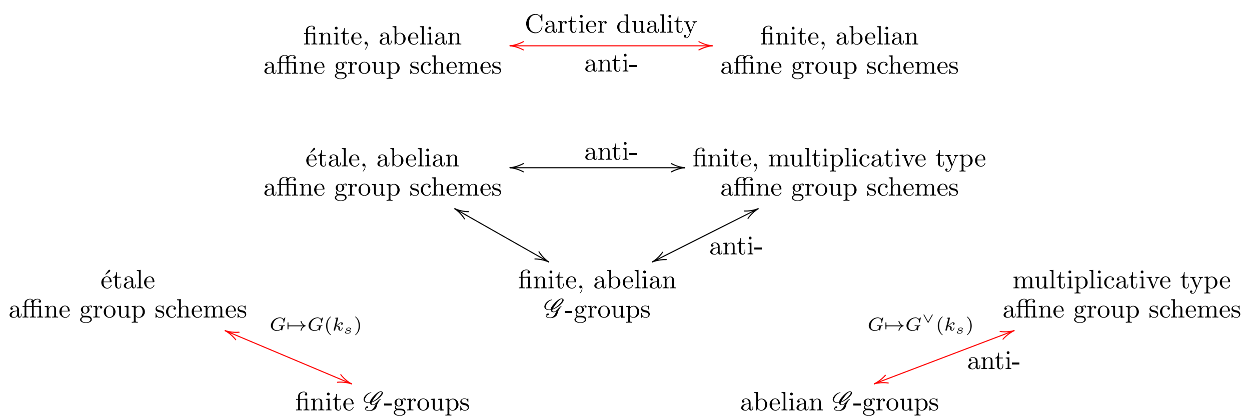
FINITE AND ABELIAN AFFINE GROUP SCHEMES

YIFEI ZHAO

Contents

The goal of these notes is to describe the simplest kind of affine group schemes: the finite ones and the abelian ones, as well as special cases of them—étale and multiplicative type affine group schemes. To keep us anchored, we set out to prove the three equivalences of categories labeled by red arrows in the following diagram. Restricting these equivalences to full subcategories yields the three-way equivalence in the middle of the diagram:



Everything here is from Waterhouse [1, pp. 1-61].

0.1. **Notations and assumptions.** In the above diagram:

(i) k denotes a field, k_s its separable closure, and $\mathscr{G}$ be the absolute Galois group of k:
$$\mathscr{G} = \mathrm{Gal}(k_s/k) = \varprojlim_{\substack{k \subset L \\ \text{finite Galois}}} \mathrm{Gal}(L/k);$$
(ii) a *$\mathscr{G}$-set* is a set X equipped with a $\mathscr{G}$-action that is *continuous* in the sense that X can be written as a union of X_α, such that on each X_α, the $\mathscr{G}$-action factors through $\mathrm{Gal}(L_\alpha/k)$ for some finite Galois extension L_α of k;
(iii) a *$\mathscr{G}$-group* is a $\mathscr{G}$-set equipped with a group structure such that $\mathscr{G}$ acts by group homomorphisms;
(iv) all schemes and morphisms between schemes are assumed to be over k.
(v) all other terms in the diagram will be explained in detail.

Date: July 9, 2015.

1. Generalities

There are a number of equivalent ways to define affine group schemes. We fix a ground *ring* k, so the discussion in this section is completely general. We also think of a scheme as a functor $k\textbf{-Alg} \to \textbf{Set}$. In particular, given a k-algebra A, the notation $\mathrm{Spec}(A)$ denotes the functor $k\textbf{-Alg}(k, -)$ rather than a locally ringed space.

1.1. **Group-valued functors I.** An *affine group scheme* over k is a representable functor

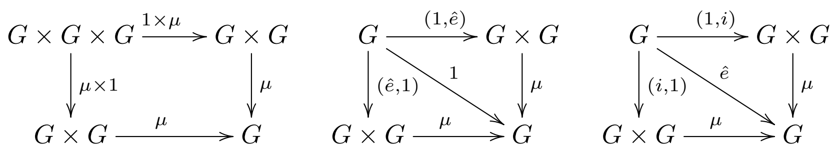
$$G : k\textbf{-Alg} \to \textbf{Set}$$

which factors through **Grp**;

1.2. **Group-valued functors II.** A functor $G : k\textbf{-Alg} \to \textbf{Set}$ factors through **Grp** if and only if there are natural transformations

$$\mu : G \times G \to G, \quad e : \mathrm{Spec}(k) \to G, \quad i : G \to G$$

such that the following diagrams commute (*associative*, *unital*, *invertible*):



where $\hat{e}$ is the composition $G \to \mathrm{Spec}(k) \xrightarrow{e} G$.

