IndCoh Seminar: Operads and Koszul duality

Justin Campbell

April 22, 2016

1 Operads

1.1 Fix a field k of characteristic zero. The DG category Vect^Σ of *symmetric sequences* is defined to be $\prod_{n\geq 1}\mathrm{Rep}(\Sigma_n)$ where Σ_n is the symmetric group on n letters.

Define Σ to be the category $\coprod_{n\geq 1} B\Sigma_n$, so we can identify $\mathrm{Fun}(\Sigma,\mathrm{Vect})\tilde{\to}\mathrm{Vect}^\Sigma$. In particular we have the Yoneda embedding

$$\Sigma \xrightarrow{\sim} \Sigma^{\mathrm{op}} \longrightarrow \mathrm{Vect}^\Sigma,$$

where the first functor is inversion of morphisms.

Observe that Σ is equivalent to the category of nonempty finite sets and bijections, and in particular has the cocartesian (nonunital) symmetric monoidal structure. Moreover, Σ is the free symmetric monoidal category on one object, meaning that for any symmetric monoidal category $\mathscr{M}$ restriction to $B\Sigma_1 = \mathrm{pt}$ determines an equivalence

$$\mathrm{Fun}^\otimes(\Sigma,\mathscr{M})\xrightarrow{\sim}\mathscr{M},$$

where $\mathrm{Fun}^\otimes$ means symmetric monoidal functors.

1.2 For us, a symmetric monoidal DG category is a commutative algebra object in the symmetric monoidal category of DG categories under Lurie's tensor product. In particular the tensor product preserves colimits in both variables.

Observe that Vect^Σ inherits a nonunital symmetric monoidal structure from Σ called *Day convolution*, defined on objects by the formula

$$(\mathscr{P}\odot\mathscr{Q})(n) = \bigoplus_{i+j=n}\mathrm{Ind}_{S_i\times S_j}^{S_n}(\mathscr{P}(i)\otimes\mathscr{Q}(j)).$$

The Yoneda embedding $\Sigma\to\mathrm{Vect}^\Sigma$ realizes Vect^Σ as the free symmetric monoidal DG category on one object, because restriction defines an equivalence

$$\mathrm{Fun}^\otimes_{\mathrm{DG}}(\mathrm{Vect}^\Sigma,\mathscr{C})\xrightarrow{\sim}\mathrm{Fun}^\otimes(\Sigma,\mathscr{C})\xrightarrow{\sim}\mathscr{C} \tag{1.2.1}$$

(the inverse to the first functor is given by left Kan extension).

In particular we have a canonical equivalence

$$\mathrm{End}^\otimes_{\mathrm{DG}}(\mathrm{Vect}^\Sigma)\xrightarrow{\sim}\mathrm{Vect}^\Sigma,$$

and hence a (non-symmetric) monoidal structure on Vect^Σ given by reversed composition, which on objects is determined by the formula

$$\mathscr{P}\star\mathscr{Q} = \bigoplus_{n\geq 1}(\mathscr{P}(n)\otimes\mathscr{Q}^{\odot n})_{\Sigma_n}.$$

Because of the equivalence (1.2.1), any symmetric monoidal DG category is a module for Vect^Σ with its $\star$-monoidal structure (the reason we chose the reversed composition), and the action is given on objects by

$$\mathscr{P}\star X = \bigoplus_{n\geq 1}(\mathscr{P}(n)\otimes X^{\otimes n})_{\Sigma_n}.$$

1.3 Having dispensed with some formalities, we can now define the main objects of interest.

Definition 1.3.1. The category Oprd of *(reduced k-linear) operads* is the full subcategory of associative $\star$-algebras $\mathscr{P}$ in Vect^{Σ} such that the unit map $k \to \mathscr{P}(1)$ is an isomorphism.

This is a rather restrictive definition of operad which only allows us to parameterize augmented, or equivalently nonunital objects. Note that our assumption on an operad $\mathscr{P}$ implies that it is canonically augmented as an associative algebra in Vect^{Σ}. In particular the unit $\mathbb{1}_\star$ for the $\star$-monoidal structure, which satisfies $\mathbb{1}_\star(1) = k$ and $\mathbb{1}_\star(n) = 0$ for all $n > 1$, is also the zero object in Oprd.

Concretely, the structure of an operad on a symmetric sequence $\mathscr{P}$ consists of *composition maps*

$$\mathscr{P}(\ell) \otimes \mathscr{P}(m_1) \otimes \cdots \otimes \mathscr{P}(m_\ell) \longrightarrow \mathscr{P}(m_1 + \cdots + m_\ell) \tag{1.3.1}$$

for all $\ell \geq 1$, $m_1, \cdots, m_\ell \geq 1$, together with a unit element in $\mathscr{P}(1)$, satisfying associativity, unitality, and Σ-equivariance conditions. One thinks of the elements of $\mathscr{P}(n)$ as n-ary operations.

1.4 For an operad $\mathscr{P}$ and a symmetric monoidal DG category $\mathscr{C}$ we can therefore speak of the category of $\mathscr{P}$-modules in $\mathscr{C}$, which is usually called the category of *$\mathscr{P}$-algebras in $\mathscr{C}$* and denoted by $\mathscr{P}\text{-alg}(\mathscr{C})$ for reasons which will become clear.

