A CRASH COURSE ON ∞-CATEGORIES

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Recall from Justin's talk that we need a theory of quasi-coherent sheaves on any prestack. Let me first show you how this is achieved formally.

0.1. **Towards** $\mathrm{QCoh}(\mathscr{X})$**.** Fix a field k. A *prestack* is a functor $(\mathrm{Aff}/k)^{\mathrm{op}} \to \mathbf{Spc}$. Here, $(\mathrm{Aff}/k)^{\mathrm{op}}$ is the ∞-category of "derived rings" over k, and **Spc** is the ∞-category of "spaces."

Given a prestack $\mathscr{X}$, we may consider the category $\mathrm{Aff}/\mathscr{X}$ of "affine derived schemes" over $\mathscr{X}$. Then QCoh can be understood as a functor

$$(\mathrm{Aff}/\mathscr{X})^{\mathrm{op}} \to \mathrm{DGCat}, \quad \mathrm{Spec}(A) \mapsto A\text{-Mod}$$

that associates to a derived ring A its "DG category" of A-modules. The category $\mathrm{QCoh}(\mathscr{X})$ is by definition the limit of this functor:

$$\mathrm{QCoh}(\mathscr{X}) := \lim_{(\mathrm{Aff}/\mathscr{X})^{\mathrm{op}}} A\text{-Mod}.$$

taken in the ∞-category DGCat of all DG categories!

0.1.1. *What does this talk do?* By next week, we will have all the necessary ingredients available in order to make the above definition. In this talk, we will put the ∞-category R-Mod for an *ordinary* ring R on firm foundation. This is useful for us because:

(i) It ties closely to the classical theory of derived categories;
(ii) Our model of derived rings is the ∞-category of commutative differential graded algebras over k, which is by definition the ∞-category of "commutative algebra objects" in $\mathrm{Vect} := k\text{-Mod}$.

1. A few words on model categories

1.1. **From homological algebra to homotopical algebra.** Let R be an ordinary ring. Homological algebra tells us to make the following constructions:

$$R \rightsquigarrow \mathrm{Ch}(R) \rightsquigarrow D(R).$$

Here $\mathrm{Ch}(R)$ denotes the category of (unbounded) complexes of R-modules with chain maps; the derived category $D(R)$ is obtained by formally inverting at all quasi-isomorphisms.[1]

Remark 1.1. Derived functors operate on the level of $D(R)$ (modulo boundedness issues). For instance, in order to compute $(-) \otimes^{\mathbb{L}}_R M$ for some R-module M, we have to first take a projective resolution, calculate the naïve tensor product, and arrive at an object well-defined (up to unique isomorphism) only in $D(R)$.

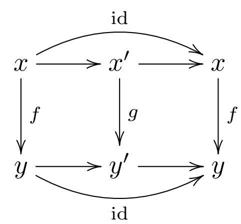
[1] Throughout this talk (and this semester for that matter!), we stick to cohomological convention.

1.1.1. *What is a model category?* The above construction of derived categories and derived functors are formalized by the notion of a model category:

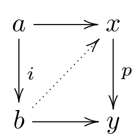
Definition 1.2 (Following Dwyer-Spalinski)**.** A *model category* is a category $\mathbf{C}$ with three distinguished classes of morphisms: *weak equivalences*, *fibrations*, and *cofibrations*. Each class is required to be closed under composition and to contain all identity morphisms.

A morphism which is both a fibration (resp. cofibration) and a weak equivalence is called an *acyclic fibration* (resp. *acyclic cofibration.*) We further require the following axioms:

(**MC1**) $\mathbf{C}$ has *finite* limits and colimits.
(**MC2**) If f, g are morphisms such that gf is defined, then if two of the three maps f, g, gf are weak equivalences, then so is the third.
(**MC3**) Suppose f is a *retract* of g, i.e., there exists a commutative diagram



Then if g is a weak equivalence, fibration, or cofibration, so is f.
(**MC4**) Given a solid commutative diagram



a lift is required to exist if either (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.
(**MC5**) Any morphism f can be factored in two ways: (i) $f = pi$ where i is a cofibration and p is an acyclic fibration, and (ii) $f = pi$ where i is an acyclic cofibration and p is a fibration.

Remark 1.3. **MC1** tells us that $\mathbf{C}$ has initial object $\emptyset$ and final object $*$. We call an object x in $\mathbf{C}$ *fibrant* if $x \to *$ is a fibration, and *cofibrant* if $\emptyset \to x$ is a cofibration.

Note that by **MC5**, each object x has a *cofibrant replacement* Qx, i.e., a cofibrant object $Qx \in \mathbf{C}$ together with an acyclic fibration $Qx \to x$. Dually, x also has a *fibrant replacement* Rx together with an acyclic cofibration $x \to Rx$. In general, we do not expect the assignment $x \rightsquigarrow Qx$ to be functorial.

Remark 1.4. The data of a model category is over-determined in the following sense: if one knows the class of weak equivalences and fibrations, then the class of cofibrations is recovered as those morphisms satisfying the left-lifting property against all acyclic fibrations.

Of course, **MC4** shows that all cofibrations have this property. For the converse, let me give you a hint—given a morphism $i : a \to b$ with the above left-lifting property, factor i into $i = p\tilde{i}$, where $\tilde{i}$ is a cofibration and p is a trivial fibration (using **MC5**). Then realize i as a retract of $\tilde{i}$, and apply **MC3**.