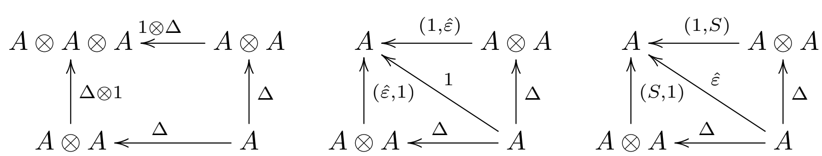
Indeed, if G factors through **Grp**, then

(i) for each k-algebra R, the map μ_R is the group multiplication on $G(R)$, and the map i_R is the inverse on $G(R)$;
(ii) $e : \mathrm{Spec}(k) \to G$ is the unique natural transformation (by Yoneda lemma) corresponding to the unit element in $G(k)$.

1.3. **Hopf k-algebras.** A *Hopf k-algebra* is a k-algebra A equipped with k-algebra homomorphisms

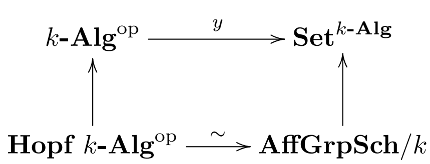
$$\Delta : A \to A \otimes A, \quad \varepsilon : A \to k, \quad S : A \to A$$

such that the following diagrams commute (*co-associative*, *co-unital*, *antipodal*):



where $\hat{\varepsilon}$ is the composition $A \xrightarrow{\varepsilon} k \to A$.

Given an affine group scheme G represented by the k-algebra A, Yoneda lemma shows that the natural transformations μ, e, and i of §1.2 correspond to k-algebra homomorphisms that turn A into a Hopf k-algebra. Conversely, for each Hopf algebra A, the functor $\mathrm{Spec}(A) = k\textbf{-Alg}(A, -)$ is an affine group scheme. Clearly, homomorphisms of Hopf k-algebras from A to B are in bijection with homomorphisms of affine group schemes from $\mathrm{Spec}(B)$ to $\mathrm{Spec}(A)$. In other words, restricting the Yoneda embedding to Hopf k-algebras we get a fully faithful, essentially surjective functor to affine group schemes over k:



So the bottom arrow is an equivalence of categories.

1.4. **A cheap trick.** We prove a useful little lemma:

Lemma 1.1. *Let $f : A \to B$ be a k-algebra morphism of Hopf k-algebras. Then f is a morphism of Hopf k-algebras whenever f commutes with Δ.*

Proof. We can reword the proof of the corresponding statement for groups line by line. However, the above generalities offer a cheap trick: if $f : A \to B$ commutes with Δ, then the corresponding natural transformation $f^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ commutes with multiplication μ. Because a map of groups is a morphism whenever it commutes with multiplication, f^* is a morphism of affine group schemes. As such, the map f must be a morphism of Hopf k-algebras. □

1.5. **An example: $\mathbb{G}_m$.** The affine group scheme $\mathbb{G}_m$ is the functor of units:

$$R \mapsto R^\times.$$

It is represented by the Hopf k-algebra $k[T, T^{-1}]$ with

$$\Delta(T) = T \otimes T, \quad \varepsilon(T) = 1, \quad S(T) = T^{-1}.$$

A *torus* is an affine group scheme isomorphic to a finite product of $\mathbb{G}_m$.

Lemma 1.2. *Let G be an affine group scheme represented by Hopf k-algebra A. There is a group isomorphism*[1]*:*

$$\mathrm{Hom}(G, \mathbb{G}_m) \cong \{a \in A^\times : \Delta(a) = a \otimes a\}$$

Elements of $\mathrm{Hom}(G, \mathbb{G}_m)$ are called *characters* of G, whereas elements of the set on the right are called *group-likes* of A.

Proof. $\mathrm{Hom}(G, \mathbb{G}_m)$ consists precisely of morphisms of Hopf k-algebras

$$k[T, T^{-1}] \to A.$$

Such morphisms are determined by their image of T, and by Lemma 1.1, correspond bijectively to group-likes in A. The group multiplication on $\mathrm{Hom}(G, \mathbb{G}_m)$ is induced by that of $\mathbb{G}_m$. In other words, if $\varphi_a, \varphi_b \in \mathrm{Hom}(G, \mathbb{G}_m)$ sends T to a respectively b. Then $\varphi_a \cdot \varphi_b$ fits in the diagram

$$\begin{array}{ccc} k[T,T^{-1}] & \xrightarrow{\Delta} & k[T,T^{-1}] \otimes k[T,T^{-1}] \\ & \searrow^{\varphi_a \cdot \varphi_b} & \downarrow{\scriptstyle (\varphi_a, \varphi_b)} \\ & & A \end{array}$$

Hence $\varphi_a \cdot \varphi_b$ maps T to ab, so $\varphi_a \cdot \varphi_b = \varphi_{ab}$. □

2. Cartier duality

In this section, we may again let k be a ring.

