IndCoh Seminar:
Ind-coherent sheaves I

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1 Finiteness conditions

1.1 Fix a cocomplete category $\mathscr{C}$ (as usual "category" means "∞-category"). This section contains a discussion of finiteness conditions on objects of categories. Not all of the material will be used in Section 2, but it will be important later.

Definition 1.1.1. An object X of $\mathscr{C}$ is called *compact* if the functor $\mathscr{C} \to \mathrm{Spc}$ corepresented by X preserves filtered colimits.

Equivalently, for all filtered diagrams $F : \mathscr{I} \to \mathscr{C}$, any morphism $X \to \operatorname{colim} F$ factors through $F(i) \to F$ for some $i \in \mathscr{I}$.

Suppose that $\mathscr{C}$ is stable. Then the functor corepresented by an object X naturally factors through the functor $\Omega^\infty : \mathrm{Sptr} \to \mathrm{Spc}$, where Sptr is the category of spectra. The resulting functor $\mathscr{C} \to \mathrm{Sptr}$ preserves all colimits, or equivalently direct sums, if and only if X is compact. If $\mathscr{C}$ has the structure of a DG category over a field k, then one can replace Sptr by Vect and Ω^∞ by the Dold-Kan functor.

Given a category $\mathscr{C}_0$ with finite colimits, its *ind-completion* $\mathrm{Ind}(\mathscr{C}_0)$ is a cocomplete category equipped with a functor $\mathscr{C}_0 \to \mathrm{Ind}(\mathscr{C}_0)$ (automatically fully faithful) with the property that for any cocomplete category $\mathscr{D}$, restriction induces an equivalence from colimit-preserving functors $\mathrm{Ind}(\mathscr{C}_0) \to \mathscr{D}$ to right exact (i.e. finite-colimit preserving) functors $\mathscr{C}_0 \to \mathscr{D}$. More precisely, Ind is the left adjoint of the forgetful functor from cocomplete categories and colimit-preserving functors to categories with finite colimits and right exact functors. Informally speaking, $\mathrm{Ind}(\mathscr{C}_0)$ is obtained from $\mathscr{C}_0$ by freely adjoining filtered colimits.

One can construct $\mathrm{Ind}(\mathscr{C}_0)$ as follows: since the category of presheaves $\mathrm{Fun}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Spc})$ is cocomplete, the Yoneda embedding extends to a functor

$$\mathrm{Ind}(\mathscr{C}_0) \longrightarrow \mathrm{Fun}(\mathscr{C}_0^{\mathrm{op}}, \mathrm{Spc}),$$

which is fully faithful with essential image consisting of functors which preserve finite limits. On functors Ind is the operation of right Kan extension. If $\mathscr{C}_0$ is stable or a (non-cocomplete) DG category then one can replace Spc with Sptr or Vect. In particular $\mathrm{Ind}(\mathscr{C}_0)$ is then stable or DG respectively.

Proposition 1.1.2. *An object of* $\mathrm{Ind}(\mathscr{C}_0)$ *is compact if and only if it is a retract of an object in* $\mathscr{C}_0$.

Proof. We prove the "only if" implication, leaving the "if" direction to the interested reader. Suppose that X is a compact object of $\mathrm{Ind}(\mathscr{C}_0)$ and write $X \tilde{\to} \operatorname{colim}_i X_i$ for some filtered diagram $\mathscr{I} \to \mathscr{C}_0$. But then by compactness the identity on X factors through some X_i, which is to say X is a retract of X_i. □

The following result will be used to define the t-structure on ind-coherent sheaves.

Proposition 1.1.3. *If* $\mathscr{C}_0$ *is a stable category with a t-structure, then* $\mathrm{Ind}(\mathscr{C}_0)$ *has a unique t-structure such that* $\mathscr{C}_0 \to \mathscr{C}$ *is t-exact and* $\tau^{\leq 0}$ *is continuous.*

Proof. Observe that the inclusion $\mathscr{C}_0^{\leq 0} \to \mathscr{C}_0$ induces a fully faithful functor

$$\mathscr{C}^{\leq 0} := \mathrm{Ind}(\mathscr{C}_0^{\leq 0}) \longrightarrow \mathrm{Ind}(\mathscr{C}_0) =: \mathscr{C}.$$

