Contents

Let k be an algebraically closed field of characteristic zero. Let X be a smooth, connected, projective curve over k. We use G to denote a reductive algebraic group over k.

1. The Hecke eigenvector problem

1.1. **Main players.** An eigenvector problem refers to a problem of the following type: given a collection of mutually commuting matrices A acting on a vector space V, can we attach an eigenvector $v \in V$ to every (collective) eigenvalue of A?

1.1.1. For the purpose of geometric Langlands,

- the collection of "matrices" is the symmetric monoidal category:
$$\mathbf{Sph}_G := \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_G)^{\mathcal{L}^+G}$$
of $\mathcal{L}^+G$-equivariant $\mathcal{D}$-modules on the affine Grassmannian Gr_G;
- the "vector space" is the derived category
$$\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_G)$$
of $\mathcal{D}$-modules on the moduli stack of G-bundles over X.

Let me explain first what the geometric objects involved are. Then we will define the (abelian and derived) category of $\mathcal{D}$-modules over them.

1.1.2. *The affine Grassmannian.* The introductory article [Zh16] is definitely worth a look. The *affine Grassmannian* Gr_G is defined as a presheaf on $\mathbf{Sch}^{\mathrm{aff}}$ as follows:
$$S = \mathrm{Spec}(R) \rightsquigarrow (\mathcal{P}_G, \alpha : \mathcal{P}_G \sim \mathcal{P}_G^{\circ})$$
where $\mathcal{P}_G$ is a G-bundle[1] over $\mathrm{Spec}(R[\![t]\!])$[2] and α is a trivialization of $\mathcal{P}_G$ over $\mathrm{Spec}(R(\!(t)\!))$. (Here and after, we use the notation $\mathcal{P}_G^{\circ}$ to denote the trivial G-torsor.)

Remark 1.1. Note that a G-torsor over $\mathrm{Spec}(R[\![t]\!])$ is equivalent to a compatible system of G-torsors over $\mathrm{Spec}(R[t]/t^n)$ for each $n \geq 0$.

We call an *ind-scheme* Y a presheaf on $\mathbf{Sch}^{\mathrm{aff}}$ representable as a filtered colimit of schemes:
$$Y = \underrightarrow{\mathrm{colim}}\, Y_i$$
such that all transition maps $f_{ij} : Y_i \to Y_j$ are closed immersions.[3] We summarize some basic properties of Gr_G:

- Gr_G is an ind-proper ind-scheme. (The "ind-proper" part requires the hypothesis that G be reductive.)

[1] Since G is smooth, fppf local triviality of G-torsors $\Longleftrightarrow$ smooth local triviality $\Longleftrightarrow$ étale local triviality.

[2] I wrongly wrote $S \times \mathrm{Spec}(k[\![t]\!])$ today. Note that $R[\![t]\!]$ is the *completed* tensor product $R \widehat{\otimes}_k k[\![t]\!]$.

[3] Some authors call these *strict* ind-schemes.

– Replacing $R[\![t]\!]$ by $R\underset{k}{\widehat{\otimes}}\mathcal{O}_x$, where $\mathcal{O}_x$ is the complete local ring of a closed point $x \in X$, we have a version of Gr_G denoted by $\mathrm{Gr}_{G,x}$.

  The Beauville-Laszlo theorem asserts that $\mathrm{Gr}_{G,x}$ also represents the presheaf:

$$S = \mathrm{Spec}(R) \rightsquigarrow (\mathcal{P}_G, \alpha : \mathcal{P}_G\big|_{S\times(X-\{x\})} \xrightarrow{\sim} \mathcal{P}_G^\circ)$$

  where $\mathcal{P}_G$ is a G-torsor over the whole curve X.

Defining the *loop group* $\mathcal{L}G$ by the presheaf:

$$S = \mathrm{Spec}(R) \rightsquigarrow \mathrm{Maps}(\mathrm{Spec}(R(\!(t)\!)), G)$$

and the *arc group* $\mathcal{L}^+G$ by

$$S = \mathrm{Spec}(R) \rightsquigarrow \mathrm{Maps}(\mathrm{Spec}(R[\![t]\!]), G),$$

we have

– $\mathcal{L}^+G$ is represented by a group scheme (albeit of infinite type), and $\mathcal{L}G$ is represented by a group ind-scheme (again, not of ind-finite type.)

  The natural (right) action of $\mathcal{L}^+G$ on $\mathcal{L}G$ realizes the projection:

$$\mathcal{L}G \to \mathrm{Gr}_G$$

  as an étale $\mathcal{L}^+G$-torsor.

Remark 1.2. The étale local triviality amounts to the following: given a G-torsor $\mathcal{P}_G$ over $\mathrm{Spec}(R[\![t]\!])$, there exists an étale map $R \to R'$ such that $\mathcal{P}_G\big|_{\mathrm{Spec}(R'[\![t]\!])}$ is trivial. Indeed, $\mathcal{P}_G\big|_{\mathrm{Spec}(R)}$ admits a section after some étale base change $R \to R'$. One then uses the smoothness of $\mathcal{P}_G$ to extend the section from $\mathrm{Spec}(R')$ to $\mathrm{Spec}(R'[\![t]\!])$.

This argument shows that the étale quotient $\mathcal{L}G/\mathcal{L}^+G$ agrees with Gr_G. Since Gr_G is an fpqc sheaf, the above quotient can indeed be formed in any topology between étale and fpqc and will agree with Gr_G.

– Gr_G admits a presentation $\mathrm{Gr}_G \cong \underrightarrow{\mathrm{colim}}\, Y_i$ by proper schemes Y_i such that the (left) $\mathcal{L}^+G$-action on each Y_i factors through a finite type quotient $\mathcal{L}^+G \to H_i$.

