

IndCoh Seminar: Infinitesimal differential geometry

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1 Filtrations in algebraic geometry

1.1 In classical algebraic geometry, many objects of differential nature come equipped with a filtration. For example, the sheaf of differential operators on a smooth variety is filtered by order of the operator. A more subtle example is the Hodge filtration on the constant sheaf. Unfortunately, both of these filtrations are usually defined by explicit formulas which are difficult to generalize to the derived setting.

To see what this means, let's recall the definitions. Fix field k of characteristic zero. For a smooth variety X over k , the sheaf of differential operators \mathcal{D}_X can be viewed as a quasicoherent \mathcal{O}_X -bimodule, i.e. an object of $\mathrm{QCoh}(X \times X)^\heartsuit$. As such it is supported on the formal completion of the diagonal in $X \times X$, and the canonical filtration on the formal completion determines a filtration on such a sheaf. Namely, a section belongs to the n^{th} piece of the filtration if it is supported on the n^{th} infinitesimal neighborhood of the diagonal.

To describe the Hodge filtration algebraically we work in the derived category of left \mathcal{D} -modules on X . We view them in the classical way, as complexes of sheaves of left \mathcal{D}_X -modules whose cohomologies are \mathcal{O}_X -quasicoherent. The de Rham complex DR_X in this context has the form

$$\mathcal{D}_X \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^1 \longrightarrow \cdots \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^d$$

where $d = \dim X$ and the differential is given by the usual explicit formula (the term $\mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^p$ is situated in cohomological degree p). Observe that although the terms of the complex have the form $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{F}$ where \mathcal{F} is a quasicoherent sheaf on X , none of the differentials are induced by maps of quasicoherent sheaves in this way. The kernel of the first differential is \mathcal{O}_X , the cohomologically normalized constant (alias IC) \mathcal{D} -module on X . That is, we have a canonical isomorphism of \mathcal{D} -modules $k_X \xrightarrow{\sim} \mathrm{DR}_X[-d]$, where k_X denotes the constant \mathcal{D} -module. The Hodge filtration on k_X is induced by the descending ‘‘stupid filtration’’ on DR_X , whose n^{th} piece is

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^n \longrightarrow \cdots \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^d.$$

Remarkably, the Hodge filtration is also induced by a certain geometric filtration. It is convenient to switch to right \mathcal{D} -modules on X , which identify with the category of right crystals $\mathrm{IndCoh}(X_{\mathrm{dR}})$, where X_{dR} denotes the de Rham prestack of X . We will see that there is a sequence of formal moduli problems

$$X = X^{(0)} \longrightarrow X^{(1)} \longrightarrow \cdots \longrightarrow X^{(n)} \longrightarrow \cdots \longrightarrow X_{\mathrm{dR}}$$

under X , where $\mathrm{colim}_n X^{(n)} \xrightarrow{\sim} X_{\mathrm{dR}}$ and each $X^{(n)}$ is a square-zero extension of $X^{(n-1)}$ by the direct image in $\mathrm{IndCoh}(X^{(n-1)})$ of $\mathrm{Sym}^n(T(X)[1])$ (here $T(X)$ denotes the *Serre* dual of Ω_X^1). This induces a filtration on the dualizing sheaf ω_X which is Verdier dual to the Hodge filtration on k_X constructed above.

1.2 What is a filtration? In a DG category \mathcal{C} there is no intrinsic notion of monomorphism, so it is necessary to weaken the definition of filtration as follows. The DG category of *filtered objects in \mathcal{C}* is defined by

$$\mathcal{C}^{\mathrm{fil}} := \mathrm{Fun}(\overrightarrow{\mathbb{Z}}, \mathcal{C})$$

where $\overrightarrow{\mathbb{Z}}$ is the category associated with the total ordering on \mathbb{Z} .