1.1.2. *Homotopy category.* To every model category $\mathbf{C}$, there is another category $\mathrm{Ho}(\mathbf{C})$ capturing the homotopy-theoretic information of $\mathbf{C}$, called the *homotopy category* of $\mathbf{C}$. We may define $\mathrm{Ho}(\mathbf{C})$ as the universal category equipped with a functor $\mathbf{C} \to \mathrm{Ho}(\mathbf{C})$ sending weak equivalences to isomorphisms.

Remark 1.5. One can also construct $\mathrm{Ho}(\mathbf{C})$ explicitly. For fibrant-cofibrant objects $x, y \in \mathbf{C}$, there is a well-defined notion of homotopy equivalence on the set $\mathrm{Hom}_{\mathbf{C}}(x, y)$. (The construction requires some work, see §4 of [DS95]).

One then define $\mathrm{Ho}(\mathbf{C})$ to have the same set of objects as $\mathbf{C}$, with hom-sets

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{C})}(x, y) := \mathrm{Hom}_{\mathbf{C}}(RQx, RQy).$$

1.1.3. *Model structures on* Ch(R). Let Ch(R) be the category of (unbounded) complexes of R-modules. The following model structure on Ch(R) is due to Quillen, and is commonly referred to as the *projective model structure*:

(Weak) Quasi-isomorphism.
(Fib) Degree-wise surjection.

The cofibrations are determined (in light of Remark 1.4).

Proposition 1.6. *Such a model structure exists on* Ch(R).

Proof. This is [Lu11, Proposition 7.1.2.8]. □

[Discuss the injective model structure and the case for fields.]

2. Towards ∞-categories

2.1. Quasi-categories. [Insert blurb.]

2.1.1. *Reminder on simplicial sets.* Let Δ denote the category whose objects are

$$[n] := \{0, 1, \cdots, n\}, \text{ for all integer } n \geq 0$$

and morphisms are non-decreasing functions $f : [m] \to [n]$ (i.e., $f(i) \leq f(j)$ if $i \leq j$).

Definition 2.1. A *simplicial set* is a contravariant functor $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$.

There is also a "low-brow" way of encoding the information contained in X. For each $n \geq 0$, we have a set X_n, called the *n-simplices* of X. Fix such n; then for each $0 \leq i \leq n$, we have

(i) a *face map* $d_i : X_n \to X_{n-1}$ corresponding to the injection $[n-1] \to [n]$ that misses the target $i \in [n]$.
(ii) a *degeneracy map* $s_i : X_n \to X_{n+1}$ corresponding to the surjection $[n+1] \to [n]$ collapsing both $i, i+1 \in [n+1]$ onto $i \in [n]$.

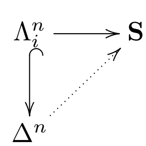
The Yoneda embedding gives simplicial sets $\Delta^n := \mathrm{Hom}_{\Delta^{\mathrm{op}}}(-, [n])$. One should think of Δ^n as the standard n-simplex. Indeed, the non-degenerate simplices of Δ^n correspond naturally to "boundary data" of the standard n-simplex.

In what follows, the category of simplicial sets is denoted by $\mathbf{Set}_\Delta$.

Remark 2.2. Since the category **Set** is complete and co-complete, so is $\mathbf{Set}_\Delta$, and the limits and colimits in $\mathbf{Set}_\Delta$ are computed pointwise.

2.1.2. *The Horn-filling conditions.* Let Λ^n_i be simplicial subset of Δ^n with ith face missing ($0 \leq i \leq n$). We will use right-lifting properties against inclusions of the form $\Lambda^n_i \to \Delta^n$ to define important classes of simplicial sets:

Definition 2.3. (i) A *space* (also *∞-groupoid* or *Kan complex*) is a simplicial set **S** such that every map $\Lambda^n_i \to \mathbf{S}$ (for $0 \leq i \leq n$) admits a lift to Δ^n:

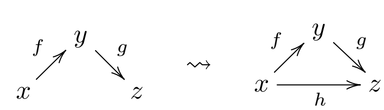


(ii) A *quasi-category* (also *∞-category* or *weak Kan complex*) is a simplicial set **C** required to satisfy the above lifting property only for $0 < i < n$.

Let me first convince you why a quasi-category **C** behaves like a category. Since the n-simplices of **C** model the n-morphisms, we should be able to compose two 1-morphisms. Suppose we have $f, g \in \mathbf{C}_1$ with faces

$$d_1 f = x, \quad d_0 f = d_1 g = y, \quad d_0 g = z.$$

We then have a map $\Lambda^2_1 \to \mathbf{C}$ sending the edges $\{0,1\} \mapsto f$, $\{1,2\} \mapsto g$. The horn-filling condition then gives a 2-*simplex* $\sigma \in \mathbf{C}_2$:



Let $h = d_1\sigma$. Then h is a "candidate" for the composition of f and g.

Remark 2.4. We call it a "candidate" because there is no uniqueness requirement on the lifting. However, we can relate every two candidates h, h' by a *homotopy*, i.e. there is a 2-simplex of the form

$$\begin{array}{ccccc} & & x & & \\ & \overset{\mathrm{id}:=s_0(x)}{\nearrow} & & \searrow^{h'} & \\ x & & \xrightarrow[h]{\qquad} & & z. \end{array}$$

Indeed, if $h = d_1\sigma$ and $h' = d_1\sigma'$, then we have a map $\Lambda^3_2 \to \mathbf{C}$, given on faces by $\{\sigma', \sigma, \bullet, s_0 f\}$. After filling in this horn, we obtain a 3-simplex τ such that $d_2\tau$ is a homotopy from h to h'.

