HIGHER ALGEBRA

Contents

In this talk, we will cover the definitions of algebra, module, and the categories of them in the ∞-categorical world. We will start most concepts by introducing their classical analogues.

1. Monoidal categories

1.1. **How does it work for ordinary categories?** Recall that a *monoid* M is a set with

(i) a map $M \times M \to M$, denotes $(x, y) \mapsto xy$, and
(ii) an object $1 \in M$ such that

$$1x = x = x1, \quad x(yz) = (xy)z.$$

The definition of a monoidal category is very similar to this, but "one category level up":

Definition 1.1. A *monoidal category* $\mathscr{C}$ is a category with $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, and $I \in \mathscr{C}$ with functorial isomorphisms

$$X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z, \quad I \otimes X \cong X \otimes I \cong X$$

such that a bunch of coherence conditions hold.

Example 1.2. (i) The ordinary category $\mathbf{Vect}_k^{\mathrm{ord}}$ of k-vector spaces, together with tensor product forms a monoidal category;
(ii) the category $\mathbf{Ab}$ of abelian groups, together with tensor product (over $\mathbb{Z}$);
(iii) the category $\mathbf{Set}$, together with products.

1.1.1. *Packaging $\mathscr{C}$ into one functor.* It turns out that we can package the entire data of a monoidal category (together with all its coherence axioms) into one functor.

We first associate to $\mathscr{C}$ a category $\mathscr{C}^{\otimes}$ with objects $(X_1, \cdots, X_n)$ $(n \geq 0)$, where each $X_i \in \mathscr{C}$, and a morphism

$$(X_1, \cdots, X_n) \to (Y_1, \cdots, Y_m)$$

consists of a non-decreasing map $\phi : [m] \to [n]$, together with a morphism for each $1 \leq i \leq m$:

$$f_i : X_{\phi(i-1)+1} \otimes X_{\phi(i-1)+2} \otimes \cdots \otimes X_{\phi(i)} \to Y_i.$$

Remark 1.3. If $\phi(i-1) = \phi(i)$, then we require a morphism $f_i : \mathbf{1} \to Y_i$.

The category $\mathscr{C}^{\otimes}$ comes with an obvious functor to the opposite category of simplices Δ^{op}:

$$\begin{array}{ccc} \mathscr{C}^{\otimes} & & (X_1, \cdots, X_n) \\ \big\downarrow{\scriptstyle p} & & \big\downarrow \\ \Delta^{\mathrm{op}} & & [n] \end{array}$$

We will note the following two straightforward facts about the functor p:

Date: February 12, 2016. This is a talk by Alex Perry, and these notes are taken by Yifei Zhao. Blame it on Yifei if you find any error (and make sure he corrects them.)

Lemma 1.4. *The functor p is a coCartesian fibration of ordinary categories.*

Proof. Suppose we are given the following solid diagram:

$$\begin{array}{ccc} (X_1, \cdots, X_n) & \dashrightarrow & (Y_1, \cdots, Y_m) \\ {\scriptstyle p}\downarrow & & \vdots \\ [n] & \xrightarrow{\phi^{\mathrm{op}}} & [m] \end{array}$$

We want to lift ϕ^{op} to a coCartesian morphism in $\mathscr{C}^{\otimes}$ with target some $(Y_1, \cdots, Y_m)$. Indeed, we simply define Y_i to be such an object that $f_i : X_{\phi(i-1)+1} \otimes \cdots \otimes X_{\phi(i)} \xrightarrow{\sim} Y_i$ is an isomorphism.

Now, given any $(Z_1, \cdots, Z_l)$ covering $[l] \in \Delta^{\mathrm{op}}$, together with a morphism $(X_1, \cdots, X_n) \to (Z_1, \cdots, Z_l)$ covering a composition:

$$[n] \xrightarrow{\phi^{\mathrm{op}}} [m] \xrightarrow{\psi^{\mathrm{op}}} [l],$$

we need to produce a map $(Y_1, \cdots, Y_m) \to (Z_1, \cdots, Z_l)$ covering ψ^{op}. Note that we already have

$$g_j : X_{\phi(\psi(j-1))+1} \otimes \cdots \otimes X_{\phi(\psi(j))} \to Z_j,$$

but the left-hand-side maps isomorphically to $Y_{\psi(j-1)+1} \otimes \cdots \otimes Y_{\psi(j)}$. □