One also has the categories of *positively*, respectively *negatively filtered objects* $\mathcal{C}^{\text{fil}, \geq 0}$ and $\mathcal{C}^{\text{fil}, \leq 0}$ obtained by replacing $\overrightarrow{\mathbb{Z}}$ with $\overrightarrow{\mathbb{Z}}^{\geq 0}$ and $\overrightarrow{\mathbb{Z}}^{\leq 0}$. The restriction functor $\mathcal{C}^{\text{fil}} \rightarrow \mathcal{C}^{\text{fil}, \geq 0}$, respectively $\mathcal{C}^{\text{fil}} \rightarrow \mathcal{C}^{\text{fil}, \leq 0}$ admits a fully faithful left adjoint, whose essential image consists of functors $\overrightarrow{\mathbb{Z}} \rightarrow \mathcal{C}$ which send the negative integers to zero, respectively are constant on the nonnegative integers.

The functor $\text{colim} : \mathcal{C}^{\text{fil}} \rightarrow \mathcal{C}$ which takes the colimit over $\overrightarrow{\mathbb{Z}}$ is will be denoted by oblv^{fil} , since we think of it as forgetting the filtration. Observe that with this definition, there are many nonzero filtrations on the zero object!

The DG category of *graded objects of* \mathcal{C} is defined by

$$\mathcal{C}^{\text{gr}} := \text{Fun}(\mathbb{Z}, \mathcal{C}) = \prod_{\mathbb{Z}} \mathcal{C}.$$

Similarly to the filtered case, we have the categories of *positively*, respectively *negatively graded objects* $\mathcal{C}^{\text{gr}, \geq 0}$ and $\mathcal{C}^{\text{gr}, \leq 0}$, fully faithful functors $\mathcal{C}^{\text{gr}, \geq 0} \rightarrow \mathcal{C}$ and $\mathcal{C}^{\text{gr}, \leq 0} \rightarrow \mathcal{C}$, and a forgetful functor $\text{oblv}^{\text{gr}} : \mathcal{C}^{\text{gr}} \rightarrow \mathcal{C}$ given by taking the colimit (in this case direct sum) over \mathbb{Z} .

The functor of restriction along $\mathbb{Z} \rightarrow \overrightarrow{\mathbb{Z}}$ is denoted by

$$\text{Rees} : \mathcal{C}^{\text{fil}} \longrightarrow \mathcal{C}^{\text{gr}}.$$

The left adjoint of Rees, given by left Kan extension along $\mathbb{Z} \rightarrow \overrightarrow{\mathbb{Z}}$, will be denoted by $(\text{gr} \rightarrow \text{fil})$. For a graded object X_{\bullet} of \mathcal{C} , the n^{th} piece of the filtration on $(\text{gr} \rightarrow \text{fil})(X_{\bullet})$ is $\bigoplus_{m \leq n} X_m$. Observe that $(\text{gr} \rightarrow \text{fil})$ admits a left inverse, namely the functor of associated graded

$$\text{ass-gr} : \mathcal{C}^{\text{fil}} \longrightarrow \mathcal{C}^{\text{gr}}.$$

Given a filtered object

$$\cdots \longrightarrow X_{n-1} \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots$$

of \mathcal{C} its associated graded has the n^{th} graded piece $\text{cofib}(X_{n-1} \rightarrow X_n)$.

We point out that a (symmetric) monoidal structure on \mathcal{C} lifts naturally to (symmetric) monoidal structures \mathcal{C}^{fil} and \mathcal{C}^{gr} which preserve the positively and negatively filtered/graded objects, and the functors ass-gr and $(\text{gr} \rightarrow \text{fil})$ have (symmetric) monoidal structures.

1.3 Now we will explain how gradings are interpreted in geometry. The category Vect^{gr} of graded vector spaces identifies with $\text{QCoh}(\text{pt}/\mathbb{G}_m)$, i.e. the category of representations of \mathbb{G}_m . For a general DG category \mathcal{C} , there is a canonical equivalence

$$\mathcal{C}^{\text{gr}} \xrightarrow{\sim} \mathcal{C} \otimes \text{Vect}^{\text{gr}} = \mathcal{C} \otimes \text{QCoh}(\text{pt}/\mathbb{G}_m).$$

Here \otimes is the natural tensor product on DG categories, which we defined as modules for the symmetric monoidal category Vect .