In this situation there is a conservative forgetful functor

$$\mathrm{oblv}_{\mathscr{P}} : \mathscr{P}\text{-alg}(\mathscr{C}) \longrightarrow \mathscr{C}$$

which preserves limits, so by the adjoint functor theorem $\mathrm{oblv}_{\mathscr{P}}$ admits a left adjoint

$$\mathrm{free}_{\mathscr{P}} : \mathscr{C} \longrightarrow \mathscr{P}\text{-alg}(\mathscr{C}).$$

In fact, this adjunction is monadic and the monad $\mathrm{oblv}_{\mathscr{P}} \circ \mathrm{free}_{\mathscr{P}}$ identifies with $X \mapsto \mathscr{P} \star X$. Said differently, we can also view $\mathscr{P}\text{-alg}(\mathscr{C})$ as the category of modules in $\mathscr{C}$ over the algebra $\mathrm{oblv}_{\mathscr{P}} \circ \mathrm{free}_{\mathscr{P}}$ in $\mathrm{End}(\mathscr{C})$, and this algebra is the image of $\mathscr{P}$ under the functor

$$\mathrm{Oprd} \subset \mathrm{AssocAlg}(\mathrm{End}^{\otimes}_{\mathrm{DG}}(\mathscr{C})) \longrightarrow \mathrm{AssocAlg}(\mathrm{End}(\mathscr{C})).$$

1.5 We will study four examples of operads. Three of them are easy to describe explicitly.

Example 1.5.1. The *trivial operad* Triv has the underlying symmetric sequence $\mathbb{1}_\star$, which has a canonical operad structure: the multiplication is determined by the tautological isomorphism

$$\mathrm{Triv} \star \mathrm{Triv} \xrightarrow{\sim} \mathrm{Triv}\,.$$

The functor $\mathrm{oblv}_{\mathrm{Triv}}$ is an equivalence $\mathrm{Triv}\text{-alg}(\mathscr{C}) \tilde{\to} \mathscr{C}$.

Example 1.5.2. The *associative operad* Assoc is defined so that $\mathrm{Assoc}(n)$ is the space of degree n monomials in n non-commuting variables $x_1, \cdots, x_n$, with Σ_n acting by permuting the variables. In particular $\mathscr{P}(n)$ is isomorphic to the regular representation of Σ_n. The composition maps (1.3.1) are given by substitution, i.e. if $f \in \mathscr{P}(\ell)$ and $g_i \in \mathscr{P}(m_i)$ for $1 \leq i \leq \ell$, then

$$(f, g_1, \cdots, g_\ell) \mapsto f(g_1, \cdots, g_\ell) \in \mathscr{P}(m_1 + \cdots + m_\ell).$$

There is a canonical equivalence of Assoc-algebras in a symmetric monoidal category $\mathscr{C}$ with augmented associative algebras in $\mathscr{C}$.

Example 1.5.3. The *commutative operad* $\mathrm{Com} \subset \mathrm{Assoc}$ is defined by $\mathrm{Com}(n) = k \cdot x_1 \cdots x_n$ where $x_1, \cdots, x_n$ are commuting variables, i.e. $\mathrm{Com}(n)$ is the trivial representation of Σ_n, with the operad structure inherited from Assoc. The category of Com-algebras in a symmetric monoidal category $\mathscr{C}$ is the category of augmented commutative algebras in $\mathscr{C}$.

1.6 One can also define operads by generators and relations, as follows. The forgetful functor

$$\mathrm{oblv}_{\mathrm{Oprd}} : \mathrm{Oprd} \subset \mathrm{AssocAlg}(\mathrm{Vect}^{\Sigma}) \longrightarrow \mathrm{Vect}^{\Sigma}$$

admits a left adjoint $\mathrm{free}_{\mathrm{Oprd}}$. Moreover, for any symmetric sequence $\mathscr{P}$ the free operad on $\mathscr{P}$ has the typical presentation as a "tensor algebra:"

$$\mathrm{free}_{\mathrm{Oprd}}(\mathscr{P}) = \bigoplus_{n\geq 1} \mathscr{P}^{\star n}.$$

Suppose we are given symmetric sequences $\mathscr{G}$ and $\mathscr{R}$ together with a morphism $\mathscr{R} \to \mathrm{oblv}_{\mathrm{Oprd}}(\mathrm{free}_{\mathrm{Oprd}}(\mathscr{G}))$. Then the operad on generators $\mathscr{G}$ with relations $\mathscr{R} \to \mathrm{oblv}_{\mathrm{Oprd}}(\mathrm{free}_{\mathrm{Oprd}}(\mathscr{G}))$ is defined to be the cofiber in Oprd of

$$\mathrm{free}_{\mathrm{Oprd}}(\mathscr{R}) \longrightarrow \mathrm{free}_{\mathrm{Oprd}}(\mathscr{G}).$$

Example 1.6.1. With notation as above, the *Lie operad* Lie is the operad on generators $\mathscr{G}$ given by $\mathscr{G}(2) = k \cdot [x_1, x_2]$ and $\mathscr{G}(n) = 0$ for $n \neq 2$, with relations $\mathscr{R}$ given by $\mathscr{R}(2) = k \cdot ([x_1, x_2] + [x_2, x_1])$,

$$\mathscr{R}(3) = k \cdot ([x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]]),$$

and $\mathscr{R}(n) = 0$ for $n \neq 2, 3$. Concretely, $\mathrm{Lie}(n) \subset \mathrm{Assoc}(n)$ consists of bracket monomials in n variables, where each variable appears exactly once. This vector space has dimension $(n-1)!$, and moreover if k is algebraically closed then

$$\mathrm{Lie}(n) \xrightarrow{\sim} \mathrm{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{\Sigma_n} k_{\zeta_n}$$

for a fixed embedding $\mathbb{Z}/n\mathbb{Z} \to \Sigma_n$ and primitive n^{th} root of unity ζ_n by which $\mathbb{Z}/n\mathbb{Z}$ acts on k.

One can present Assoc as the operad generated by one binary operation with associativity the single ternary relation. To obtain Com one also imposes the binary relation of commutativity.

1.7 Fix an operad $\mathscr{P}$. We will need a couple of facts about group objects in $\mathscr{P}\text{-alg}(\mathscr{C})$. Recall that in any category with products, a *group object* is a monoid object G with multiplication $m : G \times G \to G$ with the property that $p_1 \times m : G \times G \to G \times G$ is an isomorphism, where p_1 is the first projection.