1.1.3. *Moduli stack of G-bundles.* The stack Bun_G represents the functor (valued in groupoids):

$$S \rightsquigarrow \{G\text{-bundle } \mathcal{P}_G \text{ over } S \times X\}.$$

In other words, Bun_G is the mapping stack $\underline{\mathrm{Maps}}(X, \mathrm{B}\, G)$. One can show that Bun_G is a smooth algebraic stack with affine diagonal.

Example 1.3 ($G = T$)**.** When G is a torus T, I claim that there is a *non-canonical* isomorphism:

$$\mathrm{Bun}_T \xrightarrow{\sim} \mathrm{Pic}^0(X)^{\times r} \times \mathrm{B}\, T \times \Lambda_T \tag{1.1}$$

where $r = \mathrm{rank}(T)$ and $\Lambda_T := \mathrm{Hom}(\mathbb{G}_m, T)$ is the character lattice. One way to obtain (1.1) is by fixing a closed point $x \in X$ and an isomorphism $T \cong \mathbb{G}_m^{\times r}$ (solely for the first factor). More precisely, we obtain (1.1) in three steps:

– There is an isomorphism:

$$\mathrm{Bun}_T \xrightarrow{\sim} \mathrm{Bun}_T^0 \times \Lambda_T$$

  where Bun_T^0 is the degree-0 part of Bun_T.

  This isomorphism sends

$$\mathcal{P}_T \rightsquigarrow (\mathcal{P}_T \otimes \mathcal{O}(-\lambda x), \lambda)$$

where $\lambda = \deg(\mathcal{P}_T)$, $\mathcal{O}(-\lambda x)$ is the unique T-bundle whose induced line bundle along any $\check{\mu} : T \to \mathbb{G}_m$ is $\mathcal{O}(-\langle \lambda, \check{\mu} \rangle x)$, and $\otimes$ denotes the symmetric monoidal structure on T-bundles.

– There is an isomorphism:

$$\mathrm{Bun}_T^0 \xrightarrow{\sim} \mathrm{Bun}_{T,x}^0 \times \mathrm{B}\,T$$

where $\mathrm{Bun}_{T,x}^0$ parametrizes pairs $(\mathcal{P}_T, \alpha)$ where α is a trivialization of $\mathcal{P}_T\big|_{S\times\{x\}}$.

This isomorphism sends

$$\mathcal{P}_T \rightsquigarrow (\mathcal{P}_T \otimes \mathrm{pr}_S^*(\mathcal{P}_T\big|_x)^{-1}, \mathrm{can})$$

where can denotes the canonical trivialization.

– Finally, fixing $T \cong \mathbb{G}_m^{\times r}$ gives us an isomorphism

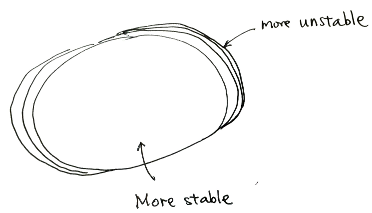
$$\mathrm{Bun}_{T,x}^0 \xrightarrow{\sim} \mathrm{Pic}^0(X)^{\times r}.$$

For $r = 1$, this is the map sending a pair $(\mathcal{L}, \alpha)$ to the isomorphism class $[\mathcal{L}]$. Its inverse is

$$[\mathcal{L}] \rightsquigarrow (\mathcal{M} \otimes \mathrm{pr}_S^*(\mathcal{M}\big|_x)^{-1}, \mathrm{can})$$

where $\mathcal{M}$ is any representative of $[\mathcal{L}]$.

Example 1.4. For G semisimple and simply connected (e.g., $G = \mathrm{SL}_2$), Bun_G looks like an onion:



It has just one connected component, and yet is not quasi-compact, i.e., the unstable peels go on *ad infinitum*. More generally, the connected components of Bun_G are in bijection with $\pi_1(G)$.[4]

1.2. **$\mathcal{D}$-modules on ind-schemes.** We now define the *abelian* category $\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_G)^{\mathcal{L}^+G}$.

1.2.1. *Some generalities.* Let $(\mathcal{C}_i, g_{ij} : \mathcal{C}_j \to \mathcal{C}_i)$ be an inverse system of categories, such that each $\mathcal{C}_i$ contains arbitrary colimits and g_{ij} preserves them.

We may form the 2-categorical limit $\lim\limits_{g_{ij}} \mathcal{C}_i$ (equipped with functors to each $\mathcal{C}_i$) characterized by the universal property that the functor of composition:

$$\mathrm{Fun}(\mathcal{D}, \lim_{g_{ij}} \mathcal{C}_i) \to \{\text{Strictly compatible system of functors } \mathcal{D} \to \mathcal{C}_i\}$$

is an equivalence of categories.

Remark 1.5. The phrase *strictly compatible* refers to the structure of natural isomorphisms between $\mathcal{D} \to \mathcal{C}_j \xrightarrow{g_{ij}} \mathcal{C}_i$ and $\mathcal{D} \to \mathcal{C}_i$. If instead we ask just for natural transformations, we would get the notion of a *lax limit.*

Here is the miracle:

[4]For a simple onion-like (but quasi-compact) algebraic stack, consider $\mathbb{A}^2/\mathbb{G}_m$ where $\mathbb{G}_m$ acts by dilation. This stack contains a proper *open* substack isomorphic to $\mathbb{P}^1$ and a peel $\{0\}/\mathbb{G}_m$.