We will set $\phi^i : [1] \to [n]$ to be the map sending $\{0, 1\}$ to $\{i-1, i\}$. The definition of $\mathscr{C}^{\otimes}_{[1]}$ shows that there is an equivalence $\mathscr{C}^{\otimes}_{[1]} \xrightarrow{\sim} \mathscr{C}$; this category receives functors: $(\phi^i)^{\mathrm{op}}_! : \mathscr{C}^{\otimes}_{[n]} \to \mathscr{C}^{\otimes}_{[1]}$. The second fact follows immediately:

Lemma 1.5. *We have an equivalence $\mathscr{C}^{\otimes}_{[n]} \xrightarrow{\sim} (\mathscr{C}^{\otimes}_{[1]})^n \cong \mathscr{C}^n$. Under this equivalence, the functor $(\phi^i)^{\mathrm{op}}_!$ corresponds to the i^{th} projection.* □

1.1.2. *Recovering the monoidal category $\mathscr{C}$.* From a monoidal category $\mathscr{C}$, we have obtained a coCartesian fibration $\mathscr{C}^{\otimes} \to \Delta^{\mathrm{op}}$. We now start with a category $\mathscr{C}^{\otimes}$ satisfying the conclusions of Lemma 1.4 and 1.5 and try to recover the category $\mathscr{C}$ together with its monoidal structure.

In what follows, let $p : \mathscr{C}^{\otimes} \to \Delta^{\mathrm{op}}$ be a coCartesian fibration such that the functors induced by each ϕ^i gives an equivalence $\mathscr{C}^{\otimes}_{[n]} \xrightarrow{\sim} (\mathscr{C}^{\otimes}_{[1]})^n$. We make the following observations:

(i) $\mathscr{C}^{\otimes}_{[0]}$ is the one-point category. This follows from Lemma 1.5 with $n = 0$. From the projection $[1] \to [0]$, we obtain a functor $\mathscr{C}^{\otimes}_{[0]} \to \mathscr{C}^{\otimes}_{[1]}$ whose image we call I.

(ii) Consider $d^1 : [1] \to [2]$ mapping $\{0, 1\}$ to $\{0, 2\}$. Then we have

$$\begin{array}{ccc} \mathscr{C}^{\otimes}_{[2]} & \xrightarrow{(d^1)^{\mathrm{op}}_!} & \mathscr{C}^{\otimes}_{[1]} \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\ \mathscr{C} \times \mathscr{C} & \longrightarrow & \mathscr{C} \end{array}$$

which recovers the monoidal structure $(X_1, X_2) \mapsto X_1 \otimes X_2$ via the horizontal functor below.

(iii) Now let's see how we recover the coherence axioms. Consider the following two distinct factorizations of the map $[1] \to [3]$:

$$\begin{array}{ccc} [3] & \xleftarrow{d^1} & [2] \\ {\scriptstyle d^2}\uparrow & & \uparrow{\scriptstyle d^1} \\ [2] & \xleftarrow{d^1} & [1] \end{array}$$

This commutative square induces a diagram of categories:

$$\begin{array}{ccccc} (X_1, X_2, X_3) & \mathscr{C}^{\otimes}_{[3]} & \longrightarrow & \mathscr{C}^{\otimes}_{[2]} & (X_1 \otimes X_2, X_3) \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ (X_1, X_2 \otimes X_3) & \mathscr{C}^{\otimes}_{[2]} & \longrightarrow & \mathscr{C}^{\otimes}_{[1]} & X_1 \otimes (X_2 \otimes X_3) \cong (X_1 \otimes X_2) \otimes X_3 \end{array}$$

which establishes the associativity constraint.
(iv) The pentagon axiom follows simply by considering factorizations of $[1] \to [4]$.

1.2. **Monoidal ∞-categories.** The idea of packaging the coherence axioms into a single functor leads to the following definition—the functor will now help us keep track of an infinite tail of coherence axioms.

