to \mathrm{cofib}(f))$, this defines the needed morphism. We leave it to the interested reader to construct a homotopy between the two maps $X \to \Omega\Sigma X$.

□

2.2 Although it is tangential to our course, we should mention that in a pointed category $\mathscr{C}$ with finite limits, the loop functor lifts to

$$\Omega : \mathscr{C} \longrightarrow \mathrm{Grp}(\mathscr{C}),$$

where $\mathrm{Grp}(\mathscr{C})$ denotes the category of groups in $\mathscr{C}$.

This can be done as follows. First, assuming only that $\mathscr{C}$ admits fiber products, we recall the Čech nerve construction. Fix an object X in $\mathscr{C}$ and consider the undercategory $\mathscr{C}_{X/}$. Then the Čech nerve is a functor

$$\mathscr{C}_{X/} \longrightarrow \mathrm{Grpd}(\mathscr{C}) \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{C}),$$

where $\mathrm{Grpd}(\mathscr{C})$ is the category of groupoids in $\mathscr{C}$ (see [HTT] Section 6.1.2). On the level of objects, this functor sends $X \to Y$ to the simplicial object defined on simplices by

$$[n] \mapsto X \times_Y \cdots \times_Y X,$$

where the fiber product has $n+1$ factors. In particular $[0] \mapsto X$, i.e. the natural target for the Čech nerve is the category $\mathrm{Grpd}_{/X}(\mathscr{C})$ of groupoids in $\mathscr{C}$ over X. The face and degeneracy maps are given by contraction and projection, respectively. For example, composition in the groupoid is given by the contraction

$$X \times_Y X \times_Y X \longrightarrow X \times_Y Y \times_Y X = X \times_Y X.$$

Back to the pointed case: observe that the forgetful functor $\mathscr{C}_{0/} \to \mathscr{C}$ is an equivalence. Then the Čech nerve is a functor

$$\mathscr{C} = \mathscr{C}_{0/} \longrightarrow \mathrm{Grpd}_{/0}(\mathscr{C}) = \mathrm{Grp}(\mathscr{C}).$$

Recall that the forgetful functor $\mathrm{Grp}(\mathscr{C}) \to \mathscr{C}$ is given by evaluation at $[1]$, so the resulting composition

$$\mathscr{C} \longrightarrow \mathrm{Grp}(\mathscr{C}) \longrightarrow \mathscr{C}$$

is $X \mapsto 0 \times_X 0 = \Omega X$ as desired.

Notice that this construction can also be reduced to the category of pointed spaces Spc_* using the Yoneda embedding (although eventually the Čech nerve must be invoked in some form). This is because by definition

$$\mathrm{Hom}_{\mathscr{C}}(X, \Omega Y) = \Omega \,\mathrm{Hom}_{\mathscr{C}}(X, Y).$$

In fact, for $\mathscr{C} = \mathrm{Spc}_*$ we have the following result, which is the prototype for all "delooping" theorems. Define Grp, the category of (homotopical) groups, to be $\mathrm{Grp}(\mathrm{Spc}) = \mathrm{Grp}(\mathrm{Spc}_*)$.

Theorem 2.2.1. *The loop functor*

$$\Omega : \mathrm{Spc}_* \longrightarrow \mathrm{Grp}$$

restricts to an equivalence on the full subcategory of connected pointed spaces.

See Theorem 5.2.6.10 in [HA] for a proof.

The inverse functor is the *classifying space* construction, a.k.a. the bar construction. It is given by

$$\mathrm{Grp} \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Spc}_*) \xrightarrow{\mathrm{colim}} \mathrm{Spc}_*\,.$$

2.3 There are several ways to formulate the condition of being stable. We take as our definition the strongest characterization, and then show that this definition is equivalent to the weakest, or rather most easily verified, version of the condition.

Definition 2.3.1. A category $\mathscr{C}$ is called *stable* if

(i) $\mathscr{C}$ is pointed,

(ii) $\mathscr{C}$ admits finite limits and finite colimits, and

(iii) a square in $\mathscr{C}$ is a pullback if and only if it is a pushout.