Lemma 1.6. *Suppose each g_{ij} admits a left adjoint g_{ij}^L. Then there is a canonical equivalence of categories:*

$$\operatorname*{colim}_{g_{ij}^L} \mathcal{C}_i \xrightarrow{\sim} \lim_{g_{ij}} \mathcal{C}_i$$

Proof. The functor is defined for each $\mathcal{C}_i \to \mathcal{C}_j$ as

$$\mathcal{C}_i \ni x \rightsquigarrow \operatorname*{colim}_{k \geq i,j} g_{jk} \circ g_{ik}^L(x) \in \mathcal{C}_j.$$

We omit checking that it induces an equivalence of categories. □

Remark 1.7. If each g_{ij} satisfies the property $1 \xrightarrow{\sim} g_{ij} \circ g_{ij}^L$, i.e., g_{ij}^L is fully faithful, then the functor $\mathcal{C}_i \to \mathcal{C}_j$ is simply the application of g_{ij}^L for $j > i$ and g_{ji} for $j < i$.

1.2.2. We assume at our disposal the theory of $\mathcal{D}$-modules over a finite type scheme.[5] In particular, for a closed immersion: $f : Y \hookrightarrow Y'$, there is a pair of adjoint functors:

$$f_! = f_* : \mathcal{D}\text{-}\mathbf{Mod}(Y) \rightleftarrows \mathcal{D}\text{-}\mathbf{Mod}(Y') : \mathrm{H}^0 f^!$$

on the level of abelian categories.

Given an ind-scheme $Y = \underrightarrow{\operatorname{colim}}\, Y_i$, we may define the abelian category of *$\mathcal{D}$-modules on Y* by:

$$\mathcal{D}\text{-}\mathbf{Mod}(Y) := \operatorname*{colim}_{(f_{ij})_*} \mathcal{D}\text{-}\mathbf{Mod}(Y_i) \xrightarrow{\sim} \lim_{\mathrm{H}^0 f_{ij}^!} \mathcal{D}\text{-}\mathbf{Mod}(Y_i)$$

where the equivalence is due to Lemma 1.6. One checks immediately that this definition is independent of the presentation $Y = \underrightarrow{\operatorname{colim}}\, Y_i$.

1.2.3. The abelian category of $\mathcal{L}^+G$-equivariant objects in $\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_G)$ is well-defined since the action of $\mathcal{L}^+G$ on a finite type Y_i factors through some $\mathcal{L}_kG$. Hence we may set:

$$\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_G)^{\mathcal{L}^+G} := \operatorname*{colim}_{(f_{ij})_*} \mathcal{D}\text{-}\mathbf{Mod}(Y_i)^{\mathcal{L}_kG} \xrightarrow{\sim} \lim_{\mathrm{H}^0 f_{ij}^!} \mathcal{D}\text{-}\mathbf{Mod}(Y_i)^{\mathcal{L}_kG}$$

Remark 1.8.
- In order for $\mathcal{D}\text{-}\mathbf{Mod}(Y_i)^{\mathcal{L}_kG}$ to be defined independently of k, one observes that the kernel

$$1 \to U \to \mathcal{L}_{k'}G \to \mathcal{L}_kG \to 1, \quad k' \geq k$$

  is a connected, unipotent group. For such H acting on any scheme Y of finite type, the categories $\mathcal{D}\text{-}\mathbf{Mod}(Y)^U$ and $\mathcal{D}\text{-}\mathbf{Mod}(Y)$ are equivalent.
- The adjunction still holds because the forgetful functor $\mathcal{D}\text{-}\mathbf{Mod}(Y)^H \to \mathcal{D}\text{-}\mathbf{Mod}(Y)$ is fully faithful when H is a connected algebraic group. (How to prove this?)

1.3. **$\mathcal{D}$-modules on algebraic stacks.** There is a way to produce the abelian category of $\mathcal{D}$-modules on certain algebraic stacks due to Beilinson and Drinfeld.

[5]Locally, a finite type scheme can be embedded as the closed subscheme of a smooth scheme $Z \hookrightarrow X$. We then define the category of $\mathcal{D}$-modules over it as $\mathcal{D}$-modules over X supported on Z. One checks that this definition is independent of the chosen embedding.

I will be using the basic operations on $\mathcal{D}$-modules as summarized in http://www-math.mit.edu/~etingof/dmodfactsheet.pdf.

1.3.1. *Good for lazybones.* A smooth algebraic stack $\mathcal{Y}$ is *good for lazybones* if for smooth morphism $Z \to \mathcal{Y}$ from a *scheme* Z, the complex:

$$\mathcal{T}_{\mathcal{Y}}\big|_Z := [\mathcal{T}_{Z/\mathcal{Y}} \to \mathcal{T}_Z] \in \mathrm{QCoh}^{[-1,0]}(Z)$$

has the property that $\mathrm{Sym}(\mathcal{T}_{\mathcal{Y}}\big|_Z)$ is concentrated in degree 0.

Remark 1.9 (What's $\mathcal{T}_{Z/\mathcal{Y}}$?)**.** One can define the relative tangent sheaf $\mathcal{T}_{Z/\mathcal{Y}}$ in two ways:

- Pick a smooth cover $U \to \mathcal{Y}$, we have a diagram of Cartesian squares:

$$\begin{tikzcd} \tilde{U} \underset{Z}{\times} \tilde{U} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \tilde{U} \arrow[r] \arrow[d] & Z \arrow[d] \\ U \underset{\mathcal{Y}}{\times} U \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r] & \mathcal{Y} \end{tikzcd}$$

  The sheaf $\mathcal{T}_{Z/\mathcal{Y}}$ is the descent of

$$(\mathcal{T}_{\tilde{U}/U}, \quad \mathrm{pr}_1^* \mathcal{T}_{\tilde{U}/U} \xrightarrow{\sim} \mathcal{T}_{\tilde{U} \underset{Z}{\times} \tilde{U}/U \underset{\mathcal{Y}}{\times} U} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{T}_{\tilde{U}/U})$$

  along the smooth cover $\tilde{U} \to Z$. In particular, $\mathcal{T}_{Z/\mathcal{Y}}$ is locally free.
- Contemplating the following diagram:

$$\begin{tikzcd} Z \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\ & Z \underset{\mathcal{Y}}{\times} Z \arrow[r] \arrow[d] & Z \arrow[d] \\ & Z \arrow[r] & \mathcal{Y} \end{tikzcd}$$

  we find a canonical isomorphism:

$$\mathcal{T}_{Z/\mathcal{Y}} \xrightarrow{\sim} \Delta^* \mathcal{T}_{Z \underset{\mathcal{Y}}{\times} Z/Z}. \tag{1.2}$$

Remark 1.10 (What's Sym?)**.** The functor Sym takes a complex of $\mathcal{O}_Z$-modules to the free <u>graded commutative</u> DG $\mathcal{O}_Z$-algebra it generates.