2.1. **Duality of Hopf k-algebras.** A Hopf k-algebra is a k-vector space that has both an algebra and a "co-algebra" structure. We rewrite the definition in a way that makes this symmetry more apparent. Indeed, a Hopf k-algebra is a k-vector space A endowed with k-linear maps

$$\begin{array}{ccccc} A & A & A & k & A \otimes A \\ m \uparrow \text{ multiplication, associative, commutative} & u \uparrow \text{ unit} & S \uparrow \text{ antipode} & \varepsilon \uparrow \text{ co-unit} & \Delta \uparrow \text{ co-multiplication, co-associative} \\ A \otimes A & k & A & A & A \end{array}$$

The condition that S, ε, Δ are algebra morphisms amounts to the following compatibility conditions (diagrams omitted):

(i) co-multiplication commutes with multiplication;

[1]The Hom on the left is taken in the category **AffGrpSch**$/k$. For notational simplicity, morphisms in this category will always be denoted by Hom.

(ii) co-multiplication preserves unit;
(ii$^\vee$) co-unit commutes with multiplication;
(iii) co-unit preserves unit;
(iv) antipode commutes with multiplication;
(v) antipode preserves unit.

Lemma 2.1. *Let A be a Hopf k-algebra.*
(v$^\vee$) antipode preserves co-unit;
(iv$^\vee$) if A is co-commutative, then antipode commutes with co-multiplication.

Proof. These facts can be translated from their group-theoretic equivalents. More precisely, $k\text{-}\mathbf{Alg}(A, k)$ is a group with identity ε (by definition!), whose inverse is given by $S\varepsilon$. Hence $S\varepsilon = \varepsilon$.

If A is co-commutative, for each k-algebra R, the group $k\text{-}\mathbf{Alg}(A, R)$ is commutative. So $(gh)^{-1} = g^{-1}h^{-1}$ for all $g, h \in k\text{-}\mathbf{Alg}(A, R)$. In other words, ΔS agrees with $(S, S)\Delta$ after post-composing with any k-algebra homomorphism $(g, h) : A \otimes A \to R$. Apply this to $(1, 1) : A \otimes A \to A \otimes A$, and we obtain $\Delta S = (S, S)\Delta$, as required. □

Lemma 2.1 shows that if Δ is co-commutative, the structure of a Hopf algebra is completely symmetric. More precisely:

Theorem 2.2 (Cartier duality)**.** *Suppose A is a co-commutative Hopf k-algebra. Then its dual $A^\vee = k\text{-}\mathbf{Mod}(A, k)$ is a co-commutative Hopf k-algebra with the following structure maps:*

$$\begin{array}{ccccc} A^\vee & A^\vee & A^\vee & k & A^\vee \otimes A^\vee \\ \Delta^\vee \uparrow \textit{multiplication} & \varepsilon^\vee \uparrow \textit{unit} & S^\vee \uparrow \textit{antipode} & u^\vee \uparrow \textit{co-unit} & m^\vee \uparrow \textit{co-multiplication} \\ A^\vee \otimes A^\vee & k & A^\vee & A^\vee & A^\vee \end{array}$$

Furthermore, if A is a finite, projective k-module, then the same holds for $A^\vee$, and $(A^\vee)^\vee \cong A$ naturally.

Proof. Clearly, the maps $\Delta^\vee, \varepsilon^\vee, S^\vee, u^\vee, m^\vee$ are k-linear. We only need to check the compatibility conditions. Indeed, condition (i) for A is equivalent to condition (i) for $A^\vee$:

$$\begin{array}{ccc} A \otimes A & \xleftarrow{m \otimes m} & A \otimes A \otimes A \otimes A \\ \uparrow \Delta & & \uparrow \Delta \otimes \Delta \\ A & \xleftarrow{m} & A \otimes A \end{array} \iff \begin{array}{ccc} A^\vee \otimes A^\vee & \xrightarrow{m^\vee \otimes m^\vee} & A^\vee \otimes A^\vee \otimes A^\vee \otimes A^\vee \\ \downarrow \Delta^\vee & & \downarrow \Delta^\vee \otimes \Delta^\vee \\ A^\vee & \xrightarrow{m^\vee} & A^\vee \otimes A^\vee \end{array}$$

Similarly, (ii) for A is (ii$^\vee$) for $A^\vee$, vice versa. (iii) for A is (iii) for $A^\vee$. (iv) for A is (iv$^\vee$) for $A^\vee$, vice versa. (v) for A is (v$^\vee$) for $A^\vee$, vice versa.