It is natural to ask whether one can characterize geometrically the condition that a graded vector space is positively or negatively graded. For this it is necessary to use the language of monoid-equivariant morphisms of prestacks in categories. We refer the reader to Gaitsgory-Rozenblyum for the precise definitions: see Chapter IV.5.1. For a monoid object in prestacks, or equivalently a functor $\mathcal{M} : \text{Aff}^{\text{op}} \rightarrow \text{Mon}(\text{Spc})$, one can speak of an action of \mathcal{M} on a prestack in categories $F : \text{Aff}^{\text{op}} \rightarrow \text{Cat}$. For a morphism of such prestacks $F_1 \rightarrow F_2$, there is notion of lax, respectively oplax, strict \mathcal{M} -equivariant structure.

A typical example of a prestack in categories is the functor $S \mapsto \text{QCoh}(S) \otimes \mathcal{C}$, where \mathcal{C} is a DG category. Given an action of \mathcal{M} on this prestack we write $\mathcal{C}^{\mathcal{M}, \text{lax}}$, respectively $\mathcal{C}^{\mathcal{M}, \text{oplax}}$, $\mathcal{C}^{\mathcal{M}}$ for the category of lax, respectively oplax, strict \mathcal{M} -equivariant objects in \mathcal{C} . If \mathcal{M} is a group then the three categories of equivariant objects coincide.

In this notation we have $\mathcal{C}^{\text{gr}} = \mathcal{C}^{\mathbb{G}_m}$ for any DG category \mathcal{C} , where \mathbb{G}_m acts trivially on \mathcal{C} . Less obviously, we have $\mathcal{C}^{\text{gr}, \geq 0} = \mathcal{C}^{\mathbb{A}^1, \text{oplax}}$ and $\mathcal{C}^{\text{gr}, \leq 0} = \mathcal{C}^{\mathbb{A}^1, \text{lax}}$, where \mathbb{A}^1 is viewed as a monoid under multiplication. That is, a graded object of \mathcal{C} is positively graded if and only if its \mathbb{G}_m -equivariant structure extends to a lax \mathbb{A}^1 -equivariant structure, and similarly for negatively graded objects and oplax \mathbb{A}^1 -equivariant structures.

1.4 It turns out that there is also an algebro-geometric model for filtrations. Let \mathbb{G}_m act on \mathbb{A}^1 by dilations and consider the quotient stack $\mathbb{A}^1/\mathbb{G}_m$. The map $\{1\} \rightarrow \mathbb{A}^1/\mathbb{G}_m$ is an open embedding, and the complementary closed substack is $\{0\}/\mathbb{G}_m \rightarrow \mathbb{A}^1/\mathbb{G}_m$.

Proposition 1.4.1. *The functor $\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \rightarrow \mathrm{Vect}$ which takes the fiber at $\{1\}$ factors through $\mathrm{oblv}^{\mathrm{fil}}$ and induces an equivalence*

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \xrightarrow{\sim} \mathrm{Vect}^{\mathrm{fil}}.$$

Proof. There is a tautological equivalence Γ from quasicoherent sheaves on \mathbb{A}^1 to $k[t]$ -modules, i.e. vector spaces equipped with an endomorphism. A \mathbb{G}_m -equivariant structure on such a sheaf \mathcal{F} corresponds to a grading on $\Gamma(\mathbb{A}^1, \mathcal{F})$ such that the given endomorphism has degree one, which is the same datum as a filtered vector space. Explicitly, we have

$$\Gamma(\mathbb{A}^1, \mathcal{F}) \xrightarrow{\sim} \bigoplus_{n \in \mathbb{Z}} \Gamma(\mathbb{A}^1, \mathcal{F}(n \cdot \{0\}))^{\mathbb{G}_m},$$

with the degree one endomorphism induced by the natural maps $\mathcal{F}(n \cdot \{0\}) \rightarrow \mathcal{F}((n+1) \cdot \{0\})$. Here we are twisting by the \mathbb{G}_m -equivariant line bundles $\mathcal{O}(n \cdot \{0\})$, which are isomorphic as sheaves but not \mathbb{G}_m -equivariantly. The graded module corresponding to $\mathcal{O}(n \cdot \{0\})$ is $t^{-n}k[t]$ with the degree grading, and in particular we see that

$$\mathcal{F} \xrightarrow{\sim} \mathcal{F}(n \cdot \{0\}) \longrightarrow \mathcal{F}((n+1) \cdot \{0\}) \xrightarrow{\sim} \mathcal{F}$$

agrees with multiplication by the coordinate.