Proposition 1.7.1. *The inclusion* $\mathrm{Grp}(\mathscr{P}\text{-alg}(\mathscr{C})) \to \mathrm{Mon}(\mathscr{P}\text{-alg}(\mathscr{C}))$ *is an equivalence.*

The loops functor

$$\Omega_{\mathscr{P}} : \mathscr{P}\text{-alg}(\mathscr{C}) \longrightarrow \mathrm{Grp}(\mathscr{P}\text{-alg}(\mathscr{C}))$$

preserves limits and therefore admits a left adjoint $B_{\mathscr{P}}$. Concretely, $B_{\mathscr{P}}$ is given by the bar construction

$$B_{\mathscr{P}}(G) = \mathrm{colim}(\cdots G \times G \substack{\to\\[-1em]\to\\[-1em]\to} G \rightrightarrows 1).$$

Proposition 1.7.2. *The functors* $\Omega_{\mathscr{P}}$ *and* $B_{\mathscr{P}}$ *are mutually inverse equivalences.*

2 Co-operads

2.1 Recall that for any monoidal category $\mathscr{C}$, one defines

$$\mathrm{CoassocCoalg}(\mathscr{C}) := \mathrm{AssocAlg}(\mathscr{C}^{\mathrm{op}})^{\mathrm{op}}.$$

Definition 2.1.1. The category Cooprd of (*reduced* k*-linear*) *co-operads* is the full subcategory of $\mathrm{CoassocCoalg}(\mathrm{Vect}^{\Sigma})$ consisting of coalgebras $\mathscr{Q}$ for which the counit map $\mathscr{Q}(1) \to k$ is an isomorphism.

In particular, our co-operads are canonically coaugmented.

2.2 One way to construct co-operads is as follows. Denote by $\mathrm{Vect}^{\Sigma}_{\mathrm{fd}} \subset \mathrm{Vect}^{\Sigma}$ the full subcategory consisting of symmetric sequences $\mathscr{P}$ such that, for all $n \geq 1$, the complex $\mathscr{P}(n)$ has finite-dimensional cohomologies in all degrees.

Proposition 2.2.1. *Termwise dualization $\mathscr{P} \mapsto \mathscr{P}^*$ is a monoidal anti-equivalence* $(\mathrm{Vect}^{\Sigma}_{\mathrm{fd}})^{\mathrm{op}} \tilde{\to} \mathrm{Vect}^{\Sigma}_{\mathrm{fd}}$.

In particular we obtain an anti-equivalence between operads and co-operads whose underlying symmetric sequence belongs to $\mathrm{Vect}^{\Sigma}_{\mathrm{fd}}$.

Example 2.2.2. The *coassociative*, *cocommutative*, and *co-Lie co-operads* are defined by $\mathrm{Coassoc} := \mathrm{Assoc}^*$, $\mathrm{Cocom} := \mathrm{Com}^*$, and $\mathrm{CoLie} := \mathrm{Lie}^*$ respectively.

2.3 Let $\mathscr{C}$ be a symmetric monoidal DG category. There are two kinds of coalgebras in $\mathscr{C}$ for a given co-operad $\mathscr{Q}$ (in positive characteristic there are four!). The category $\mathscr{Q}\text{-coalg}^{\text{ind-nilp}}(\mathscr{C})$ of *ind-nilpotent $\mathscr{Q}$-coalgebras in $\mathscr{C}$* is by definition the category of $\mathscr{Q}$-comodules in $\mathscr{C}$ with respect to the $\star$-action of Vect^{Σ} on $\mathscr{C}$. In this context we have the comonadic adjunction

$$\mathrm{oblv}^{\text{ind-nilp}}_{\mathscr{Q}} : \mathscr{Q}\text{-coalg}^{\text{ind-nilp}}(\mathscr{C}) \rightleftarrows \mathscr{C} : \mathrm{cofree}^{\text{ind-nilp}}_{\mathscr{Q}}, \tag{2.3.1}$$

and the comonad $\mathrm{oblv}^{\text{ind-nilp}}_{\mathscr{Q}} \circ \mathrm{cofree}^{\text{ind-nilp}}_{\mathscr{Q}}$ identifies with $X \mapsto \mathscr{Q} \star X$.

2.4 The notion of ind-nilpotent coalgebra over a co-operad $\mathscr{Q}$ does not specialize to the usual notion of coalgebra when $\mathscr{Q}$ is e.g. Coassoc or Cocom. For this one uses instead the lax action of Vect^{Σ} on $\mathscr{C}$ given on objects by

$$\mathscr{P} * X = \prod_{n \geq 1} (\mathscr{P}(n) \otimes X^{\otimes n})^{\Sigma_n}.$$

The category $\mathscr{Q}\text{-coalg}(\mathscr{C})$ of (usual) *$\mathscr{Q}$-coalgebras in $\mathscr{C}$* is defined to be the category of $\mathscr{Q}$-comodules in $\mathscr{C}$ with respect to the $*$-action.

This category comes with a forgetful functor

$$\mathrm{oblv}_{\mathscr{Q}} : \mathscr{Q}\text{-coalg}(\mathscr{C}) \longrightarrow \mathscr{C},$$

which is conservative and preserves colimits.

For example, $\mathrm{Coassoc}\text{-coalg}(\mathscr{C})$ is the category of coaugmented coassociative coalgebras in $\mathscr{C}$, and $\mathrm{Cocom}\text{-coalg}(\mathscr{C})$ is the category of coaugmented cocommutative coalgebras in $\mathscr{C}$.

There is a morphism $\mathscr{P} \star X \to \mathscr{P} * X$, functorial in $\mathscr{P}$ and X, which gives rise to a functor

$$\mathrm{res}^{\star \to *} : \mathscr{Q}\text{-coalg}^{\text{ind-nilp}}(\mathscr{C}) \longrightarrow \mathscr{Q}\text{-coalg}(\mathscr{C}).$$

Moreover $\mathrm{oblv}_{\mathscr{Q}} \circ \mathrm{res}^{\star \to *} = \mathrm{oblv}^{\text{ind-nilp}}_{\mathscr{Q}}$, whence $\mathrm{res}^{\star \to *}$ preserves colimits.