The second statement is a result of commutative algebra. We omit its proof. □

2.2. **Representing characters.** Given an affine group scheme G over k, one may associate a functor $G^\vee : k\text{-}\mathbf{Alg} \to \mathbf{Set}$ by

$$R \mapsto \mathrm{Hom}(G \times_k R, \mathbb{G}_m \times_k R).$$

Clearly, $G^\vee$ factors through $\mathbf{Grp}$. Note that $G^\vee(k)$ is the character group of G. However, $G^\vee$ may not always be representable by an affine group scheme.

Proposition 2.3. *Suppose G is an abelian, affine group scheme represented by Hopf k-algebra A. If A is a finite, projective k-module, then $G^\vee$ is represented by $A^\vee$, and $(G^\vee)^\vee \cong G$ naturally.*

Proof. Indeed, $G^\vee(k) = \mathrm{Hom}(G, \mathbb{G}_m)$ corresponds to the group-likes in A (Lemma 1.2). On the other hand, the isomorphism of double dual restricts:

$$\begin{array}{ccc} A & \xrightarrow{\sim} & k\text{-}\mathbf{Mod}(A^\vee, k) \\ \uparrow & & \uparrow \\ \text{Group-likes}(A) & \xrightarrow{\sim} & k\text{-}\mathbf{Alg}(A^\vee, k) \end{array}$$

Therefore we obtain an isomorphism $G^\vee(k) \xrightarrow{\sim} k\text{-}\mathbf{Alg}(A^\vee, k)$.

Now, for any k-algebra R, we have

$$G^\vee(R) \cong (G \times_k R)^\vee(R) \cong R\text{-}\mathbf{Alg}((A \otimes_k R)^\vee, R) \cong R\text{-}\mathbf{Alg}(A^\vee \otimes_k R, R) \cong k\text{-}\mathbf{Alg}(A, R)$$

as required. The isomorphism $(G^\vee)^\vee \cong G$ is clear from Theorem 2.2. □

We call an affine group scheme G *finite* if its representing Hopf k-algebra is a finite k-module.

2.3. **Examples: k^Γ and $k[\Gamma]$.** We introduce two important examples of affine group schemes:

(i) Let Γ be a finite group. Let k^Γ be the set of maps $\Gamma \to k$. Define a Hopf k-algebra structure on k^Γ by

$$\Delta(\varphi)(g \otimes h) = \varphi(gh), \quad \varepsilon(\varphi) = \varphi(e), \quad S(\varphi)(g) = \varphi(g^{-1})$$

for all $\varphi \in k^\Gamma$, $g, h \in \Gamma$, and $e \in \Gamma$ the identity element. Note that our definition of Δ makes sense only when Γ is finite. If χ_g is the characteristic function of g, then

$$\Delta(\chi_g) = \sum_{g=hh'} \chi_h \otimes \chi_{h'}.$$

The affine group scheme $\mathrm{Spec}(k^\Gamma)$ is called the *finite constant group scheme* associated to Γ.

The basis χ_g furnishes an isomorphism

$$k^\Gamma \cong \prod_{g\in\Gamma} k \quad \text{as } k\text{-algebras.}$$

For every k-algebra R with connected spectrum[2], a k-algebra morphism from $\prod_{g\in\Gamma} k$ to R is zero on exactly one copy of k. Hence

$$k\text{-}\mathbf{Alg}(k^\Gamma, R) = \Gamma$$

i.e., the R-points of $\mathrm{Spec}(k^\Gamma)$ correspond precisely to elements of Γ. This explains the terminology "constant" group scheme.

(ii) Let Γ be an abelian group. Let $k[\Gamma]$ be the group algebra of Γ (written multiplicatively). Define a Hopf k-algebra structure on $k[\Gamma]$ by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

for all $g \in \Gamma$. The affine group scheme $\mathrm{Spec}(k[\Gamma])$ is called the *diagonalizable group scheme* associated to Γ.

Note that $k[\mathbb{Z}] \cong k[T, T^{-1}]$ and this Hopf k-algebra structure coincides with that of $\mathbb{G}_m$. Hence, if Γ is a finitely generated torsion-free abelian group, $\mathrm{Spec}(k[\Gamma])$ is a torus. In general, an affine group scheme is *diagonalizable* if its representing Hopf k-algebra A is generated by group-likes as a k-module.