Remark 2.5. With more work, one can check that homotopy is an equivalence on the set $\mathrm{Hom}(x, y) = \{f \in \mathbf{C}_1 : d_1 f = x, d_0 f = y\}$. Furthermore, composition is well-defined on the equivalence classes; so we obtain an ordinary category $\mathrm{Ho}(\mathbf{C})$, which we call the *homotopy category* of $\mathbf{C}$.

2.2. **Obtaining** R-Mod **via DG nerve.** We now return to the category $\mathrm{Ch}(R)$ of complexes of R-modules. We will construct our ∞-category R-Mod out of $\mathrm{Ch}(R)$ via a general procedure called "DG nerves."

2.2.1. *What is a DG category?* DG categories are invented as an abstract model of quasi-coherent sheaves on (commutative or non-commutative) algebro-geometric objects. Its classical definition is given as follows:

Definition 2.6 ("Classical")**.** [2] A *DG category* is a category $\mathbf{C}$ enriched in chain complexes. More precisely,

(i) for every pair of objects $x, y \in \mathbf{C}$, we have a chain complex of k-modules $\mathrm{Map}_{\mathbf{C}}(x, y)^\bullet$ such that
$$\mathrm{id}_x \in \mathrm{Map}_{\mathbf{C}}(x, x)^0;$$
(ii) for every triple $x, y, z \in \mathbf{C}$, the composition map induces a morphism of chains:
$$\mathrm{Map}_{\mathbf{C}}(x, y)^\bullet \otimes \mathrm{Map}_{\mathbf{C}}(y, z)^\bullet \to \mathrm{Map}_{\mathbf{C}}(x, z)^\bullet;$$
which is required to be associative.

Example 2.7. Let R be an ordinary k-algebra. Then the category $\mathrm{Ch}(R)$ of (unbounded) complexes of R-modules form a DG category. The mapping complex is given by
$$\mathrm{Map}_{\mathrm{Ch}(R)}(M^\bullet, N^\bullet)^p = \prod_{n \in \mathbb{Z}} \mathrm{Hom}_R(M^n, N^{n+p})$$
with differentials taking $f = (f_n)_{n\in\mathbb{Z}}$ to $df = (d \circ f_n - (-1)^n f_{n+1} \circ d)_{n\in\mathbb{Z}}$.

For instance, let f be in degree zero. Then $df = 0$ if and only if f is a chain map; and $f = dh$ for some h in degree -1 if and only if f is chain-homotopic to 0.

2.2.2. *The DG nerve.* Given a DG category $\mathbf{C}$, we will construct an ∞-category $\mathrm{N}_{\mathrm{DG}}(\mathbf{C})$, whose lower-dimensional simplices can be explicitly described:

(i) The 0-simplices are objects of $\mathbf{C}$.
(ii) The 1-simplices are triples (x, y, f) where $x, y \in \mathbf{C}$, and $f \in \mathrm{Map}_{\mathbf{C}}(x, y)^0$ with
$$df = 0.$$
(iii) The 2-simplices are sextuples $(x, y, z, f, g, h, \sigma)$, where $x, y, z \in \mathbf{C}$; $f \in \mathrm{Map}_{\mathbf{C}}(x, y)^0$, $g \in \mathrm{Map}_{\mathbf{C}}(y, z)^0$, $h \in \mathrm{Map}_{\mathbf{C}}(x, z)^0$; and $\sigma \in \mathrm{Map}_{\mathbf{C}}(x, y)^{-1}$. They are required to satisfy
$$d\sigma = g \circ f - h.$$

Or for the definitionists, the following is taken from [Lu11, Construction 1.3.1.6]:

Definition 2.8. Define the simplicial set $\mathrm{N}_{\mathrm{DG}}(\mathbf{C})$ as follows: for each integer $n \geq 0$, let $\mathrm{N}_{\mathrm{DG}}(\mathbf{C})_n$ be the set of all pairs $(\{x_i\}_{0\leq i\leq n}, \{f_I\})$, where

(i) For $0 \leq i \leq n$, x_i is an object of $\mathbf{C}$;
(ii) for every subset I of $[n]$ with at least 3 elements (write $I = \{i_- < i_1 < i_2 < \cdots < i_m < i_+\}$ with $m \geq 0$), f_I is an element of $\mathrm{Map}_{\mathbf{C}}(x_{i_-}, x_{i_+})^{-m}$, satisfying the equation
$$d(f_I) = \sum_{1\leq j\leq m} (-1)^j \left(f_{I-\{i_j\}} - f_{\{i_j<\cdots<i_m<i_+\}} \circ f_{\{i_-<i_1<\cdots<i_j\}} \right). \tag{2.1}$$

[2]Next week, we will have a different definition of DG categories that automatically organizes them into an ∞-category.

