IND-COHERENT SHEAVES AND SERRE DUALITY II

1. Introduction

Let X be a smooth projective variety over a field k of dimension n. Let V be a vector bundle on X. In this case, we have an isomorphism

$$H^i(X, \mathcal{V}) \simeq H^{n-i}(X, \mathcal{V}^\vee \otimes \omega_X)^\vee$$

where ω_X is the canonical bundle on X.

There is a way of intepreting this in terms of an adjunction. The functor

$$R\Gamma : \mathrm{Coh}(X) \to \mathrm{Coh}(*) = \mathrm{Vect}_k^{\mathrm{fin}}$$

admits a right adjoint which sends $k \in \mathrm{Vect}_k^{\mathrm{fin}}$ to $\omega_X[n]$. It follows that if V is a vector bundle on X, we have an equivalence of complexes

$$\hom_{\mathrm{Coh}(X)}(V, \omega_X[n]) \simeq \hom_{\mathrm{Vect}_k}(R\Gamma(V), k).$$

Dualizing both sides, and taking cohomology in degree i, gives the claim.

More generally, whenever $p : X \to Y$ is a proper morphism of schemes (almost of finite type over a field), then we obtain a pushforward

$$p_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y),$$

which admits a cocontinuous right adjoint

$$p^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X).$$

When p is smooth, then

$$p^! = p^* \otimes \omega_{X/Y}[d]$$

where $\omega_{X/Y}$ is the relative canonical sheaf and d is the relative dimension. This adjunction gives Serre duality in a family. In general, we note that $p^!$ does not preserve Coh.

In fact, there is a way of defining $p^!$ as a functor on IndCoh for every morphism of schemes. Given a morphism $p : X \to Y$, one has a functor $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$. This has two basic properties:

(1) When p is proper, then $p^!$ is the right adjoint to p_*.
(2) When p is an open immersion, then $p^! = p^*$ (which preserves Coh).

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The upper shriek construction $p^!$, together with the lower star functoriality p_*, realizes IndCoh out of a category of schemes and *correspondences* between them.

2. Baby example: the dual numbers

2.1. Let $A = k[\epsilon]/\epsilon^2$ be the ring of dual numbers. Let's work out what $\mathrm{IndCoh}(\mathrm{Spec} A)$ is.

The observation is that $\mathrm{IndCoh}(\mathrm{Spec} A)$ is compactly generated by a single object, namely k, because every object $\mathrm{Coh}(A)$ can be built by taking extensions of shifts of k. There's a general principle (Schwede-Shipley) that a DG category $\mathcal{C}$ generated by a single compact object X is equivalent to $\mathrm{Mod}(\mathrm{End}_{\mathcal{C}}(X))$, i.e., DG modules over the DG-algebra $\mathrm{End}_{\mathcal{C}}(X)$. It follows that

$$\mathrm{IndCoh}(\mathrm{Spec} A) \simeq \mathrm{Mod}(\mathrm{End}_{k[\epsilon]/\epsilon^2}(k, k)).$$

As a DG-algebra, $\mathrm{End}_{k[\epsilon]/\epsilon^2}(k, k)$ can be modeled as the free *associative* algebra $\mathbb{Q}[t_{-1}]$ where $|t_{-1}| = -1$ (in homological grading).[1] In particular, to give a module over this DG-ring equates to giving a DG-vector space V together with a map $V \to V[-1]$.

To conclude:

Proposition 1. We have an equivalence $\mathrm{IndCoh}(\mathbb{Q}[\epsilon]/\epsilon^2) \simeq \mathrm{Mod}(k[t_{-1}])$ where $\mathbb{Q}[t_{-1}]$ is the free associative algebra on $k[-1]$.