The meaning of this goodness definition is that the cotangent stack $T^*\mathcal{Y}$, defined by

$$T^*\mathcal{Y} := \underline{\mathrm{Spec}}_{\mathcal{Y}}(\mathrm{Sym}\,\mathcal{T}_{\mathcal{Y}}),$$

stays within the realm of classical (i.e., non-*derived*) algebraic geometry.

Example 1.11. Consider the stack $\mathrm{B}\,H$ for an algebraic group H. Letting $Z = \mathrm{pt}$, and we find from (1.2) an isomorphism

$$\mathcal{T}_{\mathrm{B}\,H}\big|_{\mathrm{pt}} \xrightarrow{\sim} [\mathfrak{h} \to 0] \in \mathrm{Vect}^{[-1,0]}.$$

where $\mathfrak{h}$ is the Lie algebra of H.

Therefore, we have

$$\mathrm{Sym}(\mathcal{T}_{\mathrm{B}\,H}\big|_{\mathrm{pt}}) \xrightarrow{\sim} [\Lambda^{\mathrm{top}}\mathfrak{h} \xrightarrow{0} \cdots \xrightarrow{0} \mathfrak{h} \xrightarrow{0} k],$$

so $\mathrm{B}\,H$ is *not* good for lazybones unless H is discrete. In the language of derived geometry, this computation would show

$$T^*(\mathrm{B}\,H) \xrightarrow{\sim} (\mathrm{pt} \underset{\mathfrak{h}^*}{\times} \mathrm{pt})/H,$$

where H acts by the co-adjoint action.

Theorem 1.12 (Beilinson-Drinfeld)**.** *If G is semisimple, the stack Bun_G is good for lazybones.*

1.3.2. For a stack $\mathcal{Y}$ good for lazybones, we define the abelian category of $\mathcal{D}$-modules on $\mathcal{Y}$ as

$$\mathcal{D}\text{-}\mathbf{Mod}(\mathcal{Y}) := \text{Category of } \{\mathcal{M}\big|_Z\}_{Z\to\mathcal{Y}}$$

where $\mathcal{M}\big|_Z$ is a $\mathcal{D}$-module on the scheme Z for every smooth morphism $Z \to \mathcal{Y}$, satisfying compatibility conditions for varying Z.

The derived category of $\mathcal{D}$-modules over $\mathcal{Y}$ is defined as the derived category of $\mathcal{D}$-$\mathbf{Mod}(\mathcal{Y})$.

Remark 1.13. This definition is only sensible when $\mathcal{Y}$ is good for lazybones. For instance, if $\mathcal{Y} = \mathrm{B}\,H$, the above definition would produce the derived category of vector spaces. However, the "actual" derived category of $\mathcal{D}$-modules on $\mathrm{B}\,H$ is much richer.

1.4. **Convolution and Hecke action.**

1.4.1. The étale $\mathcal{L}^+G$-torsor $\mathcal{L}G \to \mathrm{Gr}_G$ and the $\mathcal{L}^+G$-action on Gr_G allows us to form the twist:

$$\mathcal{L}G \overset{\mathcal{L}^+G}{\times} \mathrm{Gr}_G \xrightarrow{\sim} \mathrm{Conv}_G \tag{1.3}$$

where Conv_G is the *convolution Grassmannian* classifying modifications $\mathcal{P}_G^1 \to \mathcal{P}_G^2 \to \mathcal{P}_G^\circ$. The morphism (1.3) is defined by sending

$$g \in \mathcal{L}G(S), \quad (\mathcal{P}_G, \mathcal{P}_G \overset{\alpha}{\sim} \mathcal{P}_G^\circ)) \in \mathrm{Gr}_G(S)$$

to the convolution diagram:

$$(\mathcal{P}_G \overset{\alpha}{\sim} \mathcal{P}_G^\circ \overset{g\cdot}{\sim} \mathcal{P}_G^\circ).$$

Denote by act the morphism

$$\mathrm{act} : \mathrm{Conv}_G \to \mathrm{Gr}_G, \quad (\mathcal{P}_G^1 \sim \mathcal{P}_G^2 \sim \mathcal{P}_G^\circ) \rightsquigarrow (\mathcal{P}_G^1, \mathcal{P}_G^1 \overset{\text{composition}}{\sim} \mathcal{P}_G^\circ).$$

It is a nontrivial theorem that for $\mathcal{F}_1, \mathcal{F}_2 \in \mathbf{Sph}_G$, the object $\mathrm{act}_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$, which *a prior* lives in the derived category $D(\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_G))^{\mathcal{L}^+G}$, is in fact an object of $\mathbf{Sph}_G$. The resulting operation:

$$\mathcal{F}_1 * \mathcal{F}_2 := \mathrm{act}_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2), \quad \mathbf{Sph}_G \times \mathbf{Sph}_G \to \mathbf{Sph}_G,$$

is called the *convolution product* on $\mathbf{Sph}_G$.