Proposition 2.4. *Let Γ be a finite, abelian group. Then the Hopf k-algebras k^Γ and $k[\Gamma]$ are Cartier dual to each other.*

Proof. Clearly, $k^\Gamma \cong k\text{-}\mathbf{Mod}(k[\Gamma], k)$ by mapping each function $\varphi : \Gamma \to k$ to the unique k-module morphism that agrees with φ on Γ. By Lemma 1.1, we only need to verify that co-multiplications agree. This is clear by the construction of $\Delta : k^\Gamma \to k^\Gamma \otimes k^\Gamma$. □

An apparently surprising fact is that one can recover Γ from the above constructions:

Proposition 2.5. *Let A be a Hopf k-algebra.*

(i) If A is isomorphic to a finite product of k, then $A \cong k^\Gamma$ with $\Gamma = k\text{-}\mathbf{Alg}(A, k)$.
(ii) Suppose k is a field. If A is diagonalizable, then $A \cong k[\Gamma]$ with Γ the group-likes of A.

Recall that the abelian group of group-likes of A is exactly the character group of $\mathrm{Spec}(A)$ (Lemma 1.2).

[2] Equivalently, R has no idempotents except for 0 and 1.

Proof. There is an obvious map $A \to k^\Gamma$ sending $a \in A$ to ev_a, the evaluation at a of each $g \in \Gamma$. Since A is a finite product of k, this map is bijective. In light of Lemma 1.1, it remains to show that co-multiplication is preserved. This is a tautology:

$$\Delta(\mathrm{ev}_a)(g,h) = \mathrm{ev}_a(gh) = \mathrm{ev}_a((g,h)\Delta) = (g,h)\Delta(a) = \mathrm{ev}_{\Delta(a)}(g,h)$$

for all $a \in A$ and $g, h \in \Gamma$, hence (i).

For (ii), we need a lemma on "linear independence of characters."

Lemma 2.6. *Let k be a field and A a Hopf k-algebra. Then group-likes of A are linearly independent.*

Proof. Let $\sum_{i=1}^n \lambda_i x_i = 0$ be a relation among group-likes with fewest terms. In particular, $\lambda_i \neq 0$. Since k is a field, we may rewrite the relation as $x_1 = \sum_{i=2}^n \lambda_i' x_i$ again with $\lambda_i' \neq 0$. The elements $x_2, \cdots, x_n$ are linearly independent by construction; hence, so are $x_i \otimes x_j$ $(2 \le i, j \le n)$. Now,

$$\sum_{i=2}^n \lambda_i' x_i \otimes x_i = \Delta(x_1) = x_1 \otimes x_1 = \sum_{i,j=2}^n \lambda_i' \lambda_j' x_i \otimes x_j.$$

Comparing coefficients, we find $\lambda_i'\lambda_j' = 0$ for $i \neq j$ and $\lambda_i' = 1$. This is impossible unless $n = 2$. So $x_1 = x_2$. Contradiction. □

The lemma proves that the group-likes of A form a k-basis. In other words, the Hopf k-algebra morphism $k[\Gamma] \to A$ is bijective. □

3. Étale and multiplicative type affine group schemes

In this section, let k be a *field*, k_s be its separable closure, and $\mathscr{G}$ be the absolute Galois group of k.

3.1. **Definitions.** We can generalize the two affine group schemes k^Γ and $k[\Gamma]$ introduced in §2.3 to "twisted forms" of them—group schemes that become them after a base change.

Definition 3.1. Let G be an affine group scheme represented by the Hopf k-algebra A.

(i) G is *étale* if the structure map $k \to A$ is étale;
(ii) G is *of multiplicative type* if $G \times_k k_s$ is diagonalizable.

Proposition 2.5 has the following consequences:

(i) G is étale if and only if $G \times_k k_s$ is represented by k_s^Γ, with $\Gamma = G(k_s)$ (hence finite).[3]
(ii) G is of multiplicative type if and only if $G \times_k k_s$ is represented by $k_s[\Gamma]$, with Γ the character group of $G \times_k k_s$ (hence abelian).

The group Γ obtained this way has the extra structure of a $\mathscr{G}$-action. Our remaining task is to recover G from Γ together with its $\mathscr{G}$-action.

3.2. **A Galois descent.** Recall that a *$\mathscr{G}$-set* is a set X equipped with a $\mathscr{G}$-action that is *continuous* in the sense that X can be written as a union of X_α, such that on each X_α, the $\mathscr{G}$-action factors through $\mathrm{Gal}(L_\alpha/k)$ for some finite Galois extension L_α of k. A *morphism of $\mathscr{G}$-sets* is a $\mathscr{G}$-equivariant function.

Theorem 3.2. *There is an anti-equivalence of categories:*

$$\textit{étale } k\textit{-algebras} \underset{X \mapsto (k_s^X)^{\mathscr{G}}}{\overset{A \mapsto k\text{-}\mathbf{Alg}(A, k_s)}{\rightleftarrows}} \textit{finite } \mathscr{G}\textit{-sets}$$

Here, k_s^X denotes the set of functions $f : X \to k_s$, equipped with $\mathscr{G}$-action $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$. The set $(k_s^X)^{\mathscr{G}}$ of $\mathscr{G}$-invariant functions is naturally a k-algebra.