If $\alpha : [m] \to [n]$ is a non-decreasing function, then the induced map $\mathrm{N}_{\mathrm{DG}}(\mathbf{C})_n \to \mathrm{N}_{\mathrm{DG}}(\mathbf{C})_m$ is given by

$$(\{x_i\}_{0\le i\le n}, \{f_I\}) \mapsto (\{x_{\alpha(j)}\}_{0\le j\le m}, \{g_J\}),$$

where

$$g_J = \begin{cases} f_{\alpha(J)} & \text{if } \alpha\big|_J \text{ is injective} \\ \mathrm{id}_{x_i} & \text{if } J = \{j, j'\} \text{ with } \alpha(j) = \alpha(j') = i \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.9. *The simplicial set* $\mathrm{N}_{\mathrm{DG}}(\mathbf{C})$ *is an* ∞*-category.*

Proof. Suppose we have an inner horn $\Lambda^n_j \to \mathrm{N}_{\mathrm{DG}}(\mathbf{C})$ (for $0 < j < n$). Its data amount to the collection of objects $\{x_i\}_{0\le i\le n}$, together with morphisms $f_I \in \mathrm{Map}_{\mathbf{C}}(x_{i_-}, x_{i_+})^{-m}$ for each subset

$$I = \{i_- < i_1 < \cdots < i_m < i_+\} \subset [n], \text{ with } m \ge 0$$

which is *not* $[n] - \{j\}$ or $[n]$ (i.e., missing the jth face.) These data are required to satisfy the equations (2.1). We fill this horn by defining

$$f_{[n]-\{j\}} := \sum_{0<p<n} (-1)^{p-j} f_{\{p<\cdots<n\}} \circ f_{\{0<\cdots<p\}} - \sum_{\substack{0<p<n \\ p\neq j}} (-1)^{p-j} f_{[n]-\{p\}}$$

and setting $f_{[n]} = 0$. □

2.2.3. *Definition of* R-Mod. We will now accomplish our goal of defining the ∞-category R-Mod:

Definition 2.10. Equip $\mathrm{Ch}(R)$ with the projective model structure. Let $\mathrm{Ch}(R)^\circ$ be the full subcategory of fibrant-cofibrant objects. Set $R\text{-Mod} := \mathrm{N}_{\mathrm{DG}}(\mathrm{Ch}(R)^\circ)$.

Remark 2.11. In the above definition, we may replace the projective model structure by the injective one, and yield an equivalent definition of R-Mod. However, the projective model structure is more convenient when we study algebras over R.

[Case for fields.]

3. The ∞-categories Spc and Cat

We will now construct the ∞-categories **Spc** of spaces and **Cat** of ∞-categories. Both will follow from the general paradigm:

Simplicial model category $\mathbf{A} \rightsquigarrow$ Locally fibrant simplicial category $\mathbf{A}^\circ \rightsquigarrow \infty$-category $\mathrm{N}(\mathbf{A}^\circ)$.

The ∞-category $\mathrm{N}(\mathbf{A}^\circ)$ obtained from this procedure will be called the *underlying* ∞*-category* of $\mathbf{A}$.

3.1. The "underlying ∞-category".

In order to make the above chain precise, we need a few preliminary constructions.

3.1.1. *The Kan model structure on* $\mathbf{Set}_\Delta$. We first put a model structure on the category $\mathbf{Set}_\Delta$ of simplicial sets, with distinguished classes of morphisms given by:

(Weak) A map $X \to Y$ such that the induced map on geometric realizations[3] $|X| \to |Y|$ is a weak homotopy equivalence (of topological spaces).

(Fib) Kan fibration, i.e., a map of simplicial sets admitting the right-lifting property against all horn fillings $\Lambda^n_i \to \Delta^n$ (for $0 \le i \le n$.)

(Cofib) Monomorphism, i.e., a map $X \to Y$ of simplicial sets such that each $X_n \to Y_n$ is injective.

We note that an acyclic fibration in $\mathbf{Set}_\Delta$ is exactly a map of simplicial sets satisfying the right-lifting property against all sphere inclusions $\partial\Delta^n \to \Delta^n$.

Lemma 3.1. *The model category* $\mathbf{Set}_\Delta$ *is combinatorial. [Define this term.]*

Sketch of proof. [Fill me in.] □

[3] You may find the construction of geometric realization of a simplicial set in any standard text, e.g., [GJ09]. It will not be important for us.

Remark 3.2. It is clear that the every object in $\mathbf{Set}_\Delta$ is cofibrant, and the fibrant objects are precisely the spaces. Hence the full subcategory $\mathbf{Set}_\Delta^\circ$ of fibrant-cofibrant objects is the (ordinary) category of spaces.

3.1.2. *Simplicial model categories.* Roughly speaking, a simplicial model category comes with two additional structures—a simplicial enrichment and a model structure—which interact nicely.

Definition 3.3. A *simplicial category* is a category $\mathbf{C}$ enriched in $\mathbf{Set}_\Delta$. In other words,

(i) for every pair of objects $x, y \in \mathbf{C}$, we have a simplicial set $\mathrm{Map}_{\mathbf{C}}(x, y)$;
(ii) for every triple $x, y, z \in \mathbf{C}$, the composition map induces a morphism of simplicial sets:

$$\mathrm{Map}_{\mathbf{C}}(x, y) \times \mathrm{Map}_{\mathbf{C}}(y, z) \to \mathrm{Map}_{\mathbf{C}}(x, z)$$

which is required to be associative.

A simplicial category $\mathbf{C}$ is called *locally fibrant* if the simplicial set $\mathrm{Map}_{\mathbf{C}}(x, y)$ is fibrant (i.e., a Kan complex) for every pair of objects $x, y \in \mathbf{C}$.

