IndCoh Seminar: Overview

Justin Campbell

January 29, 2016

1 Quasi-coherent sheaves in representation theory

Fix a field k of characteristic zero and a linear algebraic group G over k.

One studies G through its symmetric monoidal category of representations $\mathrm{Rep}(G)$, which has a symmetric monoidal forgetful functor

$$\mathrm{oblv} : \mathrm{Rep}(G) \longrightarrow \mathrm{Vect}\,.$$

For instance, one has $G \tilde{\to} \mathrm{Aut}^{\otimes}(\mathrm{oblv})$. So we would like a geometric model for this situation.

Let pt/G be the prestack quotient, meaning for any scheme S we have $\mathrm{Map}(S, \mathrm{pt}/G) := \mathrm{pt}/G(S)$, where the latter is the groupoid whose one object has automorphisms $G(S)$. There is a map $\sigma : \mathrm{pt} \to \mathrm{pt}/G$, and for any $S \to \mathrm{pt}/G$ we have $S \times_{\mathrm{pt}/G} \mathrm{pt} = S \times G$.

There is a well-defined symmetric monoidal category $\mathrm{QCoh}(\mathrm{pt}/G)$ of quasi-coherent sheaves on pt/G; later we will make sense of it for any prestack. In fact, there is a canonical symmetric monoidal equivalence

$$\mathrm{QCoh}(\mathrm{pt}/G) \tilde{\longrightarrow} \mathrm{Rep}(G).$$

As soon as one makes sense of the left hand side it is just descent: the functor

$$\sigma^* : \mathrm{QCoh}(\mathrm{pt}/G) \longrightarrow \mathrm{Vect}$$

factors through oblv because a descent datum along σ is a G-action, which provides the desired equivalence.

Now we should ask how this model behaves functorially. Suppose we are given a homomorphism $G \to H$, which induces a map $f : \mathrm{pt}/G \to \mathrm{pt}/H$. Then we have a commutative square

$$\begin{array}{ccc} \mathrm{Rep}(H) & \xrightarrow{\mathrm{Res}} & \mathrm{Rep}(G) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{QCoh}(\mathrm{pt}/H) & \xrightarrow{f^*} & \mathrm{QCoh}(\mathrm{pt}/G), \end{array}$$

where Res is the functor of restriction along $G \to H$. For general reasons f^* admits a right adjoint f_*. If $G \subset H$ is a subgroup, then f_* corresponds to coinduction, and for $G \to 1$ it is the functor of G-invariants.

So it seems that quasi-coherent sheaves on classifying stacks are a good geometric model for representations of algebraic groups. What about Lie algebras?

Rather, let us consider the formal completion $\hat{G}$ of G at the identity. Recall that $\hat{G}$ has a single k-point, where its tangent space is the Lie algebra $\mathfrak{g}$ of G. The prestack quotient $\mathrm{pt}/\hat{G}$ makes sense, and there is still an equivalence

$$\mathrm{QCoh}(\mathrm{pt}/\hat{G}) \tilde{\longrightarrow} \mathrm{Rep}(\hat{G}) = \mathfrak{g}\text{-mod}.$$

Now let's vary the group. Write $\hat{f} : \mathrm{pt}/\hat{G} \to \mathrm{pt}/\hat{H}$ for the map induced by $\hat{G} \to \hat{H}$. As expected $\hat{f}^*$ agrees with restriction along $\mathfrak{g} \to \mathfrak{h}$, but $\hat{f}_*$ is a very awkward functor in general. For example, when $G = 1$ we have

$$\hat{f}_* V = \Gamma(\hat{G}, \mathscr{O}_{\hat{G}}) \hat{\otimes} V$$

with the obvious $\mathfrak{g}$-action: this functor is not even continuous (meaning it does not commute with colimits).

Experience in representation theory tells us that the natural functor

$$\mathfrak{g}\text{-mod} \longrightarrow \mathfrak{h}\text{-mod}$$

is $M \mapsto U(\mathfrak{h}) \otimes_{U(\mathfrak{g})} M$, which is left adjoint to restriction. How do we model this geometrically, with a pushforward functor?