We claim that this subcategory defines a t-structure on $\mathscr{C}$. The right adjoint to the inclusion is given by $\tau^{\leq 0} := \mathrm{Ind}(\tau^{\leq 0})$, and in particular is continuous. Now suppose $X \to Y \to Z$ is an exact triangle where X and Z belong to $\mathscr{C}^{\leq 0}$. Then $Y = \mathrm{fib}(Z \to \Sigma X)$, and we can write $X = \mathrm{colim}\, X_i$ and $Z = \mathrm{colim}_j\, Z_j$ for some filtered diagrams. Since $\tau^{\leq 0}$ is a right adjoint and hence preserves limits, we have

$$\tau^{\leq 0} Y \xrightarrow{\sim} \mathrm{fib}(\mathrm{colim}_j\, \tau^{\leq 0} Z_j \to \mathrm{colim}_i\, \tau^{\leq 0} X_i).$$

But we know that $\tau^{\leq 0} X_i \tilde{\to} X_i$ for every i and similarly for the Z_j, so it follows that $\tau^{\leq 0} Y \tilde{\to} Y$.

□

For a cocomplete category $\mathscr{C}$ we denote by $\mathscr{C}_c$ the full subcategory of compact objects. We call $\mathscr{C}$ *compactly generated* if the canonical functor $\mathrm{Ind}(\mathscr{C}_c) \to \mathscr{C}$ is an equivalence. There is another way that one might formulate this notion: the subcategory *generated* by a collection of objects is the smallest cocomplete subcategory containing them.

Proposition 1.1.4. *A cocomplete category $\mathscr{C}$ is compactly generated if and only if there is a collection of compact objects which generates $\mathscr{C}$.*

Proof. The "only if" direction is simple, so we prove the "if" direction. The hypothesis is clearly equivalent to essential surjectivity of $\mathrm{Ind}(\mathscr{C}_c) \to \mathscr{C}$, so let us show that this functor is always fully faithful. Fix objects X and Y of $\mathrm{Ind}(\mathscr{C}_c)$, presented as $X = \mathrm{colim}_i\, X_i$ and $Y = \mathrm{colim}_j\, Y_j$ where $i \mapsto X_i$ and $j \mapsto Y_j$ are filtered diagrams in $\mathscr{C}_c$. We need to prove that

$$\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C}_c)}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(\operatorname*{colim}_i X_i, \operatorname*{colim}_j Y_j),$$

where the colimits on the right hand side are taken in $\mathscr{C}$. Indeed, both sides are identified with

$$\lim_i \operatorname*{colim}_j \mathrm{Hom}_{\mathscr{C}_c}(X_i, Y_j).$$

□

Example 1.1.5. An object of Vect is compact if and only if it is bounded with finite-dimensional cohomologies. More generally, for an almost finite type scheme S an object $\mathscr{F}$ of $\mathrm{QCoh}(S)$ is compact if and only if it is perfect. If S is classical then $\mathscr{F}$ is perfect if and only if it is isomorphic to a bounded complex of vector bundles. In general $\mathrm{Perf}(S)$ is the smallest full subcategory $\mathrm{QCoh}(S)$ which is stable, contains $\mathscr{O}_S$, and is closed under taking direct summands. Moreover, $\mathrm{QCoh}(S)$ is compactly generated.

The following is a very useful property of compactly generated categories in practice.

Exercise 1.1.6. Let $F : \mathscr{C} \to \mathscr{D}$ be a continuous functor between compactly generated categories. Then the right adjoint $G : \mathscr{D} \to \mathscr{C}$, which exists by the adjoint functor theorem, is continuous if and only if F preserves compact objects.

1.2 Now suppose that $\mathscr{C}$ has a symmetric monoidal structure.

Definition 1.2.1. An object X of $\mathscr{C}$ is called *dualizable* if there exists an object $X^\vee$ and morphisms $\eta : \mathbb{1} \to X \otimes X^\vee$ and $\epsilon : X^\vee \otimes X \to \mathbb{1}$ such that

$$X \overset{\eta \otimes \mathrm{id}_X}{\longrightarrow} X \otimes X^\vee \otimes X \overset{\mathrm{id}_X \otimes \epsilon}{\longrightarrow} X$$

is homotopic to id_X and

$$X^\vee \overset{\mathrm{id}_{X^\vee} \otimes \eta}{\longrightarrow} X^\vee \otimes X \otimes X^\vee \overset{\epsilon \otimes \mathrm{id}_{X^\vee}}{\longrightarrow} X^\vee$$

is homotopic to $\mathrm{id}_{X^\vee}$.