Recall that if we fix $T \hookrightarrow B \hookrightarrow G$, we obtain a canonical bijection between the set of $\mathcal{L}^+G$-orbits of Gr_G with Λ_T^+. We denote by Gr_G^λ the orbit corresponding to $\lambda \in \Lambda_T^+$. It is of dimension $\langle \lambda, 2\check{\rho}\rangle$.[6]

Remark 1.14. Our convention is that Gr_G^λ is reduced (as an $\mathcal{L}^+G$-orbit should be.) The possibly non-reduced structure on Gr_G rarely poses a problem because the theory of $\mathcal{D}$-modules does not see nilpotents.

1.4.2. *Satake.* The geometric Satake equivalence asserts the following:

- the convolution product admits a natural commutativity constraint, making $\mathbf{Sph}_G$ a symmetric monoidal category;
- there is a natural equivalence of symmetric monoidal categories

  $$\mathrm{Sat} : \mathbf{Rep}_{\check{G}} \xrightarrow{\sim} \mathbf{Sph}_G$$

  where $\check{G}$ is the Langlands dual group of G.
- under the Satake functor, the irreducible $\check{G}$-representation V^λ of highest weight λ is sent to the $\mathcal{D}$-module $\mathrm{IC}_{\mathrm{Gr}_G^\lambda}$.

Remark 1.15. The unit object of $\mathbf{Sph}_G$ is the delta $\mathcal{D}$-module δ_0, defined as the pushforward of k along the closed immersion $\mathrm{pt} \cong \mathrm{Gr}_G^0 \hookrightarrow \mathrm{Gr}_G$.

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[6]In particular, $\mathrm{Gr}_G^0 \cong \mathrm{pt}$.

1.4.3. *Hecke stack.* We fix a closed point $x \in X$. There is an étale $\mathcal{L}^+G$-torsor $\mathrm{Bun}_{G,\infty x} \to \mathrm{Bun}_G$, where $\mathrm{Bun}_{G,\infty x}$ represents the functor

$$S = \mathrm{Spec}(R) \rightsquigarrow \{(\mathcal{P}_G, \mathcal{P}_G|_{\mathrm{Spec}(R\widehat{\otimes}\mathcal{O}_x)} \xrightarrow{\sim} \mathcal{P}_G^\circ)\}$$

One can show that $\mathrm{Bun}_{G,\infty x}$ is represented by a scheme (of infinite type).

The left $\mathcal{L}^+G$-action on Gr_G allows us to form the twist

$$\mathrm{Bun}_{G,\infty x} \overset{\mathcal{L}^+G}{\times} \mathrm{Gr}_G \xrightarrow{\sim} \mathrm{Hecke}_x \tag{1.4}$$

where Hecke_x is the *Hecke stack* defined by

$$S = \mathrm{Spec}(R) \rightsquigarrow \{(\mathcal{P}_G^1, \mathcal{P}_G^1|_{S\times(X-\{x\})} \xrightarrow{\sim} \mathcal{P}_G^2)\}.$$

The morphism (1.4) sends a pair of points:

$$(\mathcal{P}_G^1, \mathcal{P}_G^1 \sim \mathcal{P}_G^\circ) \in \mathrm{Bun}_{G,\infty x}(S), \quad (\mathcal{P}'_G, \mathcal{P}'_G \sim \mathcal{P}_G^\circ) \in \mathrm{Gr}_G(S)$$

to $(\mathcal{P}_G^1, \mathcal{P}_G^2, \mathrm{can})$ where $\mathcal{P}_G^2$ is the glueing of $\mathcal{P}_G^1|_{S\times(X-\{x\})}$ with $\mathcal{P}'_G$ along the identification:

$$\mathcal{P}_G^1|_{\mathrm{Spec}(R\widehat{\otimes}\mathcal{K}_x)} \xrightarrow{\sim} \mathcal{P}_G^\circ \xleftarrow{\sim} \mathcal{P}'_G|_{\mathrm{Spec}(R\widehat{\otimes}\mathcal{K}_x)}$$

and can is the canonical identification. We have a natural functor:

$$\vec{h} : \mathrm{Hecke}_x \to \mathrm{Bun}_G, \quad (\mathcal{P}_G^1 \sim \mathcal{P}_G^2) \rightsquigarrow \mathcal{P}_G^2$$

We define the action of $\mathcal{F} \in \mathbf{Sph}_G$ on $\mathcal{M} \in D(\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_G))$ by

$$\mathcal{M} * \mathcal{F} := \vec{h}_!(\mathcal{M}\tilde{\boxtimes}\mathcal{F}).$$

Example 1.16 ($G = \mathrm{GL}_1$).

- The affine Grassmannian $\mathrm{Gr}_{\mathrm{GL}_1}$ is a disjoint union of formal schemes[7] parametrized by $\lambda \in \mathbb{Z} = \Lambda_{\mathrm{GL}_1}$. Indeed, the closed point at λ corresponds to

$$\mathrm{pt} \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{GL}_1}^\lambda \hookrightarrow \mathrm{Gr}_{\mathrm{GL}_1}, \quad (\mathcal{O}_{D_x}, \mathcal{O}_{\overset{\circ}{D}_x} \xrightarrow{t^\lambda} \mathcal{O}).$$

- Inspecting the functor (1.4), we see that $\mathrm{Hecke}_x^\lambda := \mathrm{Bun}_{\mathrm{GL}_1,\infty x} \overset{\mathcal{L}^+\mathrm{GL}_1}{\times} \mathrm{Gr}_{\mathrm{GL}_1}^\lambda$ parametrizes the data $(\mathcal{L}, \mathcal{L}(-\lambda x), \mathrm{can})$. In other words the morphisms $\overleftarrow{h}$ and $\vec{h}$ are isomorphisms.
- Suppose $\mathcal{F}^\lambda \in \mathbf{Sph}_{\mathrm{GL}_1}$ is the delta $\mathcal{D}$-module supported on $\mathrm{Gr}_{\mathrm{GL}_1}^\lambda$. Then the action of $\mathcal{F}^\lambda$ on $\mathcal{M} \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Pic}(X))$ is simply the pullback along:

$$\mathrm{Pic}(X) \xleftarrow{-\otimes\mathcal{O}(\lambda x)} \mathrm{Pic}(X).$$

1.5. **The Hecke eigenvector problem.**

1.5.1. Here is the categorical version of the eigenvector problem. Let $\mathcal{A}$ be symmetric monoidal k-linear category acting (on the right) on a k-linear category $\mathcal{V}$.