Conjecture 2.4.1. *For any $\mathscr{Q}$ and $\mathscr{C}$, the functor $\mathrm{res}^{\star \to *}$ is fully faithful.*

3 The bar construction for associative algebras

3.1 In this section $\mathscr{A}$ is a monoidal category with limits and colimits, and we assume that the unit $\mathbb{1}$ is the zero object. For example, the category of pointed spaces with its cartesian monoidal structure has this property. More generally, if $\mathscr{A}'$ is any monoidal category with limits and colimits then take $\mathscr{A} = \mathscr{A}'_{\mathbb{1}//\mathbb{1}}$. Then $\mathrm{AssocAlg}(\mathscr{A})$ is equivalent to the category of augmented associative algebras in $\mathscr{A}'$, and in particular has a zero object.

The goal of this section is to produce a pair of adjoint functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}(\mathscr{A}) \rightleftarrows \mathrm{CoassocCoalg}(\mathscr{A}) : \mathrm{Cobar}^{\mathrm{enh}}. \tag{3.1.1}$$

3.2 The first step is to define the non-enhanced versions of the bar and cobar constructions. Because $\mathbb{1}$ is final in $\mathscr{A}$, there is a functor

$$\mathrm{triv}_{\mathrm{AssocAlg}} : \mathscr{A} \longrightarrow \mathrm{AssocAlg}(\mathscr{A})$$

right inverse to the forgetful functor $\mathrm{oblv}_{\mathrm{AssocAlg}}$, which takes the trivial algebra. Since $\mathrm{oblv}_{\mathrm{AssocAlg}}$ is conservative and limit preserving, it follows that $\mathrm{triv}_{\mathrm{AssocAlg}}$ preserves limits. By definition

$$\mathrm{Bar} : \mathrm{AssocAlg}(\mathscr{A}) \longrightarrow \mathscr{A}$$

is left adjoint to the limit-preserving functor $\mathrm{triv}_{\mathrm{AssocAlg}} \circ \Omega$.

A little more concretely: recall that there is a canonically defined functor

$$\mathrm{AssocAlg}(\mathscr{A}) \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{A}),$$

which sends A to the simplicial object

$$\mathbb{1} \rightrightarrows A \overset{\rightarrow}{\rightrightarrows} A \otimes A \cdots .$$

Proposition 3.2.1. *The functor* Bar *is the composition*

$$\mathrm{AssocAlg}(\mathscr{A}) \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{A}) \overset{\mathrm{colim}}{\longrightarrow} \mathscr{A}.$$

Dually, we have

$$\mathrm{Cobar} : \mathrm{CoassocCoalg}(\mathscr{A}) \longrightarrow \mathscr{A},$$

which is right adjoint to $\mathrm{triv}_{\mathrm{CoassocCoalg}} \circ \Sigma$ and can be computed as the composition

$$\mathrm{CoassocCoalg}(\mathscr{A}) \longrightarrow \mathrm{Fun}(\Delta, \mathscr{A}) \overset{\lim}{\longrightarrow} \mathscr{A}.$$

3.3 Now we construct the "enhanced" versions of Bar and Cobar. In order to reduce to known constructions from [HA] we work with Cobar. The idea is that for any coalgebra C, the category $C\text{-comod}^{\mathrm{r}}$ of right C-comodules is left tensored over $\mathscr{A}$, i.e. it has the stucture of a $\mathscr{A}$-module category. Then we will have

$$\mathrm{Cobar}^{\mathrm{enh}}(C) = \underline{\mathrm{End}}_{\mathscr{A}, C\text{-comod}^{\mathrm{r}}}(\mathbb{1}),$$

the algebra of inner endomorphisms relative to $\mathscr{A}$ of the coaugmentation comodule.

Proposition 3.3.1. *This construction extends to a functor*

$$\mathrm{Cobar}^{\mathrm{enh}} : \mathrm{CoassocCoalg}(\mathscr{A}) \longrightarrow \mathrm{AssocAlg}(\mathscr{A}),$$

and there is a canonical isomorphism $\mathrm{oblv}_{\mathrm{AssocAlg}} \circ \mathrm{Cobar}^{\mathrm{enh}} \tilde{\to} \mathrm{Cobar}$.

Proof. First, consider the functor

$$\mathrm{CoassocCoalg}(\mathscr{A}) \longrightarrow \mathrm{Cat}_{\bullet}$$

valued in pointed categories which, on objects, sends a coalgebra C to the category $C\text{-comod}^{\mathrm{r}}$ with the coaugmentation comodule as distinguished object. This canonically lifts to take values in pointed complete $\mathscr{A}$-module categories with limit-preserving ($\mathscr{A}$-linear) functors:

$$\mathrm{CoassocCoalg}(\mathscr{A}) \longrightarrow \mathscr{A}\text{-mod}_{\bullet}^{\mathrm{cmpl}}.$$

Now compose with the functor

$$\mathscr{A}\text{-mod}_{\bullet}^{\mathrm{cmpl}} \longrightarrow \mathrm{AssocAlg}(\mathscr{A})$$

which takes inner endomorphisms relative to $\mathscr{A}$ of the distinguished object. The composition is $\mathrm{Cobar}^{\mathrm{enh}}$.