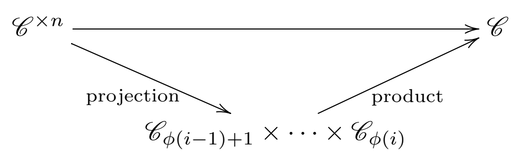
Definition 1.6. A *monoidal ∞-category* is a coCartesian fibration $\mathscr{C}^{\otimes} \to \mathrm{N}(\Delta^{\mathrm{op}})$ such that the morphism induced by ϕ^i is an isomorphism $\mathscr{C}^{\otimes}_{[n]} \xrightarrow{\sim} (\mathscr{C}^{\otimes}_{[1]})^n$.

We often abuse the notation and refer to the *underlying ∞-category* $\mathscr{C} := \mathscr{C}^{\otimes}_{[1]}$ as a monoidal ∞-category. The reader should keep in mind that the data defining a monoidal ∞-category are much larger than the ∞-category $\mathscr{C}$.

Example 1.7. (i) If $\mathscr{C}$ is an ordinary monoidal category, then $\mathrm{N}(\mathscr{C}^{\otimes})$ is a monoidal ∞-category.
(ii) The ∞-category **Vect** from last week's lecture is a monoidal ∞-category whose monoidal structure is given by tensor product of complexes.
(iii) The category $\mathbf{Perf}(S)$ of perfect complexes, where S is an ordinary scheme, is a monoidal ∞-category.
(iv) Suppose that an ∞-category $\mathscr{C}$ admits finite products. Then we will define a monoidal ∞-category $\mathscr{C}^{\times}$. Consider the functor

$$\mathrm{N}(\Delta^{\mathrm{op}}) \to \mathbf{Cat}$$

given by $[n] \mapsto \mathscr{C}^{\times n}$, and $\phi : [m] \to [n]$ gets sent to $\mathscr{C}^{\times n} \to \mathscr{C}^{\times m}$. On the i^{th} copy of $\mathscr{C}$ in $\mathscr{C}^{\times m}$, this is the map given by the composition



One can check that that this defines a functor between ∞-categories using a strict model of **Cat**.[1] The ∞-category $\mathscr{C}^{\times}$ is defined by unstraightening this functor.

2. Associative algebra

In derived algebraic geometry, we have to work with algebras inside a monoidal ∞-category, and modules over them. Before we define these objects, let's first review how algebra in ordinary monoidal categories work.

2.1. **Algebras in ordinary monoidal categories.** Let $\mathscr{C}$ be an ordinary monoidal category. An *algebra object* is an object A in $\mathscr{C}$ equipped with a multiplication morphism $A \otimes A \to A$ and a unit morphism $I \to A$ together with some compatibility conditions.

Example 2.1. (i) Algebras in $(\mathbf{Sets}, \times)$ are exactly the monoids.
(ii) Algebras in $(\mathbf{Ab}, \otimes)$ are ordinary rings.
(iii) Algebras in $(\mathbf{Vect}_k^{\mathrm{ord}}, \otimes)$ are ordinary k-algebras.

2.1.1. *Packaging an algebra into a section.* Recall that the monoidal structure on $\mathscr{C}$ determines a coCartesian fibration $\mathscr{C}^{\otimes} \to \Delta^{\mathrm{op}}$. The data of an algebra gives a section $A^{\otimes}$ of $\mathscr{C}^{\otimes} \to \Delta^{\mathrm{op}}$, which is defined by

$$A^{\otimes}[n] = (A, \cdots, A),$$

and every morphism $\phi^{\mathrm{op}} : [n] \to [m] \in \Delta^{\mathrm{op}}$ is mapped to

$$A^{\otimes}(\phi^{\mathrm{op}}) : \underbrace{(A, \cdots, A)}_{n} \to \underbrace{(A, \cdots, A)}_{m}$$

defined as the multiplication $c_{\phi(i-1)+1, \cdots, \phi(i)} : A \otimes \cdots \otimes A \to A$ on the i^{th} copy of A.

Remark 2.2. Not every section arises from algebra objects! Suppose we are given an arbitrary section $A^{\otimes} : \Delta^{\mathrm{op}} \to \mathscr{C}^{\otimes}$ given by $A^{\otimes}[n] = (A_1, \cdots, A_n)$. There is no guarantee that the A_i's will be isomorphic.

[1] This is pointed out by Jeremy.

It turns out that the condition for $A^{\otimes}$ to come from an algebra object in $\mathscr{C}$ is precisely that $A^{\otimes}$ takes ϕ^i to a coCartesian morphism of $\mathscr{C}^{\otimes}$.