Given two simplicial categories $\mathbf{C}$ and $\mathbf{D}$, we denote by $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ the set of *simplicial functors*, i.e. the functors $\mathbf{C} \to \mathbf{D}$ which respect the simplicial enrichment of $\mathbf{C}$ and $\mathbf{D}$.

Definition 3.4. Let $\mathbf{A}$ be a model category and a simplicial category. Then $\mathbf{A}$ is a *simplicial model category* if for all cofibration $i : a \to b$ and fibration $p : x \to y$, the map

$$\mathrm{Map}_{\mathbf{A}}(b, x) \xrightarrow{(i^*, p_*)} \mathrm{Map}_{\mathbf{A}}(a, x) \times_{\mathrm{Map}_{\mathbf{A}}(a,y)} \mathrm{Map}_{\mathbf{A}}(b, y)$$

is a Kan fibration, which is acyclic if i *or* p is acyclic.

The following lemma shows that the full subcategory of fibrant-cofibrant objects $\mathbf{A}^\circ$ is locally fibrant as soon as $\mathbf{A}$ is a simplicial model category.

Lemma 3.5. *Let* $\mathbf{A}$ *be a simplicial model category. Suppose* b *is a cofibrant object and* x *is a fibrant object of* $\mathbf{A}$*. Then* $\mathrm{Map}_{\mathbf{A}}(b, x)$ *is a space.*

Proof. Take a to be the initial object, and y to be the final object of $\mathbf{A}$ in the above definition. □

3.1.3. *The simplicial nerve.* The simplicial nerve construction associates to every simplicial category $\mathbf{C}$ a simplicial set $\mathrm{N}(\mathbf{C})$. In order to define $\mathrm{N}(\mathbf{C})$, we first construct a simplicial category $\mathfrak{C}[\Delta^n]$ out of each standard simplex Δ^n.

For low values of n, the resulting categories are the following:

(i) $\mathfrak{C}[\Delta^0]$ has a unique object 0 with no nontrivial morphisms.
(ii) $\mathfrak{C}[\Delta^1]$ has objects 0 and 1, together with a single nontrivial morphism from 0 to 1. In other words, $\mathrm{Map}_{\mathfrak{C}[\Delta^1]}(0, 1) = \Delta^0$ and $\mathrm{Map}_{\mathfrak{C}[\Delta^1]}(1, 0) = \emptyset$. We may picture $\mathfrak{C}[\Delta^1]$ as

$$0 \longrightarrow 1$$

(iii) $\mathfrak{C}[\Delta^2]$ has objects 0, 1, and 2, together with unique nontrivial morphisms $0 \to 1$ and $1 \to 2$. However, we have $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0, 2) = \Delta^1$, whose two *vertices* correspond to
   (a) one nontrivial morphism $0 \to 2$, and
   (b) the composition $0 \to 1 \to 2$.

   These two morphisms are connected by a unique "2-morphism," represented by the edge in $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0, 2)$.

We now generalize this construction to arbitrary values of n:

Definition 3.6. The set of objects of the simplicial category $\mathfrak{C}[\Delta^n]$ is $\{0, 1, \cdots, n\}$. Let $i \leq j$ be two objects, and $P_{i,j}$ denote the poset of subsets of $\{i, i+1, \cdots, j\}$ containing the endpoints i and j (ordered by inclusion). Define the mapping space by

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = \begin{cases} \mathrm{N}(P_{i,j}), & i \leq j \\ \emptyset, & i > j. \end{cases}$$

Here $\mathrm{N}(P_{i,j})$ is the nerve of the poset $P_{i,j}$, regarded as a(n ordinary) category [Explain].

This construction is functorial in the following sense: for any non-decreasing function $[m] \to [n]$, the corresponding morphism $\Delta^m \to \Delta^n$ induces a simplicial functor $\mathfrak{C}[\Delta^m] \to \mathfrak{C}[\Delta^n]$.

Definition 3.7. Given a simplicial category $\mathbf{C}$, define its *simplicial nerve* to be the simplicial set $\mathrm{N}(\mathbf{C})$ whose n-simplices are given by $\mathrm{Fun}(\mathfrak{C}[\Delta^n], \mathbf{C})$.

For every non-decreasing function $\alpha : [m] \to [n]$, the map $\mathrm{N}(\mathbf{C})_n \to \mathrm{N}(\mathbf{C})_m$ is given by pre-composing with the simplicial functor $\mathfrak{C}[\Delta^m] \to \mathfrak{C}[\Delta^n]$.

3.1.4. *Existence of limits and colimits.* [Fill me in.]

3.2. **The ∞-category Spc.** We will apply the above constructions to the category $\mathbf{Set}_\Delta$ itself. It is a model category (see §3.1.1), together with a natural simplicial enrichment given as follows: for every $X, Y \in \mathbf{Set}_\Delta$, we have $\mathrm{Map}(X, Y) \in \mathbf{Set}_\Delta$ defined by

$$\mathrm{Map}(X, Y)_n = \mathrm{Hom}(X \times \Delta^n, Y)$$

where Hom denotes the *set* of morphisms of simplicial sets.

Proposition 3.8. *The simplicial category $\mathbf{Set}_\Delta$ equipped with the Kan model structure is a combinatorial simplicial model category.*

Consequently, we may set $\mathbf{Spc} = \mathrm{N}(\mathbf{Set}^\circ_\Delta)$, with $\mathbf{Set}^\circ_\Delta$ is the full subcategory of cofibrant-fibrant objects (in this case, the spaces).