Suppose X is dualizable and fix $X^\vee$ and $\epsilon : X^\vee \otimes X \to \mathbb{1}$ as above. Then there is a canonical isomorphism

$$X^\vee \otimes Y \tilde{\longrightarrow} \underline{\mathrm{Hom}}_{\mathscr{C}}(X, Y),$$

where the latter object is the internal Hom which represents the functor

$$Z \mapsto \mathrm{Hom}_{\mathscr{C}}(Z \otimes X, Y).$$

In particular there is a canonical choice of dual $X^\vee = \underline{\mathrm{Hom}}_{\mathscr{C}}(X, \mathbb{1})$, and we can take ϵ to be evaluation.

Exercise 1.2.2. Show that if X is dualizable in $\mathscr{C}$ then there is a canonical isomorphism $X \tilde{\to} (X^\vee)^\vee$, and deduce that $X \mapsto X^\vee$ extends to a contravariant autoequivalence of the full category of $\mathscr{C}$ consisting of dualizable objects.

Exercise 1.2.3. Suppose that $\mathscr{C}$ is cocomplete and that the tensor product preserves colimits in each variable. Prove that if the unit object of $\mathscr{C}$ is compact, then any dualizable object is compact.

Example 1.2.4. For an almost finite type scheme S, an object of $\mathrm{QCoh}(S)$ is dualizable if and only if it is perfect, so in this case dualizability is equivalent to compactness.

Let $\mathrm{Cat}_{\mathrm{stab}}^{\mathrm{cocmpl}}$ be the category of cocomplete stable categories with continuous and exact (i.e. colimit-preserving) functors. Recall that $\mathrm{Cat}_{\mathrm{stab}}^{\mathrm{cocmpl}}$ has a canonical symmetric monoidal structure, called the *Lurie tensor product*, whose unit is Sptr. For two cocomplete stable categories $\mathscr{C}$ and $\mathscr{D}$ there is a functor $\mathscr{C} \times \mathscr{D} \to \mathscr{C} \otimes \mathscr{D}$ such that for any $\mathscr{E}$ in $\mathrm{Cat}_{\mathrm{stab}}^{\mathrm{cocmpl}}$ the induced functor

$$\mathrm{Fun}_{\mathrm{ex}}^{\mathrm{cts}}(\mathscr{C} \otimes \mathscr{D}, \mathscr{E}) \longrightarrow \mathrm{Fun}(\mathscr{C} \times \mathscr{D}, \mathscr{E})$$

is fully faithful with essential image consisting of functors $\mathscr{C} \times \mathscr{D} \to \mathscr{E}$ which are continuous and exact in each variable.

Tensor product of complexes makes Vect into a commutative algebra object in $\mathrm{Cat}_{\mathrm{stab}}^{\mathrm{cocmpl}}$, i.e. a symmetric monoidal category whose tensor product is continuous and exact in each variable. The category DGCat of DG categories can then be defined as the category of Vect-modules in $\mathrm{Cat}_{\mathrm{stab}}^{\mathrm{cocmpl}}$.

Exercise 1.2.5. For any stable categories $\mathscr{C}$ and $\mathscr{D}$ there is a canonical equivalence

$$\mathrm{Ind}(\mathscr{C} \times \mathscr{D}) \tilde{\longrightarrow} \mathrm{Ind}(\mathscr{C}) \otimes \mathrm{Ind}(\mathscr{D}).$$

A *dualizable category* is a dualizable object of $\mathrm{Cat}_{\mathrm{stab}}^{\mathrm{cocmpl}}$. Dualizable categories have favorable properties with respect to limits and colimits.

The following result produces many examples of dualizable categories.

Proposition 1.2.6. *A compactly generated stable category $\mathscr{C}$ is dualizable with dual*

$$\mathrm{Ind}((\mathscr{C}_c)^{\mathrm{op}}) \tilde{\longrightarrow} \mathrm{Fun}_{\mathrm{ex}}^{\mathrm{cts}}(\mathscr{C}, \mathrm{Sptr}).$$

Proof. The pairing ϵ is defined as the right Kan extension of

$$\mathrm{Hom} : (\mathscr{C}_c)^{\mathrm{op}} \times \mathscr{C}_c \longrightarrow \mathrm{Sptr}$$

along

$$(\mathscr{C}_c)^{\mathrm{op}} \times \mathscr{C}_c \longrightarrow \mathrm{Ind}((\mathscr{C}_c)^{\mathrm{op}} \times \mathscr{C}_c) \tilde{\longrightarrow} \mathrm{Ind}((\mathscr{C}_c)^{\mathrm{op}}) \otimes \mathscr{C}.$$