2.2. **Algebras in a monoidal ∞-category.** With these considerations in mind, we will now make the following definition:

Definition 2.3. Let $\mathscr{C}$ be a monoidal ∞-category, an *algebra in* $\mathscr{C}$ is a section of $\mathscr{C}^{\otimes} \to \mathrm{N}(\Delta^{\mathrm{op}})$ that takes each ϕ^i to a coCartesian morphism.

We obtain *the* ∞*-category* $\mathbf{Alg}(\mathscr{C})$ *of algebras in* $\mathscr{C}$ as the full subcategory of sections of $\mathscr{C}^{\otimes} \to \mathrm{N}(\Delta^{\mathrm{op}})$ satisfying the above property.

Example 2.4.
(i) The algebra objects in $\mathbf{Vect}_k$ will be called *DG algebras*; denote the ∞-category $\mathbf{Alg}(\mathbf{Vect}_k)$ by $\mathbf{DGA}_k$. This ∞-category has a strict model: the ordinary category of DG algebras comes with a model structure, whose underlying ∞-category is $\mathbf{DGA}_k$.
(ii) The ∞-category $\mathbf{Alg}(\mathbf{Spc})$ is the ∞-category of A_∞*-spaces*.

2.2.1. *Monoid objects.* Let $\mathscr{C}$ be an ∞-category with finite products. Recall that we have defined a monoidal ∞-category $\mathscr{C}^{\times}$. What are the algebra objects in $\mathscr{C}^{\times}$?

Definition 2.5. A *monoidal object* of $\mathscr{C}$ is an algebra object of $\mathscr{C}^{\times}$.

Example 2.6. For instance, $\mathrm{Alg}(\mathbf{Cat}^{\times})$ is the category of monoidal ∞-category.

3. Modules

Let $\mathscr{C}$ be a monoidal category. There is a (classical) notion of a module category $\mathscr{M}$ over $\mathscr{C}$; given an algebra object A in $\mathscr{C}$, we can define modules over A inside the category $\mathscr{M}$.

3.1. **Module categories.** The concept of a module category (over a monoidal category) upgrades the familiar notion of a module (over a ring) by one category level—we are supposed to have a functor $\mathscr{C} \times \mathscr{M} \to \mathscr{M}$ together with various coherence axioms.

3.1.1. *Packaging module categories into a functor.* Without spelling out the coherence data, let's launch directly into the presentation of $\mathscr{M}$ as a functor. This time, we will need our base category Δ^{op} to allow for a placeholder for $\mathscr{M}$:

Definition 3.1. The category Δ^+ is the ordinary category with
(i) objects $[n]$ for each n, as well as $[n]^+ = \{0, 1, \cdots, n, +\}$, and
(ii) morphisms as follows: the $[m] \to [n]$ ones are as usual (non-decreasing functions); the $[m] \to [n]^+$ ones are required to miss $+$; the $[m]^+ \to [n]^+$ ones are required to take the distinguished element $+$ to $+$.

Given a module category $\mathscr{M}$ over $\mathscr{C}$, we will build a category $\mathscr{M}^{\otimes}$ that fits into a pullback diagram:

$$\begin{array}{ccc} \mathscr{C}^{\otimes} & \longrightarrow & \mathscr{M}^{\otimes} \\ \downarrow & & \downarrow \\ \Delta^{\mathrm{op}} & \longrightarrow & \Delta^{+,\mathrm{op}} \end{array}$$

This new category $\mathscr{M}^{\otimes}$ has the following *additional* data: [I haven't specified morphisms covering $[m] \to [n]^+$.]
(i) Objects are triples $([n]^+, (X_1, \cdots, X_n), Y)$, where each $X_i \in \mathscr{C}$, and $Y \in \mathscr{M}$.
(ii) A morphism
$$([n]^+, (X_1, \cdots, X_n), Y) \to ([m]^+, (X_1', \cdots, X_m'), Y')$$
consists of one morphism $\phi : [m]^+ \to [n]^+$ in Δ^+, a bunch of morphisms in $\mathscr{C}$:
$$f_i : X_{\phi(i-1)+1} \otimes \cdots \otimes X_{\phi(i)} \to X_i' \quad (1 \le i \le m)$$
as well as a morphism $f_+ : X_{\phi(m)+1} \otimes \cdots \otimes X_n \otimes Y \to Y'$.