2 Ind-coherent sheaves

We will introduce a symmetric monoidal category $\text{IndCoh}(\text{pt}/\hat{G})$ of *ind-coherent sheaves*, which makes sense on any prestack. In this case there is a symmetric monoidal equivalence

$$\Upsilon : \mathfrak{g}\text{-mod} = \text{QCoh}(\text{pt}/\hat{G}) \xrightarrow{\sim} \text{IndCoh}(\text{pt}/\hat{G}),$$

but IndCoh has better functorial properties. Namely, there are functors

$$\text{IndCoh}(\text{pt}/\hat{G}) \underset{\hat{f}^!}{\overset{\hat{f}_*^{\text{IndCoh}}}{\rightleftarrows}} \text{IndCoh}(\text{pt}/\hat{H}),$$

which are adjoint in this case because f is proper in an appropriate sense. Under Υ, restriction along $\mathfrak{g} \to \mathfrak{h}$ corresponds to $\hat{f}^!$, and $\hat{f}_*^{\text{IndCoh}}$ is its sought-after left adjoint. In particular, if $\mathfrak{g} = 0$ then we have

$$\Upsilon(U(\mathfrak{h})) = \hat{f}_*^{\text{IndCoh}} \hat{f}^! k.$$

Although we discussed $\text{IndCoh}(\text{pt}/\hat{G})$ first for motivational purposes, in order to define IndCoh on weird prestacks like $\text{pt}/\hat{G}$ one must first set up the theory for (at least affine) schemes. Let's discuss the basic pieces of structure. To any scheme X of finite type over k we attach a symmetric monoidal category $\text{IndCoh}(X)$, and for any morphism $f : X \to Y$ there are functors

$$\text{IndCoh}(X) \underset{f^!}{\overset{f_*^{\text{IndCoh}}}{\rightleftarrows}} \text{IndCoh}(Y),$$

which are adjoint when f is proper.

When $Y = \text{pt}$ we define the *dualizing complex* $\omega_X = f^! k$, which is the unit object of $\text{IndCoh}(X)$. Serre duality is the equivalence

$$\mathbb{D}_X^{\text{Ser}} : \text{Coh}(X)^{\text{op}} \xrightarrow{\sim} \text{Coh}(X)$$

given by internal Hom into ω_X. This extends to an equivalence

$$\text{IndCoh}(X)^\vee \xrightarrow{\sim} \text{IndCoh}(X),$$

where $\text{IndCoh}(X)$ is the dual category (more on this notion later). Although f_*^{IndCoh} and $f^!$ are not adjoint in general, one always has

$$f^! \mathbb{D}_Y^{\text{Ser}} = \mathbb{D}_X^{\text{Ser}} (f_*^{\text{IndCoh}})^\vee,$$

where $(f_*^{\text{IndCoh}})^\vee$ is the functor dual to f_*^{IndCoh}.

There is a symmetric monoidal functor

$$\Upsilon_X : \text{QCoh}(X) \longrightarrow \text{IndCoh}(X),$$

which defines an action of $\text{QCoh}(X)$ on $\text{IndCoh}(X)$. There is also a functor

$$\Psi_X : \text{IndCoh}(X) \longrightarrow \text{QCoh}(X),$$

which unlike Υ_X is t-exact, but not symmetric monoidal. In fact, Ψ_X is an equivalence if and only if X is smooth.

3 Crystals

One would like to have a canonical construction of the category of $\mathscr{D}$-modules on a possibly singular scheme X. The classical approach is to locally embed X into a smooth scheme and use Kashiwara's lemma to show that the construction is independent of the choice of embedding. To get functorial operations one must constantly make choices and then check that the result does not depend on them, which quickly becomes a nuisance, if not a downright nightmare.

But there is another way. The *de Rham prestack* X_{dR} of X is defined by

$$\mathrm{Map}(S, X_{\mathrm{dR}}) := \mathrm{Map}(S_{\mathrm{red}}, X).$$

Then the category of *left crystals* on X is

$$\mathrm{Crys}^{\ell}(X) := \mathrm{QCoh}(X_{\mathrm{dR}}).$$

When X is smooth $\mathrm{Crys}^{\ell}(X)$ is canonically identified with the category $\mathscr{D}^{\ell}(X)$ of left $\mathscr{D}$-modules on X, and this equivalence fits into a commutative triangle

$$\begin{array}{ccc} \mathrm{Crys}^{\ell}(X) & \xrightarrow{\sim} & \mathscr{D}^{\ell}(X) \\ & {\scriptstyle \pi^*}\searrow \quad \swarrow{\scriptstyle \mathrm{oblv}^{\ell}} & \\ & \mathrm{Vect}, & \end{array}$$

where $\pi : X \to X_{\mathrm{dR}}$ is the natural map and oblv^{ℓ} is the forgetful functor.

Now we run into trouble: the functor oblv^{ℓ} admits a left adjoint ind^{ℓ}, but the pushforward π_* si right adjoint to π^*, and ill-behaved besides.