Under the canonical equivalence

$$\mathrm{Ind}((\mathscr{C}_c)^{\mathrm{op}}) \tilde{\longrightarrow} \mathrm{Fun}_{\mathrm{ex}}^{\mathrm{cts}}(\mathscr{C}, \mathrm{Sptr})$$

one can show that ϵ is given by evaluation. We denote this category by $\mathscr{C}^\vee$ for notational convenience, although of course we have not yet proved the duality.

We claim that for any category $\mathscr{D}$ in $\mathrm{Cat}_{\mathrm{stab}}^{\mathrm{cocmpl}}$ the functor

$$\mathscr{C}^\vee \otimes \mathscr{D} \longrightarrow \mathrm{Fun}_{\mathrm{ex}}^{\mathrm{cts}}(\mathscr{C}, \mathscr{D})$$

which corresponds to

$$\mathscr{C}^\vee \otimes \mathscr{C} \otimes \mathscr{D} \xrightarrow{\epsilon \otimes \mathrm{id}_{\mathscr{D}}} \mathrm{Sptr} \otimes \mathscr{D} = \mathscr{D}$$

is an equivalence. Once this is proved we can define η to be the unique continuous and exact functor $\mathrm{Sptr} \to \mathrm{End}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}) \tilde{\to} \mathscr{C}^\vee \otimes \mathscr{C}$ which sends the sphere spectrum to the identity, and from there it is not hard to check the necessary relations.

It suffices to prove that for any $\mathscr{E}$ in $\mathrm{Cat}^{\mathrm{cocmpl}}_{\mathrm{stab}}$ the functor

$$\mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathrm{Fun}_{\mathrm{ex}}(\mathscr{C}_c, \mathscr{D}), \mathscr{E}) \tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}, \mathscr{D}), \mathscr{E}) \longrightarrow \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}^\vee \otimes \mathscr{D}, \mathscr{E})$$

is an equivalence. First observe that passage to right adjoints and opposites defines an equivalence

$$\mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathrm{Fun}_{\mathrm{ex}}(\mathscr{C}_c, \mathscr{D}), \mathscr{E}) \tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{E}^{\mathrm{op}}, \mathrm{Fun}_{\mathrm{ex}}(\mathscr{C}_c, \mathscr{D})^{\mathrm{op}}).$$

But now we have

$$\begin{aligned} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{E}^{\mathrm{op}}, \mathrm{Fun}_{\mathrm{ex}}(\mathscr{C}_c, \mathscr{D})^{\mathrm{op}}) &\tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{E}^{\mathrm{op}}, \mathrm{Fun}_{\mathrm{ex}}((\mathscr{C}_c)^{\mathrm{op}}, \mathscr{D}^{\mathrm{op}})) \\ &\tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{E}^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}^\vee, \mathscr{D}^{\mathrm{op}})) \\ &\tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}^\vee, \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{E}^{\mathrm{op}}, \mathscr{D}^{\mathrm{op}})) \\ &\tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}^\vee, \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{D}, \mathscr{E})) \\ &\tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}^\vee \otimes \mathscr{D}, \mathscr{E}). \end{aligned}$$

□

Let $F : \mathscr{C} \to \mathscr{D}$ be a continuous exact functor between compactly generated stable categories which preserves compact objects. Write $F_c : \mathscr{C}_c \to \mathscr{D}_c$ for the resulting functor, so we obtain

$$\mathrm{Ind}(F_c^{\mathrm{op}}) : \mathscr{C}^\vee = \mathrm{Ind}((\mathscr{C}_c)^{\mathrm{op}}) \longrightarrow \mathrm{Ind}((\mathscr{D}_c)^{\mathrm{op}}) = \mathscr{D}^\vee.$$

By Exercise 1.1.6, the assumption that F preserves compact objects is equivalent to continuity of the right adjoint $G : \mathscr{D} \to \mathscr{C}$. Thus we have another functor $\mathscr{C}^\vee \to \mathscr{D}^\vee$, namely the dual $G^\vee$.

Proposition 1.2.7. *There is a canonical isomorphism* $\mathrm{Ind}(F_c^{\mathrm{op}}) \tilde{\to} G^\vee$.