Proof. Recall that an étale k-algebra A is of the form $\prod_{i=1}^n k_i$, where k_i is a finite, separable extension of k. We need to verify

[3]Indeed, this is because $k \to A$ is étale if and only if $A \otimes_k k_s$ is a finite product of k_s.

(i) Let A be an étale k-algebra; then $X = k\text{-}\mathbf{Alg}(A, k_s)$ is a finite $\mathscr{G}$-set and the evaluation map $A \to (k_s^X)^{\mathscr{G}}$ is an isomorphism. Indeed,

$$X = \coprod_{i=1}^{n} k\text{-}\mathbf{Alg}(k_i, k_s)$$

and the action of $\mathscr{G}$ on $k\text{-}\mathbf{Alg}(k_i, k_s)$ factors through $\mathrm{Gal}(L_i/k)$, where L_i is the Galois closure of $k \subset k_i$.

Note that the evaluation map fits into the diagram

$$\begin{array}{ccc}
A & \longrightarrow & (k_s^X)^{\mathscr{G}} \\
\downarrow & & \downarrow \\
A \otimes_k k_s & \xrightarrow{\mathscr{G}\text{-equivariant}} & k_s^X \\
& \searrow & \| \\
& & k_s^{k_s\text{-}\mathbf{Alg}(A \otimes_k k_s, k_s)}
\end{array}$$

The middle $\mathscr{G}$-equivariant arrow sends $a \otimes \lambda$ to the function $x \mapsto x(a)\lambda$. The skew arrow sends $a \otimes \lambda$ to the function

$$x' \mapsto x'(a \otimes \lambda) = x'(a \otimes 1)\lambda$$

for all $x' \in k_s\text{-}\mathbf{Alg}(A \otimes_k k_s, k_s)$. It is an isomorphism because $A \otimes_k k_s$ is a finite product of k_s's. Hence the middle is also an isomorphism. Now, A is the $\mathscr{G}$-fixed elements of $A \otimes_k k_s$, so the upper horizontal arrow is again an isomorphism.

(ii) This isomorphism is functorial—suppose $f : A \to A'$ is a morphism of étale k-algebras. Let X and X' be their corresponding sets of k_s-points. Then the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\sim} & (k_s^X)^{\mathscr{G}} \\
\downarrow & & \downarrow \\
A' & \xrightarrow{\sim} & (k_s^{X'})^{\mathscr{G}}
\end{array}$$

commutes. Proof omitted.

(iii) Let X be any finite $\mathscr{G}$-set and $A = (k_s^X)^{\mathscr{G}}$; then the evaluation map $X \to k\text{-}\mathbf{Alg}(A, k_s)$ is an isomorphism. Note that if $X = \coprod_{i=1}^n X_i$ as a $\mathscr{G}$-set and $A_i = (k_s^{X_i})^{\mathscr{G}}$, then

$$X \to k\text{-}\mathbf{Alg}(A, k_s) = k\text{-}\mathbf{Alg}(\prod_{i=1}^{n} A_i, k_s) = \coprod_{i=1}^{n} k\text{-}\mathbf{Alg}(A_i, k_s)$$

is the disjoint union of maps $X_i \to k\text{-}\mathbf{Alg}(A_i, k_s)$. Thus we may assume X is a single $\mathscr{G}$-orbit. In this case, the $\mathscr{G}$-action factors through some $\mathscr{G}' = \mathrm{Gal}(F/k)$ with finite, Galois extension $k \subset F$. Fix $x_0 \in X$. Let $\mathscr{H} \subset \mathscr{G}'$ be the isotropy group of x_0 and $E = F^{\mathscr{H}}$.

Claim. *There is a factorization:*

$$\begin{array}{ccccc}
A & \xrightarrow{\sim} & E & \longrightarrow & F \\
& \searrow^{\mathrm{ev}_{x_0}} & & & \downarrow \\
& & & & k_s
\end{array}$$

Since each $\varphi \in A$ is $\mathscr{G}$-equivariant, it is determined by the value $\varphi(x_0)$. Thus the map $A \to k_s$ is automatically injective.

Note that for all $\varphi \in A$ and $\sigma \in \mathrm{Gal}(k_s/F) \subset \mathscr{G}$, there holds $\sigma(\varphi(x_0)) = \varphi(\sigma(x_0)) = \varphi(x_0)$. Hence the image of A lies in F. The induced map $A \to F$ is $\mathscr{G}'$-equivariant. Now, since x_0 is $\mathscr{H}$-invariant, so is $\varphi(x_0)$. Thus $\varphi(x_0) \in E$. We obtain the factorization $A \to E$.