To see that $\mathrm{oblv}^{\mathrm{fil}}$ agrees with taking the fiber at 1 under the above equivalence, observe that for any $\overrightarrow{\mathbb{Z}}$ -shaped diagram

$$\cdots \longrightarrow X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

in a DG category, the colimit can be computed as

$$\bigoplus_{n \in \mathbb{Z}} X_n \xrightarrow[\mathrm{id}]{f} \bigoplus_{n \in \mathbb{Z}} X_n.$$

□

For any DG category \mathcal{C} we have canonical equivalences

$$\mathcal{C}^{\mathrm{fil}} \xrightarrow{\sim} \mathcal{C} \otimes \mathrm{Vect}^{\mathrm{fil}} \xrightarrow{\sim} \mathcal{C} \otimes \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m).$$

The dilation action of \mathbb{G}_m on \mathbb{A}^1 induces a \mathbb{G}_m -action on $\mathrm{QCoh}(\mathbb{A}^1)$, and a basic feature of the theory is that we have

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{A}^1)^{\mathbb{G}_m}.$$

More generally, \mathbb{G}_m acts on $\mathcal{C} \otimes \mathrm{QCoh}(\mathbb{A}^1)$ and there is a canonical equivalence

$$\mathcal{C}^{\mathrm{fil}} = \mathcal{C} \otimes \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \xrightarrow{\sim} (\mathcal{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{G}_m}.$$

Similarly to the graded case, we have $\mathcal{C}^{\mathrm{fil}, \geq 0} = (\mathcal{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1, \mathrm{oplax}}$ and $\mathcal{C}^{\mathrm{gr}, \leq 0} = (\mathcal{C} \otimes \mathrm{QCoh}(\mathbb{A}^1))^{\mathbb{A}^1, \mathrm{lax}}$.

2 Deformation to the normal bundle

2.1 First let us recall the classical construction of deformation to the normal bundle. Fix a closed embedding $f : X \rightarrow Y$ of smooth varieties and denote by $N(X/Y) := T(X/Y)[1]$ the normal bundle. We view the embedding of the zero section in the total space of $N(X/Y)$ as a “linear approximation” of f . In the context of differential geometry there is actually an isomorphism of the tubular neighborhoods around X . There is no such theorem in algebraic geometry, but in fact we can obtain $N(X/Y)$ as the degeneration of a one-parameter family whose generic fiber is isomorphic to Y . Moreover, this family has a \mathbb{G}_m -action compatible with the dilation actions on \mathbb{A}^1 and $N(X/Y)$.

The construction is the following: let \tilde{Y} be the blow-up of $Y \times \mathbb{A}^1$ along $X \times \{0\}$. Then for $t \neq 0$ the fiber \tilde{Y}_t is canonically identified with Y . The fiber \tilde{Y}_0 is reducible, with one component being $\mathrm{Bl}_X(Y)$ and the other the projective completion of $N(X/Y)$. Here the exceptional divisor in $\mathrm{Bl}_X(Y)$, i.e. the projectivization of $N(X/Y)$, is identified with the section at infinity in the projective completion of $N(X/Y)$.

The action of \mathbb{G}_m on $Y \times \mathbb{A}^1$ induces an action on \tilde{Y} , giving the aforementioned isomorphisms among fibers away from zero, acting on $N(X/Y)$ by dilation, and fixing $\mathrm{Bl}_X(Y)$. Define $Y_{\mathrm{scaled}} := \tilde{Y} \setminus \mathrm{Bl}_X(Y)$, a \mathbb{G}_m -stable open subvariety. This is the classical deformation to the normal bundle. Informally, a direction of approach to $X \times \{0\}$ inside $Y \times \mathbb{A}^1$ which avoids $Y \times \{0\}$ determines a point of $N(X/Y)$.