To show that $\mathrm{oblv}_{\mathrm{AssocAlg}} \circ \mathrm{Cobar}^{\mathrm{enh}} \tilde{\to} \mathrm{Cobar}$, recall that any right C-comodule M has a natural (i.e. functorial) resolution

$$M = \lim(M \otimes C \rightrightarrows M \otimes C \otimes C \overset{\rightarrow}{\rightrightarrows} \cdots).$$

Now apply the limit-preserving functor $M \mapsto \underline{\mathrm{Hom}}_{\mathscr{A}, C\text{-comod}^{\mathrm{r}}}(\mathbb{1}, M)$ and put $M = \mathbb{1}$ to obtain

$$\begin{aligned}\mathrm{oblv}_{\mathrm{AssocAlg}} \underline{\mathrm{End}}_{\mathscr{A}, C\text{-comod}^{\mathrm{r}}}(\mathbb{1}) &= \lim(\underline{\mathrm{Hom}}_{\mathscr{A}, C\text{-comod}^{\mathrm{r}}}(\mathbb{1}, C) \rightrightarrows \underline{\mathrm{Hom}}_{\mathscr{A}, C\text{-comod}^{\mathrm{r}}}(\mathbb{1}, C \otimes C) \overset{\rightarrow}{\rightrightarrows} \cdots) \\ &= \lim(\underline{\mathrm{Hom}}_{\mathscr{A}}(\mathbb{1}, \mathbb{1}) \rightrightarrows \underline{\mathrm{Hom}}_{\mathscr{A}}(\mathbb{1}, C) \overset{\rightarrow}{\rightrightarrows} \cdots) = \mathrm{Cobar}(C).\end{aligned}$$

□

Dually, we obtain a functor

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{AssocAlg}(\mathscr{A}) \longrightarrow \mathrm{CoassocCoalg}(\mathscr{A})$$

equipped with an isomorphism $\mathrm{oblv}_{\mathrm{CoassocCoalg}} \circ \mathrm{Bar}^{\mathrm{enh}} \xrightarrow{\sim} \mathrm{Bar}$.

3.4 Finally, we construction the adjunction (3.1.1). Recall that $\mathrm{oblv}_{\mathrm{AssocAlg}}$ preserves limits and hence admits a left ajoint $\mathrm{free}_{\mathrm{AssocAlg}}$. Moreover, the adjunction is monadic, and in particular we have an isomorphism

$$\mathrm{id}_{\mathrm{AssocAlg}(\mathscr{A})} \xrightarrow{\sim} \mathrm{colim}(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathrm{free} \circ \mathrm{oblv} \circ \mathrm{free} \circ \mathrm{oblv} \rightrightarrows \mathrm{free} \circ \mathrm{oblv}), \tag{3.4.1}$$

where here and for the rest of the section we omit some subscripts.

Dually, $\mathrm{oblv}_{\mathrm{CoassocCoalg}}$ admits a right adjoint $\mathrm{cofree}_{\mathrm{CoassocCoalg}}$ and the adjunction is comonadic, so we have an isomorphism

$$\mathrm{id}_{\mathrm{CoassocCoalg}(\mathscr{A})} \xrightarrow{\sim} \lim(\mathrm{cofree} \circ \mathrm{oblv} \rightrightarrows \mathrm{cofree} \circ \mathrm{oblv} \circ \mathrm{cofree} \circ \mathrm{oblv} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots). \tag{3.4.2}$$

Proposition 3.4.1. *The functors* $\mathrm{Bar}^{\mathrm{enh}}$ *and* $\mathrm{Cobar}^{\mathrm{enh}}$ *are adjoint.*

Proof. We need to produce an isomorphism

$$\mathrm{Hom}_{\mathrm{CoassocCoalg}(\mathscr{A})} \circ ((\mathrm{Bar}^{\mathrm{enh}})^{\mathrm{op}} \times \mathrm{id}_{\mathrm{CoassocCoalg}(\mathscr{A})}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{AssocAlg}(\mathscr{A})} \circ (\mathrm{id}_{\mathrm{AssocAlg}(\mathscr{A})} \times \mathrm{Cobar}^{\mathrm{enh}}).$$

Since both functors preserve limits, it follows from the isomorphisms () and () that we need only find an isomorphism between the two functors' compositions with

$$\mathrm{free}^{\mathrm{op}} \times \mathrm{cofree} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \longrightarrow \mathrm{AssocAlg}(\mathscr{A})^{\mathrm{op}} \times \mathrm{CoassocCoalg}(\mathscr{A}).$$

Now, computing object-wise for ease of notation, we have for any $X, Y \in \mathscr{A}$

$$\begin{aligned}\mathrm{Hom}_{\mathrm{CoassocCoalg}(\mathscr{A})}(\mathrm{Bar}^{\mathrm{enh}}(\mathrm{free}(X)), \mathrm{cofree}(Y)) &= \mathrm{Hom}_{\mathscr{A}}(\mathrm{Bar}(\mathrm{free}(X)), Y)\\ &= \mathrm{Hom}_{\mathrm{CoassocCoalg}(\mathscr{A})}(\mathrm{free}(X), \mathrm{triv}(\Omega Y))\\ &= \mathrm{Hom}_{\mathscr{A}}(X, \Omega Y).\end{aligned}$$

But then the opposite calculation shows that

$$\mathrm{Hom}_{\mathrm{AssocAlg}(\mathscr{A})}(\mathrm{free}(X), \mathrm{Cobar}^{\mathrm{enh}}(\mathrm{cofree}(Y))) = \mathrm{Hom}_{\mathscr{A}}(\Sigma X, Y).$$

□

3.5 We will also need the bar construction for modules. Let $\mathscr{C}$ be an $\mathscr{A}$-module category which admits limits and colimits and fix an augmented associative algebra A in $\mathscr{A}$. The augmentation gives rise to a functor

$$\mathrm{triv}_A : \mathscr{C} \longrightarrow A\text{-mod}(\mathscr{C}),$$

which is right inverse to oblv_A and therefore preserves limits. Its left adjoint is denoted by Bar_A.