Given an "eigenvalue", i.e., a symmetric monoidal functor:

$$\sigma : \mathcal{A} \to \mathbf{Vect},$$

we seek an "eigenvector" $v \in \mathcal{V}$ together with isomorphisms for every $a \in \mathcal{A}$:[8]

$$\gamma_a : v \cdot a \xrightarrow{\sim} \sigma(a) \otimes v$$

[7] i.e., an ind-scheme whose reduced locus is a point.

[8] Writing the action on the right and the eigenvalue on the left conjures up the following image: a sequence of innocent $\check{G}$-representations are lined up for their judgment by the Hecke eigensheaf $\mathcal{M}$, turning into local systems as they pass through one-by-one.

such that the following diagram commutes:

$$\begin{tikzcd}
(v \cdot a_1) \cdot a_2 \arrow[r, "\sim"] \arrow[d, "\gamma_{a_1} \cdot a_2"] & v \cdot (a_1 \otimes a_2) \arrow[d, "\gamma_{a_1 \otimes a_2}"] \\
(\sigma(a_1) \otimes v) \cdot a_2 \arrow[d, "\sim"] & \sigma(a_1 \otimes a_2) \otimes v \arrow[d, "\sim"] \\
\sigma(a_1) \otimes (v \cdot a_2) \arrow[r, "\mathrm{id} \otimes \gamma_{a_2}"] & \sigma(a_1) \otimes \sigma(a_2) \otimes v
\end{tikzcd}$$

Remark 1.17. It seems to me that the lower left isomorphism requires knowing something about the action, e.g., $\mathcal{A}$ acts by functors that admit left adjoints.

We are then inspired to ask for $\mathcal{M} \in \mathcal{D}$-$\mathbf{Mod}(\mathrm{Bun}_G)$ with a compatible system of isomorphisms:

$$\mathcal{M} * \mathrm{Sat}(V) \xrightarrow{\sim} \sigma(V) \otimes \mathcal{M}, \quad \forall V \in \mathbf{Rep}_{\check{G}}.$$

However, remember that we have only considered Hecke modifications at $x \in X$. For the eigenvector problem to be sensible, the eigenvalue σ should vary "continuously" in $x \in X$.

1.5.2. *Varying $x \in X$.* Varying $x \in X$ amounts to the following adjustment to the convolution and Hecke diagrams:

- Let $\mathrm{Gr}_{G,X}$ be the ind-scheme over X parametrizing triples $(x, \mathcal{P}_G, \mathcal{P}_G \sim \mathcal{P}_G^\circ)$ where $\mathcal{P}_G$ is a G-bundle over X and the trivialization is over $X - \{x\}$. There is an identification:

$$\tilde{X} \overset{\mathrm{Aut}(k[\![t]\!])}{\times} \mathrm{Gr}_G \xrightarrow{\sim} \mathrm{Gr}_{G,X} . \tag{1.5}$$

- There are analogous definitions of $\mathcal{L}_X^+ G$ and $\mathcal{L}_X G$.
- The global Hecke stack parametrizes the data $(x, \mathcal{P}_G^1, \mathcal{P}_G^2, \mathcal{P}_G^1\big|_{X-\{x\}} \xrightarrow{\sim} \mathcal{P}_G^2)$, and we have an isomorphism:

$$\mathrm{Bun}_{G,\infty} \overset{\mathcal{L}_X^+ G}{\times} \mathrm{Gr}_{G,X} \xrightarrow{\sim} \mathrm{Hecke}$$

  We have an analogous functor:

$$\vec{h} : \mathrm{Hecke} \to X \times \mathrm{Bun}_G$$

  and the Hecke action is defined by

$$\mathcal{M} * \underline{\mathcal{F}} := \vec{h}_!(\mathcal{M} \tilde{\boxtimes} \underline{\mathcal{F}})$$

  as a $\mathbf{Sph}_G$-family of functors $D(\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_G)) \to D(\mathcal{D}\text{-}\mathbf{Mod}(X \times \mathrm{Bun}_G))$.

An important observation is that every object $\mathcal{F} \in \mathbf{Sph}_G$ is automatically $\mathrm{Aut}(k[\![t]\!])$-equivariant, hence giving rise to a $\mathcal{D}$-modules $\underline{\mathcal{F}}$ over $\mathrm{Gr}_{G,X}$ as

$$\underline{\mathcal{F}} := \mathcal{O}_X \tilde{\boxtimes} \mathcal{F}$$

along (1.5).

1.5.3. Therefore, the Hecke eigenvector problem should assert the following: find $\mathcal{M} \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_G)$ with a compatible system of isomorphisms:

$$\gamma_V : \mathcal{M} * \underline{\mathrm{Sat}}(V) \xrightarrow{\sim} \underline{\sigma}(V) \boxtimes \mathcal{M},$$

for all $V \in \mathbf{Rep}_{\check{G}}$.

Remark 1.18. In particular, $\mathcal{M} * \underline{\mathrm{Sat}}(V)$ needs to live in the abelian category of $\mathcal{D}$-modules.