2.2 The version of deformation to the normal bundle which we're interested in proceeds in the following generality. Let \mathcal{X} be a prestack which is locally almost of finite type and admits deformation theory, and let $\mathcal{X} \rightarrow \mathcal{Y}$ be a formal moduli problem under \mathcal{X} . Then there is an object $\mathcal{Y}_{\mathrm{scaled}}$ in $\mathrm{PreStk}_{\mathcal{X} \times \mathbb{A}^1 // \mathcal{Y} \times \mathbb{A}^1}$, i.e. an \mathbb{A}^1 -family of objects in $\mathrm{PreStk}_{\mathcal{X} // \mathcal{Y}}$, equipped with a left-lax \mathbb{A}^1 -equivariant structure. The fiber of $\mathcal{Y}_{\mathrm{scaled}}$ over any $t \neq 0$ is canonically identified with \mathcal{Y} , and $\mathcal{Y}_{\mathrm{scaled},0}$ is $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$, the vector prestack over \mathcal{X} associated with $T(\mathcal{X}/\mathcal{Y})[1] \in \mathrm{IndCoh}(\mathcal{X})$.

The left-lax equivariant structure on $\mathcal{Y}_{\mathrm{scaled}}$ means that for $s, t \in \mathbb{A}^1$ we have maps

$$\mathcal{Y}_{\mathrm{scaled},s-t} \longrightarrow \mathcal{Y}_{\mathrm{scaled},s}$$

satisfying the appropriate associativity and unitality conditions. In particular, when $s \neq 0$ and $t = 0$ we get the composition

$$\mathcal{Y}_{\mathrm{scaled},0} = \mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1]) \longrightarrow \mathcal{X} \longrightarrow \mathcal{Y} = \mathcal{Y}_{\mathrm{scaled},s}$$

We warn the reader that $\mathrm{Vect}_{\mathcal{X}}$ takes values in formal moduli problems over \mathcal{X} , and in particular if $T(\mathcal{X}/\mathcal{Y})[1]$ is a vector bundle then $\mathrm{Vect}_{\mathcal{X}}(T(\mathcal{X}/\mathcal{Y})[1])$ is the formal completion of its total space along the zero section. If \mathcal{X} is a smooth scheme and \mathcal{Y} is its formal completion along a closed embedding into a smooth scheme, then $\mathcal{Y}_{\mathrm{scaled}}$ can be identified with the formal completion along $\mathcal{X} \times \mathbb{A}^1$ of the deformation to the normal bundle introduced in the previous subsection.

Although we will not give the full construction of $\mathcal{Y}_{\mathrm{scaled}}$ here, we'll explain what happens at the level of Lie algebroids. Recall that formal moduli problems under \mathcal{X} are equivalent to formal groupoids over \mathcal{X} , which are by definition the same as Lie algebroids on \mathcal{X} . Let \mathcal{L} be the Lie algebroid on \mathcal{X} corresponding to \mathcal{Y} : the 1-morphisms of this formal groupoid are $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$. Recall that there is a forgetful functor from Lie algebroids on \mathcal{X} to $\mathrm{IndCoh}(\mathcal{X})_{/T(\mathcal{X})}$. For each $t \in \mathbb{A}^1$, the fiber of $\mathcal{Y}_{\mathrm{scaled}}$ over t corresponds to the Lie algebroid with the same underlying ind-coherent sheaf as \mathcal{L} , with the anchor map being the original $\mathcal{L} \rightarrow T(\mathcal{X})$ scaled by t .

2.3 Let $\mathcal{X} \rightarrow \mathcal{Y}$ be as in the previous subsection. We will now explain the construction of the n^{th} infinitesimal neighborhood of \mathcal{X} in \mathcal{Y} . We are supposed to have a sequence

$$\mathcal{X} = \mathcal{X}^{(0)} \longrightarrow \mathcal{X}^{(1)} \longrightarrow \dots \longrightarrow \mathcal{X}^{(n)} \longrightarrow \dots \longrightarrow \mathcal{Y}$$

of formal moduli problems under \mathcal{X} with the property that $\mathrm{colim}_n \mathcal{X}^{(n)} \rightarrow \mathcal{Y}$ is an isomorphism. Moreover, each $\mathcal{X}^{(n)}$ is a square-zero extension of $\mathcal{X}^{(n-1)}$ by the direct image of $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$. Such a square-zero extension is determined by a map

$$\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1]) \longrightarrow T(\mathcal{X}^{(n-1)}/\mathcal{Y})|_{\mathcal{X}}. \quad (2.3.1)$$