Alternatively, recall that the augmentation gives rise to a functor

$$A\text{-mod}(\mathscr{C}) \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{C}).$$

Proposition 3.5.1. *The bar construction can be computed as the geometric realization*

$$\mathrm{Bar}_A(M) = \mathrm{colim}(\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} A \otimes M \rightrightarrows M).$$

If the monoidal structure on $\mathscr{A}$ *and the module structure on* $\mathscr{M}$ *are compatible with colimits, then the comonad* $\mathrm{Bar}_A \circ \mathrm{triv}_A$ *identifies with* $M \mapsto \mathrm{Bar}^{\mathrm{enh}}(A) \otimes M$.

4 Koszul duality

4.1 Specializing (3.1.1) to the case of Vect^{Σ}, one checks that the reducedness condition is preserved by Bar and Cobar, so we obtain adjoint functors

$$\mathrm{Bar}^{\mathrm{enh}} : \mathrm{Oprd} \rightleftarrows \mathrm{Cooprd} : \mathrm{Cobar}^{\mathrm{enh}}\,. \tag{4.1.1}$$

This is (covariant) Koszul duality for operads.

Proposition 4.1.1. *The functors in (4.1.1) are mutually inverse equivalences.*

From now on, for any operad $\mathscr{P}$ we will write

$$\mathscr{P}^{\vee} := \mathrm{Bar}^{\mathrm{enh}}(\mathscr{P}),$$

and similarly for any co-operad we put

$$\mathscr{Q}^{\vee} := \mathrm{Cobar}^{\mathrm{enh}}(\mathscr{Q}).$$

Theorem 4.1.2. *We have canonical identifications* $\mathrm{Assoc}^{\vee} = \Sigma\,\mathrm{Coassoc}$ *and* $\mathrm{Cocom}^{\vee} = \Omega\,\mathrm{Lie}$.

4.2 Fix an operad $\mathscr{P}$ and a symmetric monoidal DG category $\mathscr{C}$. The functor

$$\mathrm{triv}_{\mathscr{P}} : \mathscr{C} \longrightarrow \mathscr{P}\text{-alg}(\mathscr{C}),$$

obtained by restriction along the augmentation $\mathscr{P} \to \mathrm{Triv}$, is right inverse to $\mathrm{oblv}_{\mathscr{P}}$. Since $\mathrm{oblv}_{\mathscr{P}}$ is conservative and preserves limits, it follows that $\mathrm{triv}_{\mathscr{P}}$ also preserves limits and hence admits a left adjoint $\mathrm{coprim}_{\mathscr{P}}$.

In the notation of Section 3.5, we have $\mathrm{coprim}_{\mathscr{P}} = \mathrm{Bar}_{\mathscr{P}}$. Proposition 3.5.1 implies that the comonad $\mathrm{coprim}_{\mathscr{P}} \circ \mathrm{triv}_{\mathscr{P}}$ identifies with $X \mapsto \mathscr{P}^{\vee} \star X$. Thus $\mathrm{coprim}_{\mathscr{P}}$ lifts to

$$\mathrm{coprim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{P}} : \mathscr{P}\text{-alg}(\mathscr{C}) \longrightarrow \mathscr{P}^{\vee}\text{-coalg}^{\mathrm{ind\text{-}nilp}}(\mathscr{C}),$$

i.e. $\mathrm{oblv}^{\mathrm{ind\text{-}nilp}}_{\mathscr{P}^{\vee}} \circ \mathrm{coprim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{P}} = \mathrm{coprim}_{\mathscr{P}}$. Moreover, we have $\mathrm{coprim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{P}} \circ \mathrm{triv}_{\mathscr{P}} = \mathrm{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathscr{P}^{\vee}}$ and $\mathrm{coprim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{P}} \circ \mathrm{free}_{\mathscr{P}} = \mathrm{triv}^{\mathrm{ind\text{-}nilp}}_{\mathscr{P}^{\vee}}$.

We denote the composition $\mathrm{res}^{\star \to \circ} \circ \mathrm{coprim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{P}}$ by

$$\mathrm{coprim}^{\mathrm{enh}}_{\mathscr{P}} : \mathscr{P}\text{-alg}(\mathscr{C}) \longrightarrow \mathscr{P}^{\vee}\text{-coalg}(\mathscr{C}).$$

4.3 Now fix a co-operad $\mathscr{Q}$. The functor

$$\mathrm{triv}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}} : \mathscr{C} \longrightarrow \mathscr{Q}\text{-coalg}^{\mathrm{ind\text{-}nilp}}(\mathscr{C}),$$

obtained by corestriction along the coaugmentation $\mathrm{Triv} \to \mathscr{Q}$, is right inverse to $\mathrm{oblv}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}}$. Since the latter is conservative and preserves colimits, it follows that $\mathrm{triv}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}}$ preserves colimits, hence admits a right adjoint

$$\mathrm{prim}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}} : \mathscr{Q}\text{-coalg}^{\mathrm{ind\text{-}nilp}}(\mathscr{C}) \longrightarrow \mathscr{C}.$$

By definition, $\mathrm{prim}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}}$ can be computed as $C \mapsto \mathrm{Cobar}(\mathscr{Q}, C)$, and in particular there is a canonical morphism (but not isomorphism) of monads from $X \mapsto \mathscr{Q}^{\vee} \star X$ to $\mathrm{prim}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}} \circ \mathrm{triv}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}}$. It follows that $\mathrm{prim}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}}$ lifts to

$$\mathrm{prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{Q}} : \mathscr{Q}\text{-coalg}^{\mathrm{ind\text{-}nilp}}(\mathscr{C}) \longrightarrow \mathscr{Q}^{\vee}\text{-alg}(\mathscr{C}),$$

meaning $\mathrm{oblv}_{\mathscr{Q}^{\vee}} \circ \mathrm{prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{Q}} = \mathrm{prim}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}}$. Moreover, $\mathrm{prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{Q}} \circ \mathrm{cofree}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}} = \mathrm{triv}_{\mathscr{Q}^{\vee}}$, and there is a morphism of comonads

$$\mathrm{free}_{\mathscr{Q}^{\vee}} \longrightarrow \mathrm{prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{Q}} \circ \mathrm{triv}^{\mathrm{ind\text{-}nilp}}_{\mathscr{Q}}\,.$$

Proposition 4.3.1. *For any operad* $\mathscr{P}$*, the functors* $\mathrm{coprim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{P}}$ *and* $\mathrm{prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{P}^{\vee}}$ *are adjoint.*

Now we can state another conjecture.