Proof. [Fill me in] □

3.3. **The ∞-category Cat.** Since every ∞-category is a simplicial set, we are tempted to define another model structure on $\mathbf{Set}_\Delta$ whose fibrant-cofibrant are not the Kan complexes, but the ∞-categories, and then produce $\mathbf{Cat}$ via the above paradigm. Such a model structure exists, and is constructed by André Joyal, but the resulting category $\mathbf{Set}^{(\mathrm{Joyal})}_\Delta$ is not a simplicial model category—this is reflected by the fact that the mapping space between ∞-category is in general an ∞-category, not a Kan complex.

We can remedy this problem by defining yet another category $\mathbf{Set}^+_\Delta$ of "marked simplicial sets" which will turn out to be a simplicial model category.

3.3.1. *Marked simplicial sets.* [Fill me in.]

4. Categorical constructions

4.1. **Functors, slice categories, limits.** We now render a few familiar "external" constructions in the language of ∞-categories. Let $\mathbf{C}, \mathbf{D}$ be ∞-categories.

Definition 4.1. A *functor* $\mathbf{C} \to \mathbf{D}$ is a morphism of simplicial sets.

Lemma 4.2. *For every $X \in \mathbf{Set}_\Delta$, the simplicial set $\mathrm{Map}(X, \mathbf{C})$ is an ∞-category.*

$\mathrm{Map}(\mathbf{C}, \mathbf{D})$ will be called the *functor category*, and denoted by $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ from now on.

Sketch of proof. We want to show that $\mathrm{Map}(X, \mathbf{C})$ has the right-lifting property against inner horn inclusions $\Lambda^n_i \to \Delta^n$ $(0 < i < n)$. This is equivalent to showing that $\mathbf{C}$ has the right-lifting property against $\Lambda^n_i \times X \to \Delta^n \times X$. One then construct such lifting cell-by-cell. See [Lu09, Corollary 2.3.2.4]. □

4.1.1. *The convenient "join" construction.* Given two simplicial sets X, Y, the *join* $X \star Y$ is the simplicial set defined by

$$(X \star Y)_n = X_n \cup Y_n \cup \bigcup_{i+j=n-1} (X_i \times Y_j).$$

Alternatively, the join construction is the unique bifunctor which preserves colimits in both variables, and such that $\Delta^i \star \Delta^j \cong \Delta^{i+j+1}$.

Example 4.3. We define the *right cone* $X^\triangleright := X \star \Delta^0$, and *left cone* $X^\triangleleft := \Delta^0 \star X$.

We now use the join construction to characterize *slice categories*:

Lemma 4.4. *For every $Y \in \mathbf{Set}_\Delta$ and morphism $p : Y \to \mathbf{C}$, there exist ∞-categories $\mathbf{C}_{/p}$ and $\mathbf{C}_{p/}$ satisfying the universal properties:*

$$\mathrm{Hom}(X, \mathbf{C}_{/p}) \cong \mathrm{Hom}_p(X \star Y, \mathbf{C}), \quad \mathrm{Hom}(X, \mathbf{C}_{p/}) \cong \mathrm{Hom}_p(Y \star X, \mathbf{C})$$

where $\mathrm{Hom}_p(-,-)$ means morphisms of simplicial sets agreeing with p on Y.

Sketch of proof. [Fill me in.] □

If $Y = \Delta^0$, and p sends Y to $c \in \mathbf{C}$, we abuse the notation and denote $\mathbf{C}_{/p}$ by $\mathbf{C}_{/c}$ (and similarly for $\mathbf{C}_{p/}$.)

Definition 4.5. Let $c \in \mathbf{C}$. Then c is an *initial object* if the map $\mathbf{C}_{c/} \to \mathbf{C}$ is an acyclic fibration in $\mathbf{Set}_\Delta^{\mathrm{Kan}}$.

The notion of a *final object* is defined dually.

Definition 4.6. Let $X \in \mathbf{Set}_\Delta$ and $p : X \to \mathbf{C}$ be a map of simplicial sets. Then a *colimit* of p is an initial object in $\mathbf{C}_{p/}$.

A *limit* of p is defined dually as a final object in $\mathbf{C}_{/p}$.

4.1.2. *Kan extensions along inclusions.* Let $\mathbf{C}^0$ be a full subcategory of $\mathbf{C}$. [Define those] Suppose we have a (strictly!) commutative diagram of ∞-categories

$$\begin{array}{ccc} \mathbf{C}^0 & \xrightarrow{F_0} & \mathbf{D} \\ & \searrow \quad \nearrow_{F} & \\ & \mathbf{C} & \end{array}$$

Definition 4.7. We say that F is a *left Kan extension* of F_0 (along $\mathbf{C}^0 \to \mathbf{C}$) if for each $c \in \mathbf{C}$, the induced diagram

$$\begin{array}{ccc} \mathbf{C}^0_{/c} & \xrightarrow{F_c} & \mathbf{D} \\ & \searrow \quad \nearrow & \\ & (\mathbf{C}^0_{/c})^{\triangleright} & \end{array}$$

exhibits $F(c)$ as a colimit of F_c.