2.2. We have a natural inclusion $i : \mathrm{Spec} k \hookrightarrow \mathrm{Spec} A$. Let's work out what $i^!$ is. We have a natural functor

$$i_* : \mathrm{Vect}_k \simeq \mathrm{IndCoh}(k) \to \mathrm{IndCoh}(A) = \mathrm{Mod}(k[t_{-1}]).$$

In this setup, it carries k to $k[t_{-1}]$ and is therefore given by tensoring $V \mapsto V \otimes k[t_{-1}]$. The right adjoint is just the forgetful functor

$$\mathrm{Mod}(k[t_{-1}]) \to \mathrm{Vect}_k.$$

2.3. Similarly, we have a morphism $p : \mathrm{Spec} A \to \mathrm{Spec} k$. Let's work out $p_*, p^!$.

(1) The functor $p_* : \mathrm{Coh}(A) \to \mathrm{Coh}(k)$ carries k to itself and, in general, it follows that $\mathrm{IndCoh}(A) \simeq \mathrm{Mod}(k[t_{-1}]) \to k$ is given by sending a $k[t_{-1}]$-module M to $M \otimes_{k[t_{-1}]} k$.
(2) It follows that $p^!$ is the right adjoint, so it carries a k-vector space V to the $k[t_{-1}]$-module V, where t_{-1} acts by zero.

[1] This is not great notation. There is also a free commutative algebra on a degree -1 class; in this case it squares to zero.

Alternatively, we can work as follows for compact objects. This would work for any local artinian k-algebra A.

(1) $p_* : \mathrm{Coh}(A) \to \mathrm{Vect}_k$ is just the forgetful functor.
(2) Therefore, $p^!$ is given by the *right adjoint* to the forgetful functor, which sends a k-vector space V to $\hom_k(A, V)$ and considers that as an A-module. For $A = k[\epsilon]/\epsilon^2$, this is just A itself as an A-module; such rings are called *Gorenstein.*

3. Correspondences

3.1. The baby case. Let $\mathcal{C}$ be a category.

Definition 2. A **correspondence** between objects $X, Y \in \mathcal{C}$ is a diagram

(1)
$$X \longleftarrow Z \longrightarrow Y .$$

Definition 3. Suppose $\mathcal{C}$ has fiber products. Then we can define a new category $\mathrm{Corr}_0(\mathcal{C})$ as follows:

(1) The objects are the same as those of $\mathcal{C}$.
(2) We have $\hom_{\mathrm{Corr}_0(\mathcal{C})}(X, Y)$ to be the set of isomorphism classes of diagrams as in (1).
(3) The composition of correspondences is given by pull-back. (Draw diagram here.)

There is a functor $\mathcal{C} \to \mathrm{Corr}_0(\mathcal{C})$ which sends $X \mapsto X$ and the morphism $f : X \to Y$ to the correspondence

(2)
$$X \longleftarrow X \xrightarrow{f} Y .$$

Similarly, there is a functor $\mathcal{C}^{op} \to \mathrm{Corr}_0(\mathcal{C})$. Therefore, $\mathrm{Corr}_0(\mathcal{C})$ encodes a type of "bivariant" functoriality.

Example 1. Let $\mathcal{C}$ be the category of finite sets. Then we get the category of finitely generated free abelian monoids for $\mathrm{Corr}_0(\mathcal{C})$.

Proposition 4. Let $\mathcal{D}$ be a category. To give a functor $F : \mathrm{Corr}_0(\mathcal{C}) \to \mathcal{D}$ amounts to giving the following data:

(1) For each object $x \in \mathcal{C}$, an object $Fx \in \mathcal{D}$.
(2) For every map $f : x \to y$ in $\mathcal{C}$, a morphism $f_* : Fx \to Fy$ and $f^! : Fy \to Fx$.
(3) $f_*, f^!$ are functorial. E.g., $(g \circ f)_* = g_* \circ f_*$, etc.

(4) We have the base-change formula. Given a pullback diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array},$$

we have an equality of morphisms

$$g^! f_* = f'_* g'^! : FX \to FY'.$$

3.2. Let Sch denote the $(\infty, 1)$-category of DG schemes almost of finite type over k.

In the previous lecture, we had a functor

$$\mathrm{Sch} \to \mathrm{Cat}_k, \quad X \mapsto \mathrm{IndCoh}(X),$$

with the functoriality arising from the pushforward. For a *proper* map p, p_* preserves compact objects so that it admits a cocontinuous right adjoint $p^!$. For an open immersion p, p_* has a *left* adjoint $p^! = p^*$.

Theorem 5 (Nagata)**.** p admits a factorization $p_2 \circ p_1$ where p_1 is an open immersion and p_2 is proper.

One then defines $p^! = p_1^! \circ p_2^!$. A key point is that the construction of $p^!$ is independent of the choice of factorization. This uses the fact that "the category of all factorizations is contractible." Moreover, $p^!$ defines a contravariant functor from DG schemes to categories.