For this and other reasons one considers the category of *right crystals* on X

$$\mathrm{Crys}^{r}(X) := \mathrm{IndCoh}(X_{\mathrm{dR}}).$$

When X is smooth there is a canonical equivalence of $\mathrm{Crys}^{r}(X)$ with the category $\mathscr{D}^{r}(X)$ of right $\mathscr{D}$-modules on X. Under this equivalence the adjoint pair $(\pi_*^{\mathrm{IndCoh}}, \pi^!)$ identifies with

$$\mathrm{IndCoh}(X) \underset{\mathrm{oblv}^r}{\overset{\mathrm{ind}^r}{\rightleftarrows}} \mathscr{D}^{r}(X).$$

For arbitrary X there is an equivalence

$$\Upsilon_{X_{\mathrm{dR}}} : \mathrm{Crys}^{\ell}(X) \xrightarrow{\sim} \mathrm{Crys}^{r}(X)$$

(the functor Υ is defined for any prestack). When X is smooth this agrees with the usual equivalence $\mathscr{D}^{\ell}(X) \tilde{\to} \mathscr{D}^{r}(X)$ that sends $\mathscr{F} \mapsto \mathscr{F} \otimes_{\mathscr{O}_X} \omega_X$.

A morphism $f : X \to Y$ gives rise to $f_{\mathrm{dR}} : X_{\mathrm{dR}} \to Y_{\mathrm{dR}}$. Then the functors $f_{\mathrm{dR}}^!$ and $f_{\mathrm{dR},*}^{\mathrm{IndCoh}}$ agree with the usual !-pullback and de Rham pushforward for right $\mathscr{D}$-modules. In particular, if $Y = \mathrm{pt}$ and X is proper then $f_{\mathrm{dR},*}^{\mathrm{IndCoh}} f_{\mathrm{dR}}^! k$ computes the de Rham homology of X.

We will see that some of this good behavior is explained by the fact that X_{dR} is an *inf-scheme*. Roughly speaking, an inf-scheme is a prestack whose reduced part is a scheme (the other condition is admitting deformation theory, which we discuss a bit below). This allows us to study formal schemes (including schemes) and de Rham prestacks within the same framework, so that ind-coherent sheaves and crystals are on an equal footing.

4 Derived algebraic geometry

Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be morphisms of schemes. The basic problem we face is that if neither f nor g is flat, then the classical fiber product $(X \times_Z Y)^{\mathrm{cl}}$ loses information. Namely, if $X = \mathrm{Spec}\, A$, $Y = \mathrm{Spec}\, B$, and $Z = \mathrm{Spec}\, C$ are affine then

$$(X \times_Z Y)^{\mathrm{cl}} = \mathrm{Spec}\, H^0(A \otimes_C B)$$

(the tensor product is, of course, derived).

What kind of gadget is $A \otimes_C B$? It is a connective complex of vector spaces with a commutative algebra structure, i.e. a connective commutative DG algebra (here "commutative" means "obeying the Koszul sign rule"). In derived algebraic geometry this is the sort of object one allows as a coordinate ring, and we have

$$X \times_Z Y = \mathrm{Spec}(A \otimes_C B).$$

This is our first example of an affine DG scheme. One should think of it as some kind of infinitesimal thickening of $(X \times_Z Y)^{\mathrm{cl}}$, in much the same way as a non-reduced scheme structure. In general fiber products of DG schemes are glued from the affine case, just like the classical case.

For example, if $X, Y \subset Z$ are closed subschemes, then the "derived intersection" $X \times_Z Y$ is very natural from the perspective of intersection theory. In this setting the statement of Bézout's theorem becomes true for an identical pair of plane curves, similarly to how taking non-reduced intersections allows one to drop transversality as a hypothesis.

Say $X = Y = \{0\} \to \mathbb{A}^1 = Z$. Then we have

$$\{0\} \times_{\mathbb{A}^1} \{0\} = k[t]/(t^2)$$

where $\deg t = -1$. Also

$$\{0\} \times_{\{0\} \times_{\mathbb{A}^1} \{0\}} \{0\} = k[t]$$

where $\deg t = -2$, and so on.

What does all of this have to do with coherent sheaves? Given a cartesian square

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z, \end{array}$$

we have base change isomorphisms

$$g^* f_* \tilde{\to} g'^{,*} f'_* \text{ and } g^! f_*^{\mathrm{IndCoh}} \tilde{\to} g'^{,!} f'^{,\mathrm{IndCoh}}_*$$

of functors $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ and $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ respectively. It is essential that we used the derived fiber product here! Otherwise both formulas are already false for the example $\{0\} \times_{\mathbb{A}^1} \{0\}$.