Proof. Observe that

$$\mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}^\vee, \mathscr{D}^\vee) \tilde{\longrightarrow} \mathrm{Fun}^{\mathrm{cts}}_{\mathrm{ex}}(\mathscr{C}^\vee \otimes \mathscr{D}, \mathrm{Sptr}) \tilde{\longrightarrow} \mathrm{Fun}_{\mathrm{ex}}(\mathscr{C}_c^{\mathrm{op}} \times \mathscr{D}_c, \mathrm{Sptr}).$$

One checks that $\mathrm{Ind}(F_c^{\mathrm{op}})$ corresponds to the functor

$$\mathscr{C}_c^{\mathrm{op}} \times \mathscr{D}_c \longrightarrow \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \xrightarrow{F^{\mathrm{op}} \times \mathrm{id}_{\mathscr{D}}} \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \xrightarrow{\mathrm{Hom}_{\mathscr{D}}} \mathrm{Sptr},$$

while $G^\vee$ corresponds to

$$\mathscr{C}_c^{\mathrm{op}} \times \mathscr{D}_c \longrightarrow \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \xrightarrow{\mathrm{id}_{\mathscr{C}^{\mathrm{op}}} \times G} \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \xrightarrow{\mathrm{Hom}_{\mathscr{C}}} \mathrm{Sptr}\,.$$

The adjunction of F and G identifies these functors.

□

2 Ind-coherent sheaves

2.1 In this section we begin to set up the theory of ind-coherent sheaves. We will define the pushforward and pullback functors, but stop short of discussing base change and Serre duality.

Let S be a (derived) scheme. Recall that the DG category of quasi-coherent sheaves on S is defined by

$$\mathrm{QCoh}(S) := \lim_{\mathrm{Spec}\, A \to S} A\text{-mod},$$

where the limit runs over affine open subschemes of S. Observe that $\mathrm{QCoh}(S)$ has a natural t-stucture: an object $\mathscr{F}$ belongs to $\mathrm{QCoh}(S)^{\leq 0}$ if, for every open embedding $f : \mathrm{Spec}\, A \to S$, the pullback $f^*\mathscr{F}$ belongs to $A\text{-mod}^{\leq 0}$. This t-structure is compatible with filtered colimits, i.e. the truncation functor $\tau^{\leq 0}$ is continuous.

We take as given the the functor

$$\mathrm{QCoh}^* : \mathrm{Sch}^{\mathrm{op}} \longrightarrow \mathrm{DGCat},$$

where Sch denotes the category of schemes. By passing to right adjoints we obtain

$$\mathrm{QCoh}_* : \mathrm{Sch} \longrightarrow \mathrm{DGCat}\,.$$

Assume from now on that S is almost of finite type, and in particular quasi-compact. Then the (non-cocomplete) full subcategory $\mathrm{Perf}(S)$ (see Example 1.1.5) compactly generates $\mathrm{QCoh}(S)$, i.e.

$$\mathrm{Ind}(\mathrm{Perf}(S)) \xrightarrow{\sim} \mathrm{QCoh}(S).$$

There is another subcategory of "small" objects in $\mathrm{QCoh}(S)$, namely the *coherent complexes* $\mathrm{Coh}(S)$. An object $\mathscr{F}$ of $\mathrm{QCoh}(S)$ belongs to $\mathrm{Coh}(S)$ if it is cohomologically bounded and all its cohomology sheaves are locally finitely generated. Observe that $\mathscr{O}_S$ is coherent if and only if S is eventually coconnective, so in that case $\mathrm{Perf}(S) \subset \mathrm{Coh}(S)$ because $\mathrm{Coh}(S)$ is stable and closed under taking direct summands. This inclusion is an equivalence if and only if S is a smooth classical scheme (for S classical this is a theorem of Serre).

Definition 2.1.1. The category of *ind-coherent sheaves* on S is

$$\mathrm{IndCoh}(S) := \mathrm{Ind}(\mathrm{Coh}(S)).$$

By Proposition 1.1.3 there is a unique t-structure on $\mathrm{IndCoh}(S)$ which is compatible with filtered colimits and extends the t-structure on $\mathrm{Coh}(S)$.

Right Kan extension of the inclusion $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$ produces a t-exact functor

$$\Psi_S : \mathrm{IndCoh}(S) \longrightarrow \mathrm{QCoh}(S),$$

which is an equivalence if and only if S is a smooth classical scheme. This functor admits a left adjoint

$$\Xi_S : \mathrm{QCoh}(S) \longrightarrow \mathrm{IndCoh}(S),$$

if and only if S is eventually coconnective. In that case Ξ_S is fully faithful and Ψ_S is essentially surjective.