We would like $\underline{\sigma}$ to be an exact functor $\mathbf{Rep}_{\check{G}} \to \mathcal{D}\text{-}\mathbf{Mod}(X)$ which respects the symmetric monoidal structures. Such data are precisely pairs $(\mathcal{P}_{\check{G}}, \nabla)$ where $\mathcal{P}_{\check{G}}$ is a $\check{G}$-bundle over X and ∇ is a connection on $\mathcal{P}_{\check{G}}$; the corresponding functor is the associated bundle construction:

$$\mathbf{Rep}_{\check{G}} \ni V \rightsquigarrow V_{\mathcal{P}_{\check{G}}} \text{ with connection induced from } \nabla.$$

The family γ_V needs to be compatible with factorization structure. We will state this property below in §1.6.2.

1.6. Factorization structure and statement of the Hecke eigen-property.

1.6.1. Note that $\underline{\mathrm{Sat}} : \mathbf{Rep}_{\check{G}} \to \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_{G,X})^{\mathcal{L}^+_X G}$ fits into a family of functors parametrized by $I \in \mathbf{fSet}$:

- The functor
$$\underline{\mathrm{Sat}}^I : \mathbf{Rep}_{\check{G}^I} \to \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_{G,X^I})^{\mathcal{L}^+_{X^I} G}$$
sends $\boxtimes_{i \in I} V_i$ to the $\mathcal{D}$-module:
$$\underline{\mathrm{Sat}}^I(\boxtimes_{i \in I} V_i) := j_{!*}(\boxtimes_{i \in I} \underline{\mathrm{Sat}}(V_i)\big|_{X^I - \Delta})$$
using the identification:
$$\mathrm{Gr}^{\times I}_{G,X}\big|_{X^I - \Delta_{I \to \{1\}}} \xrightarrow{\sim} \mathrm{Gr}_{G,X^I}\big|_{X^I - \Delta_{I \to \{1\}}}.$$
Remark 1.19. Here and after, the notation $\Delta_{I \to J}$ means the image along the diagonal embedding $X^J \to X^I$.
- ("*Well-definedness* over $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ and *unitality*") Whenever we have a map $I \to J$ in $\mathbf{fSet}$, we have a (2-)commutative diagram:
$$\begin{tikzcd} \mathbf{Rep}_{\check{G}^I} \arrow[r] \arrow[d] & \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_{G,X^I})^{\mathcal{L}^+_{X^I} G} \arrow[d] \\ \mathbf{Rep}_{\check{G}^J} \arrow[r] & \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_{G,X^J})^{\mathcal{L}^+_{X^J} G} \end{tikzcd}$$
- ("*Factorization*") Whenever the map $\alpha : I \twoheadrightarrow J$ is surjective, we have an identification:
$$\underline{\mathrm{Sat}}^I(\boxtimes_{j \in J} V_{\alpha^{-1}(j)})\big|_{X^I - \Delta_\alpha} \xrightarrow{\sim} \boxtimes_{j \in J} \underline{\mathrm{Sat}}^{\alpha^{-1}(j)}(V_{\alpha^{-1}(j)})\big|_{X^I - \Delta_\alpha} \tag{1.6}$$
of $\mathcal{D}$-modules over (the identified spaces):
$$\mathrm{Gr}_{G,X^I}\big|_{X^I - \Delta_\alpha} \xrightarrow{\sim} \prod_{j \in J} \mathrm{Gr}_{G,X^{\alpha^{-1}(j)}}\big|_{X^I - \Delta_\alpha},$$
for any $V_{\alpha^{-1}(j)} \in \mathbf{Rep}_{\check{G}^{\alpha^{-1}(j)}}$ where $j \in J$.
The isomorphisms (1.6) are required to satisfy a cocycle condition for $I \twoheadrightarrow J \twoheadrightarrow K$. This structure is clear from the definition of the functors $\underline{\mathrm{Sat}}^I$.

Remark 1.20. We will define the object $\mathrm{Gr}_{G,\mathrm{Ran}(X)}$ later in the semester and explain the general meaning of "factorization."

1.6.2. Given a $\check{G}$-local system $\sigma = (\mathcal{P}_{\check{G}}, \nabla)$, a *Hecke eigensheaf* with respect to σ is an object $\mathcal{M} \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Bun}_G)$ together with isomorphisms:

$$\gamma_{V_I} : \mathcal{M} * \underline{\mathrm{Sat}}^I(V_I) \xrightarrow{\sim} (V_I)_\sigma \boxtimes \mathcal{M}, \quad \text{in } \mathcal{D}\text{-}\mathbf{Mod}(X^I \times \mathrm{Bun}_G) \tag{1.7}$$

parametrized by all $I \in \mathbf{fSet}$ and $V_I \in \mathbf{Rep}_{\check{G}^I}$. They are subject to the following compatibility:

– For every map $\alpha : I \to J$ in $\mathbf{fSet}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M} * \underline{\mathrm{Sat}}^I(V^I)\big|_{X^J \times \mathrm{Bun}_G} \arrow[r, "\gamma_{V_I}"] \arrow[d, "\sim"] & (V_I)_\sigma\big|_{X^J} \boxtimes \mathcal{M} \arrow[d, "\sim"] \\
\mathcal{M} * \underline{\mathrm{Sat}}^J(\mathrm{Res}^{\check{G}^J}_{\check{G}^I} V^I) \arrow[r, "\gamma_{V_J}"] & (V_J)_\sigma \boxtimes \mathcal{M}
\end{tikzcd}$$

Remark 1.21. Here, the "restriction" $\big|_{X^J}$ means applying the functor $\Delta^\bullet_\alpha$ to $\mathcal{D}$-modules.