Condition (iii) might appear bizarre on a first encounter, but in some sense it encodes the derived version of the first isomorphism theorem. Indeed, it implies that for any morphism $f : X \to Y$ the canonical map

$$X \longrightarrow \operatorname{fib}(Y \to \operatorname{cofib}(f)) \tag{2.3.1}$$

is an isomorphism, and likewise for

$$\operatorname{cofib}(\operatorname{fib}(f) \to X) \to Y. \tag{2.3.2}$$

Proposition 2.3.2. *A pointed category is stable if and only if it admits cofibers and suspension is an equivalence.*

Proof. The "only if" direction is much simpler. Recall that a stable category $\mathscr{C}$ admits all finite colimits by hypothesis, including cofibers. For any object X the pushout square

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma X \end{array}$$

is also a pullback square, meaning $X \to \Omega\Sigma X$ is an isomorphism. Similarly $\Sigma\Omega X \to X$ is an isomorphism.

Conversely, suppose that $\mathscr{C}$ admits cofibers and Σ is an equivalence. Then it follows immediately from Lemma 2.1.3 that 2.3.1 is an isomorphism, whence the commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{cofib}(f) & = & \operatorname{cofib}(f) \end{array}$$

induces a morphism $\Omega \operatorname{cofib}(f) \to X$. Now consider the diagram

$$\begin{array}{ccccc} \Omega \operatorname{cofib}(f) & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & Y & \longrightarrow & \operatorname{cofib}(f). \end{array}$$

Since the outer square and the square on the right are pullbacks, so is the square on the left, i.e. $\Omega \operatorname{cofib}(f) \xrightarrow{\sim} \operatorname{fib}(f)$. In particular $\mathscr{C}$ admits fibers.

Now observe that

$$X \amalg Y \xrightarrow{\sim} \operatorname{cofib}(\Omega X \xrightarrow{0} Y),$$

so $\mathscr{C}$ has finite coproducts. Moreover, we have

$$X \times Y \xrightarrow{\sim} \operatorname{fib}(X \xrightarrow{0} \Sigma Y) \xrightarrow{\sim} \Omega \operatorname{cofib}(X \xrightarrow{0} \Sigma Y) \xrightarrow{\sim} X \amalg Y,$$

which means $\mathscr{C}$ also admits finite products and that these coincide with finite coproducts. We will say $\mathscr{C}$ admits finite *direct sums* and denote them by $X \oplus Y$.

Thus we can speak of the sum of morphisms, given by

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}_{\mathscr{C}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y \oplus Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(X,Y),$$

where the second map is given by composition with the map $Y \oplus Y \to Y$ which is the identity on both summands. This makes $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ into a commutative group in spaces (a.k.a. a connective spectrum) with identity given by the zero map. The inversion map

$$\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}}(X,Y)$$

can be constructed as follows: observe that $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is identified with the space of commutative squares

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma Y. \end{array}$$

The inversion map is given by transposition of such squares.

For any pushout square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & W, \end{array} \tag{2.3.3}$$

we have an isomorphism $\mathrm{cofib}(X \to Y \oplus Z) \to W$, where $X \to Y \oplus Z$ is given by $(f, -g)$. Therefore pushouts exist, so $\mathscr{C}$ admits all finite colimits by Corollary 4.2.4 in [HTT]. The isomorphism (2.3.1) tells us that $X \to \mathrm{fib}(Y \oplus Z \to W)$ is an isomorphism, whence (2.3.3) is a pullback. The same argument carried out in the opposite category $\mathscr{C}^{\mathrm{op}}$ shows that $\mathscr{C}$ admits finite limits and that any pullback square is also a pushout. □

Note that since a category $\mathscr{C}$ is stable if and only if $\mathscr{C}^{\mathrm{op}}$ is, it follows from Proposition 2.3.2 that a pointed category is stable if and only if it admits fibers and the loop functor is an equivalence.

Definition 2.3.3. A functor is called *left exact* if it preserves finite limits, *right exact* if it preserves finite colimits, or *exact* if it is both left and right exact.

Exact functors are the appropriate morphisms of stable categories.