In fact, these infinitesimal neighborhoods are the fiber over 1 in an \mathbb{A}^1 -oplax equivariant family

$$\mathcal{X} \times \mathbb{A}^1 = \mathcal{X}_{\mathrm{scaled}}^{(0)} \longrightarrow \mathcal{X}_{\mathrm{scaled}}^{(1)} \longrightarrow \dots \longrightarrow \mathcal{X}_{\mathrm{scaled}}^{(n)} \longrightarrow \dots \longrightarrow \mathcal{Y}_{\mathrm{scaled}}.$$

The ‘scaled’ version of the desired map (2.3.1) is

$$\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1]) \longrightarrow T(\mathcal{X}_{\mathrm{scaled}}^{(n-1)}/\mathcal{Y})|_{\mathcal{X} \times \mathbb{A}^1}$$

in $\mathrm{IndCoh}(\mathcal{X} \times \mathbb{A}^1)^{\mathbb{A}^1, \mathrm{oplax}} = \mathrm{IndCoh}(\mathcal{X})^{\mathrm{fil}, \geq 0}$, where $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$ is viewed as a graded object placed in degree n , then as a filtered object by applying $(\mathrm{gr} \rightarrow \mathrm{fil})$. To produce the map one shows that $T(\mathcal{X}_{\mathrm{scaled}}^{(n-1)}/\mathcal{Y})|_{\mathcal{X} \times \mathbb{A}^1}$ belongs to $\mathrm{IndCoh}(\mathcal{X})^{\mathrm{fil}, \geq n}$, and then identifies the degree n piece of the associated graded with $\mathrm{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$.

3 Examples and applications

3.1 First, we return to the example of the PBW filtration on differential operators. Recall that in the classical setting of a smooth scheme, differential operators are the universal enveloping algebra of the tangent Lie algebroid. We will explain how to construct the PBW filtration on the enveloping algebra of an arbitrary Lie algebroid.

Let \mathcal{X} be as in the previous section and fix a Lie algebroid \mathcal{L} on \mathcal{X} , which corresponds tautologically to a formal groupoid \mathcal{G} over \mathcal{X} . Such a groupoid is (non-tautologically) equivalent to a formal moduli problem $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ under \mathcal{X} , which should be viewed as the quotient of \mathcal{X} by the action of \mathcal{G} . Define the category of \mathcal{L} -modules as

$$\mathcal{L}\text{-mod}(\text{IndCoh}(\mathcal{X})) := \text{IndCoh}(\mathcal{Y}),$$

i.e. ind-coherent sheaves on \mathcal{X} equivariant for the formal groupoid corresponding to \mathcal{L} . Since π is an inf-schematic nil-isomorphism, the functors π_*^{IndCoh} and $\pi^!$ are adjoint, and we rename them as

$$\text{IndCoh}(\mathcal{X}) \begin{array}{c} \xrightarrow{\text{ind}_{\mathcal{L}}} \\ \xleftarrow{\text{oblv}_{\mathcal{L}}} \end{array} \mathcal{L}\text{-mod}(\text{IndCoh}(\mathcal{X})). \quad (3.1.1)$$

For example, if $\mathcal{L} = T(\mathcal{X})$, then $\mathcal{L}\text{-mod}(\text{IndCoh}(\mathcal{X}))$ is the category of right crystals on \mathcal{X} .

The universal enveloping algebra $U(\mathcal{L})$ is by definition the monad on $\text{IndCoh}(\mathcal{X})$ induced by the adjunction (3.1.1). Note that $\text{oblv}_{\mathcal{L}} = \pi^!$ is continuous, so that $U(\mathcal{L})$ is an associative algebra in $\text{End}_{\text{cts}}(\text{IndCoh}(\mathcal{X}))$. This is not as esoteric as it sounds: under a mild finiteness hypothesis on \mathcal{X} (namely, that $\text{QCoh}(\mathcal{X})$ is dualizable) the functor