Lemma 4.8. *Given a functor $F_0 : \mathbf{C}^0 \to \mathbf{D}$, the left Kan extension F exists if and only if for each $c \in \mathbf{C}$, the composed functor*

$$\mathbf{C}^0_{/c} \to \mathbf{C}^0 \xrightarrow{F_0} \mathbf{D}$$

admits a colimit.

Proof. [Fill me in. This is Lemma 4.3.2.13. Write about propositions that follow.] □

4.2. **Straightening and unstraightening.** In ordinary category theory, the Grothendieck construction gives an equivalence between Cartesian fibrations over a base category $\mathbf{C}$ and (2-)functors from $\mathbf{C}^{\mathrm{op}}$ to the category of categories.

There is an ∞-categorical analogue of this construction due to Jacob Lurie, which will be very important for us because "straightening" a Cartesian fibration turns out to be essentially the only way we can write down a functor from a given ∞-category to **Cat**.

4.2.1. *Cartesian fibration.* We first define a relative version of ∞-categories called "inner fibrations."

Definition 4.9. Let $p : X \to S$ be a map of simplicial sets. Then p is an *inner fibration* if it satisfies the right-lifting property against inclusions of inner horns $\Lambda^n_i \to \Delta^n$ $(0 < i < n)$.

In particular, for every 0-simplex $s \in S$, the fiber $\{s\} \times_S X$ is an ∞-category. So one may think of an inner fibration as a family of ∞-categories parametrized by the base S.

Definition 4.10. Let $p : X \to S$ be an inner fibration of simplicial sets. Let $f : x \to y$ be an arrow in X. Then f is *Cartesian* (or *p-Cartesian*) if the induced map

$$X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$$

is an acyclic fibration (in $\mathbf{Set}_\Delta$; see §3.1.1.)

There is also a characterization of Cartesian arrows directly in terms of lifting conditions:

Lemma 4.11. *The arrow f is Cartesian if and only if for all diagrams ($n \geq 2$):*

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{\sigma} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & S \end{array}$$

such that $\sigma\big|_{\Delta^{\{n-1,n\}}} = f$, a lifting (dotted arrow) exists.

[Explain]

Definition 4.12. A map of simplicial sets $p : X \to S$ is a *Cartesian fibration* if

(i) p is an inner fibration, and
(ii) for every arrow $t : a \to b$ of S and every vertex $y \in X$ with $p(y) = b$, there exists a Cartesian arrow $f : x \to y$ such that $p(f) = t$.

A *strict transform* between Cartesian fibrations $p : X \to S$ and $p' : X' \to S$ is a functor $F : X \to X'$ covering S, such that $F(f)$ is a Cartesian arrow in X' whenever f is a Cartesian arrow in X.

Remark 4.13. Cartesian fibrations over S are organized into an ∞-category $(\mathbf{Cart}_{/S})_{\text{strict}}$, which is a 1-full subcategory [Explain] of $\mathbf{Cat}_{/S}$ whose objects are Cartesian fibrations over S, and whose 1-morphisms are strict transforms.

4.2.2. *coCartesian fibration.* The notion of a Cartesian arrow has an obvious dual:

Definition 4.14. Let $p : X \to S$ be an inner fibration of simplicial sets. Let $f : x \to y$ be an arrow in X. Then f is *coCartesian* (or *p-coCartesian*) if the induced map

$$X_{f/} \to X_{x/} \times_{S_{p(x)/}} S_{p(f)/}$$

is an acyclic fibration.

We leave to the reader the definition of coCartesian fibrations and the ∞-category $(\mathbf{coCart}_{/S})_{\text{strict}}$.

4.2.3. *Straightening and unstraightening.* Let S be an ∞-category[4]. There is a model structure on the category $(\mathbf{Set}^+_\Delta)_{/S}$ of marked simplicial sets over S (considered as a marked simplicial set with all arrows marked), called the "Cartesian model structure." The fibrant-cofibrant objects of $(\mathbf{Set}^+_\Delta)_{/S}$ are exactly the Cartesian fibrations over S.

On the other hand, there is the category $\mathrm{Fun}(\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{Set}^+_\Delta)$ of simplicial functors $\mathfrak{C}[S]^{\mathrm{op}} \to \mathbf{Set}^+_\Delta$, equipped with the "projective model structure" whose underlying ∞-category is $\mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Cat})$. In this language, Lurie's theorem [Lu09, Theorem 3.2.0.1] can be stated as

Theorem 4.15. *There is a Quillen equivalence:*

$$(\mathbf{Set}^+_\Delta)_{/S} \underset{\mathrm{Un}}{\overset{\mathrm{St}}{\rightleftarrows}} \mathrm{Fun}(\mathfrak{C}[S]^{\mathrm{op}}, \mathbf{Set}^+_\Delta)$$

The procedure of taking "underlying ∞-categories" (explained in §3.1) then shows

Corollary 4.16. *There is an equivalence of ∞-categories*

$$(\mathbf{Cart}_{/S})_{\text{strict}} \underset{\mathrm{Un}}{\overset{\mathrm{St}}{\rightleftarrows}} \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Cat})$$

Remark 4.17. (i) There is a dual theorem which expresses the equivalence between $(\mathbf{coCart}_{/S})_{\text{strict}}$ and $\mathrm{Fun}(S, \mathbf{Cat})$.
(ii) Since the straightening functor is roughly speaking the process of "taking fibers" of a Cartesian fibration [I don't know how to prove a precise version of this.], the above equivalence restricts to an equivalence between Cartesian fibrations over S *in groupoids* and the ∞-category $\mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Spc})$.