Proposition 2.3.4. *For a functor $F : \mathscr{C} \to \mathscr{D}$ between stable categories, the following are equivalent:*

(i) F is left exact,

(ii) F is right exact,

(iii) F is exact.

Proof. We prove that (i) implies (iii), since it follows by passing to opposite categories that (ii) implies (iii), and the other implications are then tautological. Now in particular F commutes with finite products, i.e. sums, so it suffices to show that F preserves cofibers. Given $f : X \to Y$ in $\mathscr{C}$, the isomorphism (2.3.1) and left exactness of F imply that

$$F(X) \xrightarrow{\sim} \mathrm{fib}(F(Y) \to F(\mathrm{cofib}(f))).$$

But then the isomorphism (2.3.2) for $F(f)$ says that $\mathrm{cofib}(F(f)) \xrightarrow{\sim} F(\mathrm{cofib}(f))$. □

2.4 The following result shows that stable categories are an enhancement of triangulated categories.

Proposition 2.4.1. *The homotopy category of a stable category has a canonical triangulated structure. An exact functor between stable categories is canonically triangulated on the level of homotopy categories.*

Proof. Let $\mathscr{C}$ be stable. The translation (a.k.a. shift) automorphism of $\operatorname{Ho}\mathscr{C}$ is Σ. A distinguished triangle is a square of the form

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z, \end{array} \tag{2.4.1}$$

which is a pushout (equivalently, a pullback). The connecting morphism

$$Z = \operatorname{cofib}(X \to Y) \longrightarrow \operatorname{cofib}(X \to 0) = \Sigma X$$

is already determined by this datum.

The constructions given in the proof of Proposition 2.3.2 show that $\mathscr{C}$ is *additive*, i.e. $\mathscr{C}$ has finite direct sums and the resulting commutative monoid structures on Hom spaces in $\mathscr{C}$ are in fact group structures. It follows that $\operatorname{Ho}\mathscr{C}$ is additive.

Mapping cones in $\operatorname{Ho}\mathscr{C}$ are just cofibers. We explain how to "rotate" distinguished triangles, and refer the reader to [HA], Theorem 1.1.2.15, for the verification of the remaining axioms. The octahedral axiom in particular is a nuisance; it is the first in an infinite chain of compatibilities entailed by the isomorphisms (2.3.1) and (2.3.2).

Suppose we are given a distinguished triangle (2.4.1). It fits into a larger diagram

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & \Sigma X, \end{array}$$

where the outer square and hence the square on the right are pushouts. So the right hand square is the desired rotated distinguished triangle, with connecting morphism $\Sigma X \to \Sigma Y$.

□

3 t-structures

3.1 A t-structure on a stable category is the same as a t-structure on its homotopy category, so here higher categories don't really offer anything new. However, for the reader's convenience we give the definition below, avoiding the notion of triangulated category entirely. Actually our definition is a compact reformulation of the standard one.

Definition 3.1.1. A *t-structure* on a stable category $\mathscr{C}$ is a full subcategory $\mathscr{C}^{\leq 0}$, whose objects are called *connective*, such that

(i) the inclusion $\mathscr{C}^{\leq 0} \to \mathscr{C}$ admits a right adjoint $\tau^{\leq 0}$, and

(ii) the subcategory $\mathscr{C}^{\leq 0}$ is closed under extensions.

Condition (ii) means that if the first and third terms of an exact triple are connective, so is the second. Define the full subcategory $\mathscr{C}^{\geq 1} \subset \mathscr{C}$ as the kernel of $\tau^{\leq 0}$. Then the objects of

$$\mathscr{C}^{\geq 0} := \Sigma \mathscr{C}^{\geq 1}$$

are called *coconnective*. For any $n \in \mathbb{Z}$ define

$$\mathscr{C}^{\leq n} := \Omega^n \mathscr{C}^{\leq 0} \text{ and } \mathscr{C}^{\geq n} := \Omega^n \mathscr{C}^{\geq 0}.$$

Observe that the inclusion $\mathscr{C}^{\leq n} \to \mathscr{C}$ has the right adjoint $\tau^{\leq n} = \Omega^n \tau^{\leq 0} \Sigma^n$.