$$\text{IndCoh}(\mathcal{X} \times \mathcal{X}) \longrightarrow \text{End}_{\text{cts}}(\text{IndCoh}(\mathcal{X}))$$

which sends \mathcal{H} to the endofunctor

$$\mathcal{F} \mapsto \pi_{2,*}^{\text{IndCoh}}(\mathcal{H} \otimes \pi_1^!(\mathcal{F}))$$

is an equivalence. If \mathcal{L} is a classical Lie algebroid on a smooth scheme X , then the usual enveloping algebra is a quasicoherent sheaf on $X \times X$ supported on the formal completion of the diagonal. This matches up with our $U(\mathcal{L})$ under the equivalence

$$\Upsilon : \text{QCoh}(\widehat{X \times X}_{\Delta}) \xrightarrow{\sim} \text{IndCoh}(\widehat{X \times X}_{\Delta}).$$

3.2 The desired filtered version $U(\mathcal{L})^{\text{fil}}$ is a lift of $U(\mathcal{L})$ to an associative algebra in $\text{End}_{\text{cts}}(\text{IndCoh}(\mathcal{X}))^{\text{fil}, \geq 0}$. Moreover, the associated graded of $U(\mathcal{L})^{\text{fil}}$ should identify with the graded monad $\text{free}_{\text{Com}}(\mathcal{L}) \otimes -$, where we abusively write \mathcal{L} for ind-coherent sheaf underlying the Lie algebroid \mathcal{L} .

The filtration on $U(\mathcal{L})$ is compatible with the formation of infinitesimal neighborhoods in the following sense. Let $\mathcal{G}_1 = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ be the space of 1-morphisms in the groupoid \mathcal{G} . The unit section $\mathcal{X} \rightarrow \mathcal{G}_1$ is a formal moduli problem under \mathcal{X} , and we write ${}^i\mathcal{X}^{(n)}$ for the n^{th} infinitesimal neighborhood of \mathcal{X} in \mathcal{G}_1 . Letting $p_1^{(n)}, p_2^{(n)} : {}^i\mathcal{X}^{(n)} \rightarrow \mathcal{X}$ denote the projections, there is an isomorphism

$$\text{oblv}_{\text{Assoc}}(U(\mathcal{L})^{\text{fil}}) \xrightarrow{\sim} \text{colim}_n (p_2^{(n)})_*^{\text{IndCoh}} \circ (p_1^{(n)})^!$$

in $\text{End}_{\text{cts}}(\text{IndCoh}(\mathcal{X}))^{\text{fil}, \geq 0}$.

As for the actual construction of the filtration, we will explain how to obtain the desired algebra $U(\mathcal{L})^{\text{fil}}$ in $\text{End}_{\text{cts}}(\text{IndCoh}(\mathcal{X}))^{\text{fil}}$ using $\mathcal{Y}_{\text{scaled}}$, omitting the verification that the filtration is positive. Observe that we have canonical equivalences

$$\begin{aligned} \text{End}_{\text{cts}}(\text{IndCoh}(\mathcal{X}))^{\text{fil}} &\xrightarrow{\sim} \text{End}_{\text{cts}}(\text{IndCoh}(\mathcal{X})) \otimes \text{QCoh}(\mathbb{A}^1/\mathbb{G}_m) \\ &\xrightarrow{\sim} \text{Fun}_{\text{cts}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X} \times \mathbb{A}^1/\mathbb{G}_m)). \end{aligned}$$

Now consider the map $f_{\text{scaled}} : \mathcal{X} \times \mathbb{A}^1 \rightarrow \mathcal{Y}_{\text{scaled}}$, which gives rise to a $\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$ -linear functor

$$(f_{\text{scaled}}/\mathbb{G}_m)^! : \text{IndCoh}(\mathcal{Y}_{\text{scaled}}/\mathbb{G}_m) \longrightarrow \text{IndCoh}(\mathcal{X} \times \mathbb{A}^1/\mathbb{G}_m).$$

The left adjoint $(f_{\text{scaled}/\mathbb{G}_m})_*^{\text{IndCoh}}$ is also $\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)$ -linear for general reasons, which implies that $(f_{\text{scaled}/\mathbb{G}_m})^! \circ (f_{\text{scaled}/\mathbb{G}_m})_*^{\text{IndCoh}}$ has the structure of an algebra in

$$\text{End}_{\text{QCoh}(\mathbb{A}^1/\mathbb{G}_m)}(\text{IndCoh}(\mathcal{X} \times \mathbb{A}^1/\mathbb{G}_m)) \xrightarrow{\sim} \text{Func}_{\text{cts}}(\text{IndCoh}(\mathcal{X}), \text{IndCoh}(\mathcal{X} \times \mathbb{A}^1/\mathbb{G}_m)).$$

This is the desired $U(\mathcal{L})^{\text{fil}}$.