Lemma 2.1.2. *Let $\mathscr{F}$ be an object of $\mathrm{QCoh}(S)^-$ whose cohomology sheaves are finitely generated. Then for any $n \in \mathbb{Z}$ there exists $\mathscr{F}_0$ in $\mathrm{Perf}(S)$ and a morphism $\mathscr{F}_0 \to \mathscr{F}$ whose cofiber belongs to $\mathrm{QCoh}(S)^{\leq n}$.*

Proof. Let m be the largest integer such that $H^m(\mathscr{F}) \neq 0$. Since $\mathscr{F}$ is a filtered colimit of objects in $\mathrm{Perf}(S)$ and $H^m(\mathscr{F})$ is finitely generated, we can find a perfect complex $\mathscr{G}_1$ and a map $\mathscr{G}_1 \to \mathscr{F}$ such that $H^m(\mathscr{G}_1) \to H^m(\mathscr{F})$ is surjective. Truncating if necessary, we can assume that $\mathscr{G}_1$ belongs to $\mathrm{Perf}(S)^{\leq m}$. The surjectivity implies that the cofiber of this morphism belongs to $\mathrm{QCoh}(S)^{\leq m-1}$. Now apply the same procedure to the fiber of $\mathscr{G}_1 \to \mathscr{F}$ to obtain $\mathscr{G}_1'$, and set

$$\mathscr{G}_2 = \mathrm{cofib}(\mathscr{G}_1' \to \mathscr{G}_1).$$

One checks that the canonical map $\mathscr{G}_2 \to \mathscr{F}$ has cofiber belonging to $\mathrm{QCoh}(S)^{\leq m-2}$. Iterating this procedure we find that if $k \geq m - n$ then we can take $\mathscr{F}_0 = \mathscr{G}_k$.

□

Proposition 2.1.3. *The functor Ψ_S induces an equivalence*

$$\mathrm{IndCoh}(S)^+ \xrightarrow{\sim} \mathrm{QCoh}(S)^+$$

on eventually coconnective objects.

Proof. Using shifts, we reduce to proving that

$$\Psi_S : \mathrm{IndCoh}(S)^{\geq 0} \xrightarrow{\sim} \mathrm{QCoh}(S)^{\geq 0}.$$

This functor is essentially surjective because any object of $\mathrm{QCoh}(S)^{\geq 0}$ can be written as a filtered colimit of objects in $\mathrm{Coh}(S)^{\geq 0}$.

As for fully faithfulness, we will prove that

$$\mathrm{Hom}_{\mathrm{IndCoh}(S)}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{QCoh}(S)}(\Psi_S(\mathscr{F}), \Psi_S(\mathscr{G}))$$

for any $\mathscr{G}$ in $\mathrm{IndCoh}(S)^{\geq 0}$ and any $\mathscr{F}$ in $\mathrm{IndCoh}(S)$, and moreover we can assume that $\mathscr{F}$ lies in $\mathrm{Coh}(S)$. As previously mentioned $\mathscr{G}$ can be written as a filtered colimit of objects in $\mathrm{Coh}(S)^{\geq 0}$, so it suffices to show that the functor $\mathrm{QCoh}(S)^{\leq 0} \to \mathrm{Vect}^{\leq 0}$ given by

$$\mathscr{G} \mapsto \tau^{\leq 0} \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathscr{F}, \mathscr{G})$$

commutes with filtered colimits. Apply the lemma to obtain $\mathscr{F}_0 \to \mathscr{F}$ where $\mathscr{F}_0$ is perfect and the cofiber belongs to $\mathrm{Coh}(S)^{\leq -1}$. This implies that

$$\tau^{\leq 0} \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \tau^{\leq 0} \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathscr{F}_0, \mathscr{G}),$$

and since $\mathscr{F}_0$ is compact in $\mathrm{QCoh}(S)$ we are done.

□

It follows that the kernel of Ψ_S is the full subcategory $\mathrm{IndCoh}(S)_{\mathrm{nil}}$ consisting of objects $\mathscr{F}$ satisfying $H^n(\mathscr{F}) = 0$ for all $n \in \mathbb{Z}$.