For each $e \in E$, let $\psi \in A$ be the $\mathscr{G}'$-equivariant function with $\psi(x_0) = e$. It is well-defined since e is $\mathscr{H}$-invariant.

Using the claim, we have a series of morphisms of $\mathscr{G}$-sets (whose action factors through $\mathscr{G}'$):

$$X \to k\text{-}\mathbf{Alg}(A, k_s) \xrightarrow{\sim} k\text{-}\mathbf{Alg}(E, k_s) \xrightarrow{\sim} k\text{-}\mathbf{Alg}(E, F)$$

Note that both X are k-$\mathbf{Alg}(E, F)$ are isomorphic to $\mathscr{G}'/\mathscr{H}$. Hence the first map is an isomorphism.

(iv) This isomorphism is functorial. Statement and proof omitted. □

3.3. **Recovering étale affine group schemes.** A *$\mathscr{G}$-group* is a $\mathscr{G}$-set equipped with a group structure such that $\mathscr{G}$ acts by group morphisms. A *morphism of $\mathscr{G}$-groups* is a $\mathscr{G}$-equivariant group morphism.

Theorem 3.3. *There is an equivalence of categories:*

$$\begin{matrix}\textit{étale affine group} \\ \textit{schemes over } k\end{matrix} \underset{\Gamma \mapsto \mathrm{Spec}((k_s^\Gamma)^{\mathscr{G}})}{\overset{G \mapsto G(k_s)}{\rightleftarrows}} \textit{finite } \mathscr{G}\textit{-groups}$$

Proof. This equivalence is a restriction of the one in Theorem 3.2. We need to verify:

(i) A has a Hopf k-algebra structure $\implies$ k-$\mathbf{Alg}(A, k_s)$ has a $\mathscr{G}$-group structure. The group structure is apparent. We only need to show that $\mathscr{G}$ acts by group morphisms. Note that for $\sigma \in \mathscr{G}$ and $g, h \in k\text{-alg}(A, k_s)$, there holds

$$\sigma \circ (g, h) = (\sigma \circ g, \sigma \circ h) = (\sigma(g), \sigma(h)).$$

Pre-composing with Δ, we find $\sigma(gh) = \sigma(g)\sigma(h)$.

(ii) Γ has a $\mathscr{G}$-group structure $\implies$ $(k_s^\Gamma)^{\mathscr{G}}$ has a Hopf k-algebra structure. The co-multiplication, co-unit, and antipode are given on k_s^Γ by

$$\Delta(\varphi)(g, h) = \varphi(gh), \quad \varepsilon(\varphi) = \varphi(e), \quad S(\varphi)(g) = \varphi(g^{-1})$$

for all $\varphi \in k_s^\Gamma$, $g, h \in \Gamma$, and $e \in \Gamma$ the unit element. They commute with $\mathscr{G}$-action, hence induce a Hopf k-algebra structure on $(k_s^\Gamma)^{\mathscr{G}}$.

(iii) The constructions in (i) and (ii) are inverse to each other. Given an étale Hopf k-algebra A, let $\Gamma = k\text{-}\mathbf{Alg}(A, k_s)$. We ought to show that the evaluation map $A \to (k_s^\Gamma)^{\mathscr{G}}$ commutes with co-multiplication. Indeed,

$$\mathrm{ev}_{\Delta a}(g, h) = (g, h)(\Delta a) = (gh)(a) = \Delta(\mathrm{ev}_a)(g, h).$$

for all $a \in A$ and $g, h \in \Gamma$.

Given a finite $\mathscr{G}$-group Γ, let $A = (k_s^\Gamma)^{\mathscr{G}}$. We ought to show that the evaluation map $\Gamma \to k\text{-}\mathbf{Alg}(A, k_s)$ is a $\mathscr{G}$-equivariant group morphism. Indeed,

$$\mathrm{ev}_{gh}(a) = a(gh) = \Delta(a)(g, h) = (\mathrm{ev}_g, \mathrm{ev}_h)\Delta(a) = (\mathrm{ev}_g \cdot \mathrm{ev}_h)(a)$$

for all $a \in A$ and $g, h \in \Gamma$, and

$$\mathrm{ev}_{\sigma(g)}(a) = a(\sigma(g)) = \sigma(a(g))$$

for all $\sigma \in \mathscr{G}$.