So now we have for every map $p : X \to Y$ the following two pieces of data:

- $p_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ (defined previously).
- $p^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$.

A priori, these two are not related to one another (unless p is either an open immersion or a proper map). However, we have two functors and they satisfy a base-change relation.

Theorem 6. Given a cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array},$$

we have a natural isomorphism of functors

$$g^! f_* \simeq f'_* g'^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y').$$

Suppose g is proper. Then we can obtain a natural map in one direction,

$$f'_* g'^! \to g^! f_*$$

is adjoint to a map

$$g_* f'_* g'^! = f_* g'_* g'^! \to f_*$$

which is the counit map. This map is the one that gives the base-change isomorphism.

3.3. Let $\mathcal{C}$ be an $(\infty, 1)$-category with fiber products.

Construction. Let S be a collection of morphisms in $\mathcal{C}$ that is stable under base-change and composition. There is an $(\infty, 2)$-category $\mathrm{Corr}_S(\mathcal{C})$. Informally:

- The objects are the same as those of $\mathcal{C}$.
- The $(\infty, 1)$-category $\hom_{\mathrm{Corr}_S(\mathcal{C})}(X, Y)$ is the ∞-category of correspondences from X to Y of the form

(3)
$$X \leftarrow Z \rightarrow Y \quad .$$

  We allow morphisms of diagrams where the map between "Z's" belongs to S.

Theorem 7. We have a functor $\mathrm{Corr}_{\mathrm{proper}}(\mathrm{Sch}) \to \mathrm{Cat}_k$ sending $X \mapsto \mathrm{IndCoh}(X)$. The functoriality in both directions is given by $p_*, p^!$.

The use of the $(\infty, 2)$-category encodes the fact that when p is proper, then $p_*, p^!$ are adjoint.

4. The symmetric monoidal structure

4.1. Recall that Cat_k is a symmetric monoidal ∞-category via the Lurie tensor product. Given $\mathcal{C}, \mathcal{D}$, the tensor product $\mathcal{C} \otimes \mathcal{D}$ (always k-linear) is the universal DG category receiving a k-bilinear functor $\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_k \mathcal{D}$.

For example, if R, R' are DG algebras then $\mathrm{Mod}(R) \otimes \mathrm{Mod}(R') = \mathrm{Mod}(R \otimes R')$.

If $Y, Y' \in \mathrm{Sch}$, then we have an equivalence

$$\mathrm{IndCoh}(Y) \otimes \mathrm{IndCoh}(Y') \simeq \mathrm{IndCoh}(Y \times Y').$$

This comes from the natural functor

$$\boxtimes : \mathrm{Coh}(Y) \otimes \mathrm{Coh}(Y') \to \mathrm{Coh}(Y \times Y').$$

4.2. The $(\infty, 2)$-category Corr(Sch) of schemes and correspondences (and proper morphisms between correspondences) is naturally a symmetric monoidal $(\infty, 2)$-category. This arises from taking the product of schemes and the product of correspondences, etc.

Theorem 8. The functor Corr(Sch) $\to \mathrm{Cat}_k$ has the structure of a symmetric monoidal functor.

It follows that for example IndCoh(X) is naturally a symmetric monoidal ∞-category. This is not obvious from the definition, because Coh(X) is not a symmetric monoidal ∞-category in general. The tensor structure arises from the "shriek" tensor product. Given $\mathcal{F}, \mathcal{G} \in \mathrm{IndCoh}(X)$, one forms $\mathcal{F} \boxtimes \mathcal{G} \in \mathrm{IndCoh}(X \times X)$ and forms $\Delta^!(\mathcal{F} \boxtimes \mathcal{G})$ where $\Delta : X \hookrightarrow X \times X$ is the diagonal. In other words, we are using the fact that every scheme X is canonically a commutative algebra in correspondences.

The unit is given by the dualizing complex.

4.3. Duality. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Let $X \in \mathcal{C}$ be an object.

A **dual** of X is an object Y equipped with maps

$$\mathrm{coev} : 1 \to Y \otimes X, \quad \mathrm{ev} : X \otimes Y \to 1,$$

such that

$$X \simeq X \otimes 1 \stackrel{1_X \otimes \mathrm{coev}}{\to} X \otimes Y \otimes X \stackrel{\mathrm{coev} \otimes 1_X}{\to} X$$

is homotopic to the identity, and similarly for a map $Y \to Y$.