Proposition 3.1.2. *For any $n \in \mathbb{Z}$ the inclusion $\mathscr{C}^{\geq n} \to \mathscr{C}$ admits a left adjoint $\tau^{\geq n}$. Moreover,*

(i) we have $\mathscr{C}^{\leq 0} \subset \mathscr{C}^{\leq 1}$ and $\mathscr{C}^{\geq 1} \subset \mathscr{C}^{\geq 0}$,

(ii) for any $X \in \mathscr{C}^{\leq 0}$ and $Y \in \mathscr{C}^{\geq 1}$

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) = 0,$$

(iii) for any $X \in \mathscr{C}$, the triple

$$\tau^{\leq 0} X \longrightarrow X \longrightarrow \tau^{\geq 1} X$$

is exact.

Proof. If we show that $\mathscr{C}^{\geq 0} \to \mathscr{C}$ admits a left adjoint the same follows for all $n \in \mathbb{Z}$. Put

$$\tau^{\geq 0} X := \mathrm{cofib}(\tau^{\leq -1} X \to X),$$

which clearly extends to an endofunctor of $\mathscr{C}$. We need to prove that $\tau^{\geq 0} X$ belongs to $\mathscr{C}^{\geq 0}$, from which it follows immediately that $\tau^{\geq 0}$ is the desired left adjoint. It suffices to prove that

$$\pi_0 \mathrm{Hom}_{\mathscr{C}}(Y, \tau^{\geq 0} X) = 0$$

for all $Y \in \mathscr{C}^{\leq -1}$, since then for all $n \geq 0$ we have

$$\pi_n \mathrm{Hom}_{\mathscr{C}}(Y, \tau^{\geq -0} X) = \pi_0 \Omega^n \mathrm{Hom}_{\mathscr{C}}(Y, \tau^{\geq 0} X) = \pi_0 \mathrm{Hom}_{\mathscr{C}}(\Sigma^n Y, \tau^{\geq 0} X) = 0$$

because $\Sigma^n Y \in \mathscr{C}^{\leq -1}$. Now consider the exact triple

$$\tau^{\leq -1} X \longrightarrow X \longrightarrow \tau^{\geq 0} X,$$

which yields

$$\mathrm{Hom}_{\mathscr{C}}(\Omega Y, \tau^{\leq -1} X) = \mathrm{fib}(\mathrm{Hom}_{\mathscr{C}}(\Omega Y, X) \to \mathrm{Hom}_{\mathscr{C}}(\Omega Y, \tau^{\geq 0} X)).$$

The resulting long exact sequence of homotopy groups (all abelian groups here, even π_0) is

$$\begin{aligned} \cdots \longrightarrow \pi_0 \mathrm{Hom}_{\mathscr{C}}(Y, \tau^{\leq -1} X) &\longrightarrow \pi_0 \mathrm{Hom}_{\mathscr{C}}(Y, X) \longrightarrow \pi_0 \mathrm{Hom}_{\mathscr{C}}(Y, \tau^{\geq 0} X) \\ \longrightarrow \pi_0 \mathrm{Hom}_{\mathscr{C}}(\Omega Y, \tau^{\leq -1} X) &\longrightarrow \pi_0 \mathrm{Hom}_{\mathscr{C}}(\Omega Y, X) \longrightarrow \pi_0 \mathrm{Hom}_{\mathscr{C}}(Y, \tau^{\geq 0} X). \end{aligned}$$

The first map is an isomorphism because $Y \in \mathscr{C}^{\leq -1}$, so it suffices to show that

$$\pi_0 \mathrm{Hom}_{\mathscr{C}}(\Omega Y, \tau^{\leq -1} X) \longrightarrow \pi_0 \mathrm{Hom}_{\mathscr{C}}(\Omega Y, X)$$

is injective. For this fix $\Omega Y \to \tau^{\leq -1} X$ and a nullhomotopy of the composition with $\tau^{\leq -1} X \to X$. Thus we obtain a map

$$\mathrm{cofib}(\Omega Y \to \tau^{\leq -1} X) \to X,$$

and since $\mathscr{C}^{\leq -1}$ is closed under extensions this canonically factors through $\tau^{\leq -1} X \to X$. This means we found a retraction of

$$\tau^{\leq -1} X \to \mathrm{cofib}(\Omega Y \to \tau^{\leq -1} X),$$

i.e. a nullhomotopy of $\Omega Y \to \tau^{\leq -1} X$.