3.3 The Hodge filtration on a smooth scheme generalizes to this setting as follows. Notice that $\omega_{\mathcal{X}}$ has a canonical structure of \mathcal{L} -module because $\pi^! \omega_{\mathcal{Y}} = \omega_{\mathcal{X}}$. Since

$$\text{colim}_n \mathcal{X}^{(n)} \xrightarrow{\sim} \mathcal{Y},$$

we obtain a filtration on $\omega_{\mathcal{Y}}$ given by

$$\text{colim}_n (f^{(n)})_*^{\text{IndCoh}} \omega_{\mathcal{X}^{(n)}} \xrightarrow{\sim} \omega_{\mathcal{Y}},$$

where $f^{(n)} : \mathcal{X}^{(n)} \rightarrow \mathcal{Y}$ is the natural map. Recall that $\text{oblv}(\mathcal{L}) = T(\mathcal{X}/\mathcal{Y})$. Using the fact that $\mathcal{X}^{(n-1)} \rightarrow \mathcal{X}^{(n)}$ is a square-zero extension by the direct image of $\text{Sym}^n(T(\mathcal{X}/\mathcal{Y})[1])$, one can identify the n^{th} associated graded of the Hodge filtration with $\text{ind}_{\mathcal{L}} \text{Sym}^n(\text{oblv}(\mathcal{L})[1])$. When \mathcal{X} is a smooth scheme and $\mathcal{L} = T(\mathcal{X})$, i.e. $\mathcal{Y} = \mathcal{X}_{\text{dR}}$, this agrees with the Hodge filtration previously defined.

Let's do an example which isn't a smooth scheme. Namely, fix a connected reductive group G and let $\mathcal{X} = \text{pt}/G$. Recall that the tangent complex $T(\text{pt}/G)$ identifies with $\mathfrak{g}[1]$, where \mathfrak{g} denotes the adjoint representation, under the equivalence

$$\text{Rep}(G) \xrightarrow{\sim} \text{QCoh}(\text{pt}/G) \xrightarrow{\Upsilon} \text{IndCoh}(\text{pt}/G).$$

It is not so easy to say what algebraic structure on $\mathfrak{g}[1]$ makes it into a Lie algebroid: it only carries a Lie bracket after shift by $[-1]$, i.e. looping.

Example 3.3.1. Let's use the formalism developed above to compute the homology of pt/G . It turns out that for the map $p : \text{pt}/G \rightarrow \text{pt}$, the functor $p_{\text{dR}}^!$ admits a left adjoint $p_!$. Since $p^!$ corresponds to triv_G under the equivalence (3.3), its left adjoint $p_! \circ \text{ind}_{T(\text{pt}/G)}$ identifies with the functor of coinvariants coinv_G . The Hodge filtration on $\omega_{(\text{pt}/G)_{\text{dR}}}$ has $\text{ind}_{T(\text{pt}/G)} \text{Sym}^n(T(\text{pt}/G)[1])$ as its n^{th} associated graded, so the induced filtration on $H_{\bullet}^{\text{dR}}(\text{pt}/G) = p_! \omega_{(\text{pt}/G)_{\text{dR}}}$ has n^{th} associated graded $\text{Sym}^n(\mathfrak{g}[2])_G$. Since G is reductive coinv_G is t -exact, which implies that the associated spectral sequence degenerates at E_2 and hence

$$H_{\bullet}^{\text{dR}}(\text{pt}/G) \xrightarrow{\sim} \text{Sym}(\mathfrak{g}[2])_G.$$

Dualizing, we obtain the well-known Chern-Weil isomorphism.