Lemma 3.2. *The functor* $\mathscr{M}^{\otimes} \to \Delta^{+,\mathrm{op}}$ *is coCartesian.* □

Lemma 3.3. *The functor* $\mathscr{M}^{\otimes}_{[n]^+} \to \mathscr{M}^{\otimes}_{[n]} \times \mathscr{M}^{\otimes}_{[0]^+}$ *induced by the maps* $[n] \to [n]^+$ *(sending each* i *to* i*) and* $[0]^+ \to [n]^+$ *(sending* 0 *to* n *and* $+ \mapsto +$*) is an equivalence.* □

3.1.2. *Generalization to ∞-categories.* We leave the straightforward ∞-categorical generalization of this concept to the reader.

Example 3.4. (1) Given a morphism $X \to S$ of classical schemes, the category $\mathbf{Perf}(X)$ is a module category over $\mathbf{Perf}(S)$.

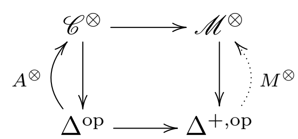
(2) Suppose $\mathscr{C}^{\otimes}$ is a monoidal ∞-category. It is a module over itself via the retraction $\Delta \to \Delta^{+} \to \Delta$. Here, the first map is obvious, and the second map is given by

$$[n] \mapsto [n], \quad [n]^{+} \mapsto [n+1].$$

This is, in fact, the most important case for us. We will frequently look for modules over a k-algebra A (i.e., an object in $\mathbf{Alg}(\mathbf{Vect}_k)$) inside the category $\mathbf{Vect}_k$ itself.

3.2. **Modules over an algebra.** Suppose $\mathscr{C}$ is a monoidal ∞-category. Let A be an object in $\mathbf{Alg}(\mathscr{C})$. Say we also have a module ∞-category $\mathscr{M}$ over $\mathscr{C}$.

The above data are summarized by the following solid diagram:



Definition 3.5. A *module object* in $\mathscr{M}$ over A is a section $M^{\otimes}$ as pictured such that $M^{\otimes}$ restricts to $A^{\otimes}$ over Δ^{op}. [There's a comment about whether *more* morphisms should be sent to coCartesian ones.]

We obtain the ∞-category A-$\mathbf{Mod}(\mathscr{M})$. We will simply write A-$\mathbf{Mod}$ if $\mathscr{C} = \mathscr{M}$.

Example 3.6. (i) Let $\mathscr{C} = \mathbf{Vect}_k^{\mathrm{ord}}$, and $A \in \mathbf{Alg}(\mathbf{Vect}_k)$ be an ordinary DG algebra. Then A-$\mathbf{Mod}$ again has a strict model as the underlying ∞-category of the model category of ordinary DG modules over A.

(ii) One can repeat the construction in Example 2.6 for modules ∞-categories.

4. Symmetric stuff

4.1. **Symmetric monoidal categories.** Let $\mathscr{C}$ be an ordinary category. Then for it to be *symmetric monoidal*, we want (in addition to being monoidal) a functorial isomorphism

$$s_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X.$$

The symmetric case requires us to change the base "operad" again:

Definition 4.1. The category Fin_* has objects $\langle n \rangle = \{1, \cdots, n\} \cup \{*\}$ and morphisms the pointed maps.

4.1.1. *Packaging into a functor.* We define a category $\mathscr{C}^{\otimes}$ over Fin_*. An object of $\mathscr{C}^{\otimes} \to \mathrm{Fin}_*$ will be $(X_1, \cdots, X_n)$. A morphism consists of $\alpha : \langle n \rangle \to \langle m \rangle$ together with a map $\bigotimes_{j \in \alpha^{-1}(i)} X_j \to Y_i$. Furthermore, we require equivalences

$$\mathscr{C}^{\otimes}_{\langle n \rangle} \xrightarrow{\sim} (\mathscr{C}^{\otimes}_{\langle 1 \rangle})^{\times n} \xrightarrow{\sim} \mathscr{C}^{n}$$

for the functors corresponding to $\alpha^i : \langle n \rangle \to \langle 1 \rangle$ where

$$\alpha^i(j) = \begin{cases} 1, i = j \\ *, \text{otherwise} \end{cases}$$

4.1.2. *Time is out.* And the rest is left to *you*.