Since the inclusions of $\mathscr{C}^{\leq 0}$ and $\mathscr{C}^{\geq 0}$ are left and right adjoints respectively, they preserve colimits and limits respectively, which yields (i). The definition of $\mathscr{C}^{\geq 1}$ immediately implies (ii). Finally, we obtain (iii) from the characterization of $\tau^{\geq 0}$ given above.

□

This proposition shows that a t-structure in our sense gives rise to a t-structure as it usually defined. In fact the converse holds, see Proposition 1.2.1.16 in [HA].

3.2 Given a stable category $\mathscr{C}$ with a t-structure, the full subcategory $\mathscr{C}^\heartsuit = \mathscr{C}^{\leq 0} \cap \mathscr{C}^{\geq 0}$ is called the *heart* of the t-structure. It is not difficult to check that $\mathscr{C}^\heartsuit$ is abelian.

One can show that that $\tau^{\leq m}\tau^{\geq n} \tilde{\to} \tau^{\geq n}\tau^{\leq m}$ for any $m, n \in \mathbb{Z}$. Define the n^{th} *cohomology functor* by

$$H^n := \Sigma^n \tau^{\geq n} \tau^{\leq n} : \mathscr{C} \longrightarrow \mathscr{C}^\heartsuit.$$

Fix an exact triple $X \to Y \to Z$. Since $\tau^{\leq n}$ and $\tau^{\geq n}$ are right and left exact respectively, the sequence $H^n(X) \to H^n(Y) \to H^n(Z)$ is exact in the middle. By rotating the exact triple, we obtain a long exact sequence.

Definition 3.2.1. An exact functor between stable categories with t-structures is called *left t-exact* if it preserves coconnective objects, *right t-exact* if it preserves coconnective objects, or *t-exact* if it is both left and right t-exact.

In particular, a t-exact functor $F : \mathscr{C} \to \mathscr{D}$ determines a functor $\mathscr{C}^\heartsuit \to \mathscr{D}^\heartsuit$.

Example 3.2.2. Let $\mathscr{A}$ be a Grothendieck abelian category, i.e. a cocomplete abelian category in which filtered colimits are exact. Then, as in Section 1.3.5 of [HA], one constructs the (unbounded) derived category $D(\mathscr{A})$, which comes with a canonical t-structure defined by truncation of complexes. We have $D(\mathscr{A})^\heartsuit = \mathscr{A}$. If $F : \mathscr{A} \to \mathscr{B}$ is a left, respectively right exact functor between Grothendieck abelian categories which preserves filtered colimits, the resulting derived functor $D(\mathscr{A}) \to D(\mathscr{B})$ is left, respectively right t-exact.

Example 3.2.3. Let A be a DG algebra or commutative DG algebra, i.e. an associative or commutative algebra object in the derived category Vect of vector spaces. Then in general the category A-mod of left A-modules does not inherit a t-structure from Vect. Suppose we define $A\text{-mod}^{\leq 0}$ to be the full subcategory consisting of modules whose underlying vector space is connective. This subcategory is always closed under extensions, but $A\text{-mod}^{\leq 0}$ admits a right adjoint if and only if A is connective, meaning the vector space underlying A is connective. In that case the homomorphism $A \to H^0(A)$ gives rise to an equivalence

$$H^0(A)\text{-mod}^\heartsuit \tilde{\longrightarrow} A\text{-mod}^\heartsuit.$$

In particular, the canonical functor

$$D(A\text{-mod}^\heartsuit) \longrightarrow A\text{-mod}$$

is an equivalence if and only if A is classical, i.e. $A \tilde{\to} H^0(A)$.