An object is said to be **dualizable** if it has a dual. The operation of duality induces an anti-equivalence on the dualizable objects of $\mathcal{C}$. Given $f : X \to X'$, one obtains a dual morphism. Let Y be the dual of X and let Y' be the dual of X'. Given f, we obtain f^{dual} via

$$Y' \to Y' \otimes 1 \to Y' \otimes X \otimes Y \to Y' \otimes X' \otimes Y \to Y.$$

A symmetric monoidal functor preserves dualizable objects and duals.

In Corr(Sch), every scheme is dualizable: in fact, every object is self-dual. The duality morphisms are the tautological ones. The unit is $* = \mathrm{Spec} k$ and the coevaluation and evaluation maps come from the diagonal. The self-dual of a right-way morphism is the wrong-way morphism.

It follows that IndCoh(S), in the category of DG categories, is self-dual. Moreover, f_* and $f^!$ are dual in the sense of DG categories.

4.4. We now need to unwind what duality of compactly generated categories is.

Proposition 9. Let $\mathcal{C}$ be a small DG category. Then $\mathrm{Ind}(\mathcal{C})$ is dualizable and $\mathrm{Ind}(\mathcal{C}^{op})$ is the dual.

Proof. The Yoneda embedding gives the coevaluation map. There is a bilinear functor

$$\mathcal{C} \times \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Ind}(\mathcal{C})$$

given by the Yoneda functor, and the unit corresponds to the identity (or inclusion) functor. □

Unwinding this, we get an identification

$$\mathrm{Coh}(S) \simeq \mathrm{Coh}(S)^{op},$$

which is a form of Grothendieck duality in this setting. The associated pairing

$$\mathrm{Coh}(S) \times \mathrm{Coh}(S) \to \mathrm{Vect}_k$$

can be identified as follows: take $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(S)$, form their box product $\mathcal{F} \otimes \mathcal{G}$, take $\Delta^!(\mathcal{F} \boxtimes \mathcal{G})$ on X (the shriek tensor product) and then take the global sections of that, i.e., apply the lower star to a point. Equivalently, it is

$$\mathrm{hom}_{\mathrm{Coh}(S \times S)}(\Delta_* \mathcal{O}_S, \mathcal{F} \boxtimes \mathcal{G}).$$

Definition 10. Let S be a scheme and let $p : S \to \mathrm{Spec} k$. Then $p^!(k) \in \mathrm{IndCoh}(S)$ is called the **dualizing complex** ω_S.

5. Dualizing complexes

There is another point of view on the anti-equivalence of $\mathrm{Coh}(S)$ that one obtains. For simplicity, we take $S = \mathrm{Spec} A$.

Definition 11. A **dualizing module** over a k-algebra A almost of finite type is an object $K \in \mathrm{Coh}(A)$ such that:

(1) There exists n such that for every discrete A-module L, we have that $\mathrm{hom}_A(L, K)$ is concentrated in degrees $[-n, n]$ (i.e., K has finite injective dimension over A).
(2) The natural map $A \to \mathrm{hom}_A(K, K)$ is an equivalence.

Example 2. Suppose A is a local ring. If A itself is a dualizing module for itself, then A is said to be **Gorenstein** (i.e., it has finite injective dimension over itself). A complete intersection local ring is an example of a Gorenstein ring.

Theorem 12. Given a dualizing module $K \in \mathrm{Coh}(A)$, we obtain an anti-equivalence

$$\mathrm{Coh}(A) \simeq \mathrm{Coh}(A)^{op}, \quad M \mapsto \mathrm{hom}_A(M, K).$$

Proposition 13. Let $A \to B$ be a morphism of k-algebras almost of finite type. Suppose B is almost perfect as an A-module. Then if $K \in \mathrm{Coh}(A)$ is a dualizing module, the object $\mathrm{hom}_A(B, K) \in \mathrm{Coh}(B)$ is a dualizing module.

Proposition 14. Let A be an almost finite type algebra over k. Let $p : \mathrm{Spec} A \to \mathrm{Spec} k$ be the forgetful functor. Then $p^! k \in \mathrm{IndCoh}(A)$ actually belongs to $\mathrm{Coh}(A)$ and is a dualizing module.

Proof. □