Example 2.1.4. Let $A := k[\epsilon]/(\epsilon^2)$ be the algebra of dual numbers and $D := \operatorname{Spec} A$. Observe that δ_0 lies in $\mathrm{Coh}(D)$ but not $\mathrm{Perf}(D)$, because it has the projective resolution

$$\cdots \xrightarrow{\epsilon} A \xrightarrow{\epsilon} A \longrightarrow 0 \longrightarrow \cdots$$

and therefore $H^n(i^*\delta_0) = k$ for all $i \leq 0$.

The short exact sequence

$$0 \longrightarrow \delta_0 \longrightarrow \mathscr{O}_D \longrightarrow \delta_0 \longrightarrow 0$$

yields a nonzero morphism $\delta_0 \to \delta_0[1]$. Shifting this, we obtain a directed system

$$\delta_0 \longrightarrow \delta_0[1] \longrightarrow \delta_0[2] \longrightarrow \cdots,$$

which defines an object $\mathscr{F}_{\mathrm{nil}}$ in $\mathrm{IndCoh}(D)$. Clearly $\Psi_D(\mathscr{F}_{\mathrm{nil}}) = 0$ because $H^n(\mathscr{F}_{\mathrm{nil}}) = 0$ for all $n \in \mathbb{Z}$, i.e. $\mathscr{F}_{\mathrm{nil}}$ belongs to $\mathrm{IndCoh}(D)_{\mathrm{nil}}$ (the cohomology "escapes to $-\infty$").

Exercise 2.1.5. Check that $\mathscr{F} \neq 0$ in $\mathrm{IndCoh}(D)$.

It turns out that direct image of ind-coherent sheaves is easier to define than inverse image. Let $f : S \to T$ be a morphism of almost finite type schemes. The pushforward functor f_* is left t-exact and in particular induces a functor $\mathrm{QCoh}(S)^+ \to \mathrm{QCoh}(T)^+$. We define the IndCoh direct image

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(S) \to \mathrm{IndCoh}(T)$$

to be the right Kan extension of

$$\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)^+ \xrightarrow{f_*} \mathrm{QCoh}(T)^+ \xrightarrow{\sim} \mathrm{IndCoh}(T)^+ \subset \mathrm{IndCoh}(T).$$

When $T = \operatorname{Spec} k$ we write

$$\Gamma^{\mathrm{IndCoh}}(S, \mathscr{F}) = f_*^{\mathrm{IndCoh}} \mathscr{F}.$$

Since the operation of right Kan extension is functorial, we obtain a functor

$$\mathrm{IndCoh}_* : \mathrm{Sch}_{\mathrm{aft}} \longrightarrow \mathrm{DGCat},$$

where the subscript aft means almost of finite type.

Exercise 2.1.6. In the notation of Example 2.1.4, compute $\Gamma^{\mathrm{IndCoh}}(D, \mathscr{F})$.

2.2 The natural inverse image functor for ind-coherent sheaves is !-pullback. Namely, for any morphism $f : S \to T$ of schemes almost of finite type, we will construct a functor

$$f^! : \mathrm{IndCoh}(T) \longrightarrow \mathrm{IndCoh}(S).$$

Eventually, this will be upgraded to a functor

$$\mathrm{IndCoh}^! : \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{op}} \longrightarrow \mathrm{DGCat}\,.$$

First let us define, for f eventually coconnective, the $*$-pullback functor f^*_{IndCoh}. That hypothesis is equivalent to requiring that f^* sends $\mathrm{QCoh}(T)^+$ into $\mathrm{QCoh}(S)^+$, which implies that it sends $\mathrm{Coh}(T)$ into $\mathrm{Coh}(S)$. We define $f^*_{\mathrm{IndCoh}} : \mathrm{IndCoh}(T) \to \mathrm{IndCoh}(S)$ to be the right Kan extension of

$$\mathrm{Coh}(T) \xrightarrow{f^*} \mathrm{Coh}(S) \subset \mathrm{IndCoh}(S).$$

Now if f is an open embedding (more generally, étale) then in particular it is eventually coconnective, and we define $f^! := f^*_{\mathrm{IndCoh}}$. By functoriality of right Kan extensions this extends to

$$\mathrm{IndCoh}^! : (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{open}})^{\mathrm{op}} \longrightarrow \mathrm{DGCat},$$

where the superscript indicates that we only allow open embeddings. If f is proper (meaning it is proper on the level of classical schemes), then f_*^{IndCoh} sends $\mathrm{Coh}(X)$ into $\mathrm{Coh}(Y)$, so by Exercises 1.1.2 and 1.1.6 it admits a continuous right adjoint, which we also call $f^!$. Since passing to right adjoints is functorial, we obtain a functor

$$\mathrm{IndCoh}^! : (\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{proper}})^{\mathrm{op}} \longrightarrow \mathrm{DGCat},$$

where the superscript indicates that we only allow proper morphisms.