Conjecture 4.3.2. *The functor* $\mathrm{prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{Q}}$ *is fully faithful.*

4.4 The coaugmentation on $\mathscr{Q}$ defines a functor

$$\mathrm{triv}_{\mathscr{Q}} : \mathscr{C} \longrightarrow \mathscr{Q}\text{-coalg}(\mathscr{C})$$

which is right inverse to $\mathrm{oblv}_{\mathscr{Q}}$. Moreover, $\mathrm{triv}_{\mathscr{Q}}$ preserves colimits and therefore admits a right adjoint

$$\mathrm{prim}_{\mathscr{Q}} : \mathscr{Q}\text{-coalg}(\mathscr{C}) \longrightarrow \mathscr{C}.$$

The functor $\mathrm{res}^{\star\to *}$ preserves colimits and therefore admits a right adjoint $(\mathrm{res}^{\star\to *})^R$. Define

$$\mathrm{prim}^{\mathrm{enh}}_{\mathscr{Q}} : \mathscr{Q}\text{-coalg}(\mathscr{C}) \longrightarrow \mathscr{Q}^{\vee}\text{-alg}(\mathscr{C})$$

to be the composition $\mathrm{prim}^{\mathrm{enh,ind\text{-}nilp}}_{\mathscr{Q}} \circ (\mathrm{res}^{\star\to *})^R$. By definition $\mathrm{prim}^{\mathrm{enh}}_{\mathscr{Q}}$ is right adjoint to $\mathrm{coprim}^{\mathrm{enh}}_{\mathscr{Q}^{\vee}}$.

The following is a variation on the previously stated conjectures.

Conjecture 4.4.1. *For any operad $\mathscr{P}$, the unit and counit of the adjunction between $\mathrm{coprim}^{\mathrm{enh}}_{\mathscr{P}}$ and $\mathrm{prim}^{\mathrm{enh}}_{\mathscr{P}^{\vee}}$ induce isomorphisms when evaluated on the essential images of $\mathrm{prim}^{\mathrm{enh}}_{\mathscr{P}^{\vee}}$ and $\mathrm{coprim}^{\mathrm{enh}}_{\mathscr{P}}$ respectively.*

5 Lie algebras and cocommutative coalgebras

5.1 Now we specialize to the case $\mathscr{P} = \mathrm{Lie}$, so that $\mathscr{P}^{\vee} = \Sigma\,\mathrm{Cocom}$. As before $\mathscr{C}$ is a fixed symmetric monoidal DG category. Put

$$\mathrm{LieAlg}(\mathscr{C}) := \mathrm{Lie}\text{-alg}(\mathscr{C}),$$

and observe that

$$\mathrm{Cocom}\text{-coalg}(\mathscr{C}) = \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathscr{C}),$$

the latter being the category of coaugmented cocommutative coalgebras in $\mathscr{C}$. In this case we use the notation $\mathrm{Chev}^{\mathrm{enh}} := \Sigma \circ \mathrm{coprim}^{\mathrm{enh}}_{\mathrm{Lie}}$ and $\mathrm{coChev}^{\mathrm{enh}} := \Omega \circ \mathrm{prim}^{\mathrm{enh}}_{\mathrm{Cocom}}$, so we have an adjunction

$$\mathrm{Chev}^{\mathrm{enh}} : \mathrm{LieAlg}(\mathscr{C}) \rightleftarrows \mathrm{CocomCoalg}^{\mathrm{aug}}(\mathscr{C}) : \mathrm{coChev}^{\mathrm{enh}}.$$

5.2 The functor $\mathrm{Chev}^{\mathrm{enh}}$ has a canonical oplax symmetric monoidal structure, where we equip $\mathrm{LieAlg}(\mathscr{C})$ and $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathscr{C})$ with the cartesian symmetric monoidal structure. Recall that $\mathrm{oblv}_{\mathrm{Cocom}}$ has a canonical symmetric monoidal structure, i.e. the cartesian monoidal structure on $\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathscr{C})$ is given by the tensor product in $\mathscr{C}$.

Proposition 5.2.1. *The oplax symmetric monoidal structure on $\mathrm{Chev}^{\mathrm{enh}}$ is strict.*

In particular $\mathrm{Chev}^{\mathrm{enh}}$ lifts to

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) : \mathrm{Grp}(\mathrm{LieAlg}(\mathscr{C})) \longrightarrow \mathrm{Grp}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathscr{C})) =: \mathrm{CocomHopf}(\mathscr{C}).$$

Theorem 5.2.2. *The composition*

$$\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}} : \mathrm{LieAlg}(\mathscr{C}) \longrightarrow \mathrm{CocomHopf}(\mathscr{C})$$

is fully faithful.

We will see that $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega$ can be identified with a more familiar functor.