5 Deformation theory and Lie theory

Another attractive feature of DAG is that it is the natural home of deformation theory. We will say what it means for a prestack $\mathscr{X}$ (in the DAG sense) to *admit deformation theory*. An important consequence of this is that $\mathscr{X}$ admits a *tangent complex* $\mathscr{T}_{\mathscr{X}}$, which is an object of $\mathrm{IndCoh}(\mathscr{X})$. When $\mathscr{X}$ is a smooth classical scheme, this is just $\Upsilon_{\mathscr{X}}$ applied to the ordinary tangent bundle of $\mathscr{X}$. This formalism also applies in more interesting cases, e.g. $\mathscr{T}_{\mathrm{pt}/G} = \Upsilon_{\mathrm{pt}/G}(\mathfrak{g}[1])$, where $\mathfrak{g}$ denotes the adjoint representation.

To see why DAG is relevant, consider the DG scheme $D = k[t]/(t^2)$, where $\deg t = -n$ for some $n \geq 0$. Now if $x : \mathrm{Spec}\, k \to \mathscr{X}$, then $H^n(x^! \mathscr{T}_{\mathscr{X}})$ is identified with the vector space of extensions of x to $D \to \mathscr{X}$ (taken up to isomorphism).

The strongest results apply to morphisms which are completely controlled by their deformation theory. More precisely, if $\mathscr{X}$ is any prestack, then a *formal moduli problem under* $\mathscr{X}$ is a nil-isomorphism $\mathscr{X} \to \mathscr{Y}$, meaning it induces an isomorphism on the reduced parts, where $\mathscr{Y}$ is a prestack with deformation theory. Now $\mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ has the structure of a groupoid, and in fact it is a *formal moduli problem over* $\mathscr{X}$, meaning (either projection) $\mathscr{X} \times_{\mathscr{Y}} \mathscr{X} \to \mathscr{X}$ is a nil-isomorphism and the fibers are inf-schemes. So we call $\mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ a *formal groupoid* over $\mathscr{X}$.

Theorem 5.1. *If $\mathscr{X}$ admits deformation theory, then the functor from formal moduli problems under $\mathscr{X}$ to formal groupoids over $\mathscr{X}$ given by $\mathscr{Y} \mapsto \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is an equivalence.*

Such a theorem is only possible in the context of DAG! The inverse functor is given by taking the quotient of $\mathscr{X}$ by a formal groupoid, and the main point of the theorem is that this operation is well-defined on prestacks with deformation theory. As a corollary, one obtains the pointed version.

Corollary 5.1.1. *If $\mathscr{X}$ admits deformation theory, then the functor from pointed formal moduli problems over $\mathscr{X}$ to formal groups over $\mathscr{X}$ given by $\mathscr{Y} \mapsto \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is an equivalence.*

What about Lie theory? It is not so easy to work with Lie algebras in the derived setting, since one generally cannot write explicit formulae. The most convenient way to access the Lie operad is as the (shifted) Koszul dual of the cocommutative co-operad. We will see that this gives a conceptual explanation for the Lie algebra structure on the tangent space to a group at the identity. In particular, we have the following result.

Theorem 5.2. *For any prestack $\mathscr{X}$ the functor $\mathscr{G} \mapsto 1^! \mathscr{T}_{\mathscr{G}}$ lifts to an equivalence from formal groups over $\mathscr{X}$ to Lie algebras in* $\mathrm{IndCoh}(\mathscr{X})$.

One would also like to have a "linearized" version of formal groupoids: this leads to the notion of Lie algebroid. Unfortunately, it seems to be impossible to give an algebraic definition of Lie algebroids, so we will simply define Lie algebroids to *be* formal groupoids, which seems reasonable in light of Theorem 5.2. For example, the tangent Lie algebroid $\mathscr{T}_{\mathscr{X}}$ is identified with the formal groupoid given by completing $\mathscr{X} \times \mathscr{X}$ along the diagonal. Under the equivalence of Theorem 5.1 this groupoid corresponds to the formal moduli problem $\mathscr{X} \to \mathscr{X}_{\mathrm{dR}}$.

Finally, we will discuss some more advanced applications of this material to geometry. For example, we will define, for any formal moduli problem $\mathscr{X} \to \mathscr{Y}$ under a prestack $\mathscr{X}$ with deformation theory, the sequence of infinitesimal neighborhoods of $\mathscr{X}$ in $\mathscr{Y}$, and prove that $\mathscr{Y}$ is the colimit of (i.e. is exhausted by) these neighborhoods. By applying this construction to $\mathscr{X} \to \mathscr{X}_{\mathrm{dR}}$ we can put a filtration on $\omega_{\mathscr{X}}$ which agrees with the Hodge filtration in classical cases. Recall that in classical differential geometry one defines this filtration by writing down the de Rham resolution, whose differential is defined by an explicit formula, and then using the stupid filtration. This approach is doomed to fail in DAG. Instead, one has to use a geometric construction, namely deformation to the normal cone.