Now recall the following well-known theorem of Nagata.

Theorem 2.2.1. *Any morphism between (separated) classical schemes of finite type factorizes into an open embedding followed by a proper morphism.*

Exercise 2.2.2. Find such a factorization for the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, xy)$.

In fact, Nagata's theorem immediately implies the same statement for derived schemes. For any $S \to T$ the classical theorem yields a factorization

$$S^{\mathrm{cl}} \longrightarrow Z' \longrightarrow T^{\mathrm{cl}}.$$

Define $Z := Z' \amalg_{S^{\mathrm{cl}}} S$, which fits into the desired factorization $S \to Z \to T$.

So we can define $f^!$ for an arbitrary morphism f, but now it is not clear that this definition is independent of the chosen Nagata factorization. One can resolve this issue by proving that the category of Nagata factorizations of a given morphism is contractible. This implies that there is a unique functor

$$\mathrm{IndCoh}^! : \mathrm{Sch}_{\mathrm{aft}}^{\mathrm{op}} \longrightarrow \mathrm{DGCat}$$

which restricts to the same-named functors on $(\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{open}})^{\mathrm{op}}$ and $(\mathrm{Sch}_{\mathrm{aft}}^{\mathrm{proper}})^{\mathrm{op}}$.

Example 2.2.3. Let us return to the situation of Example 2.1.4. Let $i : \mathrm{Spec}\, k \to D$ be the unique point, so for any $\mathscr{F}$ in $\mathrm{IndCoh}(D)$ we have

$$i^! \mathscr{F} = \mathrm{Hom}_{\mathrm{IndCoh}(D)}(\delta_0, \mathscr{F}).$$

Thus $i^!$ lifts to a functor

$$i^!_{\mathrm{enh}} : \mathrm{IndCoh}(D) \longrightarrow B\text{-mod}^r$$

to right B-modules, where $B = \mathrm{End}_{\mathrm{Coh}(D)}(\delta_0)$. It is not hard to check that $i^!$ is conservative, and since it is continuous and admits a left adjoint the Barr-Beck theorem implies that $i^!_{\mathrm{enh}}$ is an equivalence. In other words, δ_0 is a compact generator for $\mathrm{IndCoh}(D)$, so by derived Morita theory this category is identified with right modules over B. Using the projective resolution from Example 2.1.4, one shows that $B = k[\zeta]$ is the free DG algebra on a single generator ζ in degree 1. This algebra is noncommutative, but there is a canonical isomorphism $B \tilde{\to} B^{\mathrm{op}}$ given by $\zeta \mapsto -\zeta$.

Let us try to describe the adjoint functors

$$A\text{-mod} = \mathrm{QCoh}(D) \underset{\Psi_D}{\overset{\Xi_D}{\rightleftarrows}} \mathrm{IndCoh}(D) = B\text{-mod}^r$$

in terms of the algebra of A and B. One computes $i^! \Xi_D(\mathscr{O}_D) = k$, which means Ξ_D sends A to the augmentation B-module k. By continuity it follows that $\Xi_D(M) = k \otimes_A M$ for any A-module M, where we use the isomorphism $B \tilde{\to} B^{\mathrm{op}}$ to get a right B-action. We know that Ξ is fully faithful, which implies that

$$A = \mathrm{End}_A(A) \xrightarrow{\sim} \mathrm{End}_B(k).$$

Alternatively, one can show $A = \mathrm{End}_B(k)$ directly using the exact triangle

$$B[-1] \longrightarrow B \longrightarrow k.$$

It is straightforward to check that Ψ_D sends $M \mapsto M \otimes_B k$.

Exercise 2.2.4. Show that the essential image of Ξ_D consists of B-modules on which ζ acts locally nilpotently. Characterize $\mathrm{IndCoh}(D)_{\mathrm{nil}}$ and show that the object $\mathscr{F}_{\mathrm{nil}}$ from Exercise 2.1.4 is a compact generator for $\mathrm{IndCoh}(D)_{\mathrm{nil}}$.