(iv) Hopf k-algebra morphisms $A \to B$ correspond to $\mathscr{G}$-equivariant group morphisms $k\text{-}\mathbf{Alg}(B, k_s) \to k\text{-}\mathbf{Alg}(A, k_s)$. Proof omitted. □

3.4. **Recovering multiplicative type affine group schemes.**

Theorem 3.4. *There is an anti-equivalence of categories:*

$$\begin{matrix}\textit{multiplicative type} \\ \textit{affine group schemes} \\ \textit{over } k\end{matrix} \underset{\Gamma \mapsto \mathrm{Spec}(k_s[\Gamma]^{\mathscr{G}})}{\overset{G \mapsto G^\vee(k_s)}{\rightleftarrows}} \textit{abelian } \mathscr{G}\textit{-groups}$$

For an abelian $\mathscr{G}$-group, the Hopf k-algebra structure on $k_s[\Gamma]^{\mathscr{G}}$ is induced by the $\mathscr{G}$-equivariant co-multiplication, co-unit, and antipode maps on $k_s[\Gamma]$:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}.$$

Proof. We need to verify:

(i) Let A represent a multiplicative type affine group scheme G over k; then $\Gamma = \text{Group-likes}(A \otimes_k k_s)$ is an abelian $\mathscr{G}$-group and the natural map $A \to k_s[\Gamma]^{\mathscr{G}}$ is an isomorphism. Indeed, $\mathscr{G}$ acts on the second factor of the abelian group $\Gamma \subset A \otimes_k k_s$ by group morphisms. The action is continuous since

$$A \otimes_k k_s = \bigcup_{\substack{k \subset L \\ \text{finite Galois}}} A \otimes_k L$$

and the $\mathscr{G}$-action on $A \otimes_k L$ factors through $\text{Gal}(L/k)$. Now, the natural map $A \to k_s[\Gamma]^{\mathscr{G}}$ fits into the diagram

$$\begin{array}{ccc} A & \longrightarrow & k_s[\Gamma]^{\mathscr{G}} \\ \downarrow & & \downarrow \\ A \otimes_k k_s & \xrightarrow{\sim} & k_s[\Gamma] \end{array}$$

where the lower arrow is a $\mathscr{G}$-equivariant isomorphism of Hopf k_s-algebras. As A occurs as the $\mathscr{G}$-fixed elements of $A \otimes_k k_s$, the upper arrow must be an isomorphism of Hopf k-algebras.

(ii) This isomorphism is functorial—suppose $f : A \to A'$ is a morphism of Hopf k-algebras representing multiplicative type affine group schemes. Let Γ and Γ' be the group-likes of $A \otimes_k k_s$ and $A' \otimes_k k_s$. Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{\sim} & k_s[\Gamma]^{\mathscr{G}} \\ \downarrow{\scriptstyle f} & & \downarrow \\ A' & \xrightarrow{\sim} & k_s[\Gamma']^{\mathscr{G}} \end{array}$$

commutes. Proof omitted.

(iii) Let Γ be an abelian $\mathscr{G}$-group; then the natural map $A \otimes_k k_s \to k_s[\Gamma]$ is an isomorphism of Hopf k_s-algebras. Since the $\mathscr{G}$-action on Γ is continuous, it acts on each orbit $\Gamma_\alpha \subset \Gamma$ through some finite Galois group $\text{Gal}(L_\alpha/k)$. In particular, Γ_α is finite, so $k_s[\Gamma_\alpha]$ is a finite-dimension $\mathscr{G}$-submodule of $k_s[\Gamma]$. We have a diagram corresponding to the orbit decomposition of Γ:

$$\begin{array}{ccc} A \otimes_k k_s & \longrightarrow & k_s[\Gamma] \\ \| & & \| \\ \bigoplus_\alpha A_\alpha \otimes_k k_s & \longrightarrow & \bigoplus_\alpha k_s[\Gamma_\alpha] \end{array}$$

Note that the lower arrow is a direct sum of maps of Hopf k_s-algebras $A_\alpha \otimes_k k_s \to k_s[\Gamma_\alpha]$. By Galois descent on finite-dimensional vector spaces[4], these maps are isomorphisms. Hence $A \otimes_k k_s \to k_s[\Gamma]$ is again an isomorphism.

(iv) This isomorphism is functorial. Statement and proof omitted. □

References

[1] Waterhouse, William C. *Introduction to affine group schemes.* Vol. 66. Springer Science & Business Media, 2012.

[4]There is an equivalence of categories between finite-dimensional k-vector spaces and twisted finite-dimensional $\text{Gal}(L/k)$-modules, where $k \subset L$ is a finite Galois extension. I wrote a complete proof of this in *Tensor Product of Semistable Vector Bundles over Curves in Characteristic Zero*, Theorem 2.9.

It is a bit cheap on my part for not writing out the proof of this fact here—it overlaps largely with Theorem 3.2. Perhaps I should replace the whole discussion on Galois descent in these notes by a general, abstract version (and prove that instead).