5.3 Since $\mathrm{coChev}^{\mathrm{enh}}$ is right adjoint to the symmetric monoidal functor $\mathrm{Chev}^{\mathrm{enh}}$, it has a lax symmetric monoidal structure. In particular, $\mathrm{coChev}^{\mathrm{enh}}$ lifts to

$$\mathrm{Mon}(\mathrm{coChev}^{\mathrm{enh}}) : \mathrm{CocomBialg}(\mathscr{C}) := \mathrm{Mon}(\mathrm{CocomCoalg}^{\mathrm{aug}}(\mathscr{C})) \longrightarrow \mathrm{Mon}(\mathrm{LieAlg}(\mathscr{C})) = \mathrm{Grp}(\mathrm{LieAlg}(\mathscr{C})).$$

Observe that $\mathrm{Mon}(\mathrm{coChev}^{\mathrm{enh}})$ is right adjoint to the functor

$$\mathrm{Grp}(\mathrm{LieAlg}(\mathscr{C})) \overset{\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}})}{\longrightarrow} \mathrm{CocomHopf}(\mathscr{C}) \subset \mathrm{CocomBialg}(\mathscr{C}),$$

which we abusively denote by $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}})$. By Proposition 1.7.2, $\mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}$ is left adjoint to

$$B_{\mathrm{Lie}} \circ \mathrm{Mon}(\mathrm{coChev}^{\mathrm{enh}}) : \mathrm{CocomBialg}(\mathscr{C}) \longrightarrow \mathrm{LieAlg}(\mathscr{C}).$$

Proposition 5.3.1. *There is a canonical isomorphism*

$$\mathrm{oblv}_{\mathrm{Lie}} \circ B_{\mathrm{Lie}} \circ \mathrm{Mon}(\mathrm{coChev}^{\mathrm{enh}}) \xrightarrow{\sim} \mathrm{prim}_{\mathrm{Cocom}} \circ \mathrm{oblv}_{\mathrm{Mon}}$$

of functors $\mathrm{CocomBialg}(\mathscr{C}) \to \mathscr{C}$.

That is to say, the space of primitives in a cocommutative bialgebra is canonically a Lie algebra. This will be used later to produce the Lie algebra structure on the tangent space of a group at the identity.

5.4 Now we explain how the universal enveloping algebra fits into this picture. There is a canonical map of operads Lie $\to$ Assoc, which gives rise to a limit-preserving functor

$$\mathrm{res}^{\mathrm{Assoc}\to\mathrm{Lie}} : \mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C}) \longrightarrow \mathrm{LieAlg}(\mathscr{C}).$$

By definition, the functor U of universal enveloping algebra is left adjoint to $\mathrm{res}^{\mathrm{Assoc}\to\mathrm{Lie}}$.

The functor $\mathrm{res}^{\mathrm{Assoc}\to\mathrm{Lie}}$ has a lax symmetric monoidal structure, where $\mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C})$ is symmetric monoidal under the tensor product and $\mathrm{LieAlg}(\mathscr{C})$ has the cartesian symmetric monoidal structure. It follows that U has an oplax symmetric monoidal structure, and hence induces a functor

$$\mathrm{LieAlg}(\mathscr{C}) \xrightarrow{\sim} \mathrm{CocomCoalg}(\mathrm{LieAlg}(\mathscr{C})) \longrightarrow \mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C})). \tag{5.4.1}$$

Here the first functor is inverse to the functor that forgets the cocommutative coalgebra structure, which is always an equivalence for a cartesian monoidal structure. That is, the comultiplication on a Lie algebra $\mathfrak{g}$ is just the diagonal map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$.

It can be shown that the lax symmetric monoidal structure on U is strict, but we will not need this.

Proposition 5.4.1. *There is a canonical equivalence*

$$\mathrm{CocomCoalg}(\mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C})) \xrightarrow{\sim} \mathrm{CocomBialg}(\mathscr{C}).$$

Composing the functor (5.4.1) with the equivalence in Proposition 5.4.1, we obtain a functor

$$U^{\mathrm{Hopf}} : \mathrm{LieAlg}(\mathscr{C}) \longrightarrow \mathrm{CocomBialg}(\mathscr{C}).$$

Theorem 5.4.2. *There is a canonical isomorphism of functors*

$$U^{\mathrm{Hopf}} \xrightarrow{\sim} \mathrm{Grp}(\mathrm{Chev}^{\mathrm{enh}}) \circ \Omega_{\mathrm{Lie}}.$$

In particular, U^{Hopf} *takes values in* $\mathrm{CocomHopf}(\mathscr{C})$.

5.5 Finally, we state a nonstandard formulation of the Poincaré-Birkhoff-Witt (henceforth abbreviated to PBW) theorem and deduce the usual formulation.

Theorem 5.5.1. *There is a canonical isomorphism*

$$U \circ \mathrm{triv}_{\mathrm{Lie}} \xrightarrow{\sim} \mathrm{res}^{\mathrm{Com}\to\mathrm{Assoc}} \circ \mathrm{free}_{\mathrm{Com}}$$

of functors $\mathscr{C} \to \mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C})$.

One can show that U lifts canonically to a functor

$$U^{\mathrm{fil}} : \mathrm{LieAlg}(\mathscr{C}) \longrightarrow \mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C}^{\mathrm{fil},\geq 0}),$$

where $\mathscr{C}^{\mathrm{fil},\geq 0}$ is the category of nonnegatively filtered objects in $\mathscr{C}$. Let

$$\text{ass-gr} : \mathscr{C}^{\mathrm{fil},\geq 0} \longrightarrow \mathscr{C}^{\mathrm{gr},\geq 0}$$

denote the symmetric monoidal functor of associated graded. Composing U^{fil} with $\mathrm{AssocAlg}^{\mathrm{aug}}(\text{ass-gr})$, we obtain

$$U^{\mathrm{gr}} : \mathrm{LieAlg}(\mathscr{C}) \longrightarrow \mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C}^{\mathrm{gr},\geq 0}).$$

Corollary 5.5.1.1. *The functor* U^{gr} *is canonically isomorphic to the composition*

$$\mathrm{LieAlg}(\mathscr{C}) \xrightarrow{\mathrm{oblv}_{\mathrm{Lie}}} \mathscr{C} \xrightarrow{\deg=1} \mathscr{C}^{\mathrm{gr},\geq 0} \longrightarrow \mathrm{AssocAlg}^{\mathrm{aug}}(\mathscr{C}^{\mathrm{gr},\geq 0}),$$

where the last functor is $\mathrm{res}^{\mathrm{Com}\to\mathrm{Assoc}} \circ \mathrm{free}_{\mathrm{Com}}$.