[4] Many results here work more generally for any simplicial set S, though we won't need them. See [Lu09, §3.2] for more.

4.3. **Adjunction.** We will approach adjoint functors via straightening and unstraightening. This point of view actually proves quite fruitful (even for practical purposes later into the semester).

Consider a functor $F : \mathbf{C} \to \mathbf{D}$ of ∞-categories. It can be understood as an object in $\mathrm{Fun}(\Delta^1, \mathbf{Cat})$. Apply unstraightening, we obtain a coCartesian fibration $\mathbf{M} \to \Delta^1$.

Definition 4.18. The functor F *admits a right adjoint* if the coCartesian fibration $\mathbf{M} \to \Delta^1$ is also Cartesian.

The functor $G : \mathbf{D} \to \mathbf{C}$ right adjoint to F is then obtained via straightening the Cartesian fibration $\mathbf{M} \to \Delta^1$. I will now convince you that this agrees with the usual notion of adjunction (i.e., existence of unit transform, etc.)

Proposition 4.19. *The functor G is right adjoint to F if and only if there exists a morphism $\eta : \mathrm{id}_{\mathbf{C}} \to GF$ such that the composition*

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{D}}(Fc, d) & \longrightarrow & \mathrm{Map}_{\mathbf{D}}(GFc, Gd) \\ & \searrow & \downarrow{\scriptstyle \eta^*} \\ & & \mathrm{Map}_{\mathbf{C}}(c, Gd) \end{array}$$

is a weak equivalence (of simplicial sets) for all objects $c \in \mathbf{C}$ and $d \in \mathbf{D}$.

Sketch of proof. I will tell you how to construct η assuming that the fibration $\mathbf{M} \to \Delta^1$ is both Cartesian and coCartesian.

First, there is an explicit description of straightening and unstraightening when the base is Δ^1. If a coCartesian fibration $\mathbf{M} \to \Delta^1$ arises from $F : \mathbf{C} \to \mathbf{D}$ via unstraightening, then the fibers

$$\mathbf{M}_{\{0\}} \cong \mathbf{C}, \quad \mathbf{M}_{\{1\}} \cong \mathbf{D}$$

and there exists a functor $\mathscr{F} : \mathbf{C} \times \Delta^1 \to \mathbf{M}$ covering $\mathbf{M} \to \Delta^1$ such that

$$\mathscr{F}\big|_{\mathbf{C}\times\{0\}} = \mathrm{id}_{\mathbf{C}}, \quad \mathscr{F}\big|_{\mathbf{C}\times\{1\}} = F,$$

and for each $c \in \mathbf{C}$, $\mathscr{F}\big|_{\{c\}\times\Delta^1}$ sends Δ^1 to a coCartesian arrow in $\mathbf{M}$. Dualizing, if $\mathbf{M} \to \Delta^1$ is also Cartesian, then we have a functor $\mathscr{G} : \mathbf{D} \times \Delta^1 \to \mathbf{M}$ satisfying the analogous properties.

We extend $\mathscr{F}$ to a functor $\mathbf{C} \times \Lambda^2_2 \to \mathbf{M}$ given on the two edges via

$$\begin{array}{ccc} & \mathbf{C} \times \{1\} & \\ & & \searrow{\scriptstyle \mathscr{G}\circ(F\times\mathrm{id}_{\Delta^1})} \\ \mathbf{C} \times \{0\} & \xrightarrow{\quad\mathscr{F}\quad} & \mathbf{C} \times \{2\} \end{array}$$

which makes sense since both functors evaluate to F on the shared vertex $\mathbf{C} \times \{2\}$. Finally, the fact that $\mathbf{M} \to \Delta^1$ is Cartesian allows us to extend the functor $\mathbf{C} \times \Lambda^2_2 \to \mathbf{M}$ to $\mathbf{C} \times \Delta^2 \to \mathbf{M}$. The new edge $\mathbf{C} \times \Delta^{\{0,1\}}$ gives rise to the unit transform $\eta \to GF$. Details can be found in [Lu09, §5.2.2]. ☐

4.4. **Yoneda.** We will now use straightening to write down the Yoneda functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Spc}$. Consider first the strict transform of Cartesian fibrations:

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, \mathbf{C}) & \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} & \mathbf{C} \times \mathbf{C} \\ {\scriptstyle \mathrm{ev}_0}\searrow & & \swarrow{\scriptstyle \mathrm{pr}_1} \\ & \mathbf{C} & \end{array}$$

Apply straightening, and we obtain a natural transformation of functors from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{Cat}$:

$$(c \mapsto \mathbf{C}_{c/}) \rightsquigarrow (c \mapsto \mathbf{C}).$$

Note that a natural transformation is an object in

$$\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Cat})) \cong \mathrm{Fun}(\Delta^1 \times \mathbf{C}, \mathbf{Cat}) \cong \mathrm{Fun}(\mathbf{C}, \mathrm{Fun}(\Delta^1, \mathbf{Cat})).$$

As an object in the rightmost ∞-category, it is the functor that associates to each $c \in \mathbf{C}$ the functor $\mathbf{C}_{c/} \to \mathbf{C}$. Note that the latter functor is a coCartesian fibration in groupoid. Applying straightening again, we obtain the Yoneda functor $\mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Spc}$.

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