– For a surjection $\alpha : I \twoheadrightarrow J$ and $V_{\alpha^{-1}(j)} \in \mathbf{Rep}_{\check{G}^{\alpha^{-1}(j)}}$ for all $j \in J$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M} * \underline{\mathrm{Sat}}^I(\boxtimes_{j\in J} V_{\alpha^{-1}(j)})\big|_{(X^I - \Delta_\alpha)\times \mathrm{Bun}_G} \arrow[r, "\gamma_{\boxtimes_{j\in J} V_{\alpha^{-1}(j)}}"] \arrow[d, "\sim"] & (\boxtimes_{j\in J} V_{\alpha^{-1}(j)})_\sigma\big|_{(X^I - \Delta_\alpha)} \boxtimes \mathcal{M} \arrow[d, "\sim"] \\
\mathcal{M} * \underline{\mathrm{Sat}}^{\alpha^{-1}(1)} * \cdots * \underline{\mathrm{Sat}}^{\alpha^{-1}(|J|)}\big|_{(X^I - \Delta_\alpha)\times \mathrm{Bun}_G} \arrow[r] & \boxtimes_{j\in J}(V_{\alpha^{-1}(j)})_\sigma\big|_{X^I - \Delta_\alpha} \boxtimes \mathcal{M}
\end{tikzcd}$$

where the bottom arrow is the successive application of each $\gamma_{V_{\alpha^{-1}(j)}}$.

Remark 1.22. The first compatibility shows that γ_{triv} for the trivial $\check{G}$-representation is the identity map on $\mathcal{O}_X \boxtimes \mathcal{M}$. This follows from considering $\emptyset \to \{1\}$.

2. Proofs for GL_1

2.1. **The characteristic-zero situation.** Recall our cheat of changing $\mathrm{Bun}_{\mathrm{GL}_1}$ into $\mathrm{Pic}(X)$. For $\lambda \in \mathbb{Z}$, we have

$$\mathrm{Hecke}^\lambda_{\mathrm{GL}_1, X} \xrightarrow{\sim} X \times \mathrm{Pic}(X)$$

and the morphism $\overleftarrow{h}$ and $\overrightarrow{h}$ sends a pair $(x, \mathcal{L})$ to $\mathcal{L}$, respectively $\mathcal{L}(-\lambda x)$. (See Example 1.16.)

2.1.1. Given a 1-dimensional local system σ over X, the data (1.7) for a Hecke eigensheaf $\mathcal{M} \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Pic}(X))$ amount to the isomorphisms:

$$(\mathrm{add}^{\{\lambda_i\}_{i\in I}})^\bullet \mathcal{M} \xrightarrow{\sim} (\boxtimes_i \sigma^{\lambda_i}) \boxtimes \mathcal{M}, \quad \text{in } \mathcal{D}\text{-}\mathbf{Mod}(X^I \times \mathrm{Pic}(X)) \tag{2.1}$$

where $\{\lambda_i\}_{i\in I}$ is any sequence of integers, and $\mathrm{add}^{\{\lambda_i\}_{i\in I}}$ is the morphism:

$$\mathrm{add}^{\{\lambda_i\}_{i\in I}} : X^I \times \mathrm{Pic}(X) \to \mathrm{Pic}(X)$$

sending

$$(\{x_i\}_{i\in I}, \mathcal{L}) \rightsquigarrow \mathcal{L}(\Sigma_{i\in I} \lambda_i x_i).$$

2.1.2. I claim that it suffices to find $\mathcal{M} \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Pic}(X))$ with

- the structure of a multiplicative local system on $\mathrm{Pic}(X)$,
- an isomorphism $(J^1)^\bullet \mathcal{M} \xrightarrow{\sim} \sigma$ along the Abel Jacobi map:

$$J^1 : X \to \mathrm{Pic}^1(X), \quad x \rightsquigarrow \mathcal{O}(x).$$

Indeed, one can deduce from these properties that we have an isomorphism

$$(J^\lambda)^\bullet \mathcal{M} \xrightarrow{\sim} \sigma^\lambda$$

for each $J^\lambda : X \to \mathrm{Pic}^\lambda(X)$ sending x to $\mathcal{O}(\lambda x)$.

Now, the isomorphisms (2.1) are obtained by pulling back the isomorphism over $\mathrm{Pic}(X)^{I \sqcup \{1\}}$:

$$\mathrm{mult}^\bullet \mathcal{M} \xrightarrow{\sim} \boxtimes_{i \in I \sqcup \{1\}} \mathcal{M}_i$$

along

$$(\Pi_{i \in I} J^{\lambda_i}) \times \mathrm{id} : X^I \times \mathrm{Pic}(X) \to \mathrm{Pic}(X)^{I \sqcup \{1\}}$$

and the compatibility with factorization structure can be easily checked.

2.1.3. How do we produce $\mathcal{M} \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Pic}(X))$ with the properties asserted above? Observe that J^1 induces an isomorphism:

$$\pi_1(X^{\mathrm{an}})_{\mathrm{ab}} \xrightarrow{\sim} \pi_1(\mathrm{Pic}^1(X)^{\mathrm{an}}).$$

For a projective scheme Y, finite dimensional representations of $\pi_1(Y^{\mathrm{an}})$ are equivalent to vector bundles over Y together with a flat connection; this equivalence is natural in Y.[9] Hence, the local system σ gives rise to $\mathcal{M}^1 \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Pic}^1(X))$ with

$$(J^1)^\bullet \mathcal{M}^1 \xrightarrow{\sim} \sigma.$$

We would like to spread $\mathcal{M}^1$ over all connected components of $\mathrm{Pic}(X)$. Fixing $x \in X$, we define:

$$\mathcal{M}^0 := \mathrm{add}_x^\bullet \mathcal{M} \otimes \sigma|_x^{\otimes -1} \in \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Pic}^0(X))$$

where $\mathrm{add}_x : \mathrm{Pic}^0(X) \to \mathrm{Pic}^1(X)$ is the map $\mathcal{L} \rightsquigarrow \mathcal{L}(x)$.

Claim 2.1. $\mathcal{M}^0$ is a multiplicative local system over $\mathrm{Pic}^0(X)$.

Proof.

- Since $\mathcal{M}^0\big|_{\{0\}}$ is canonically trivialized, the theorem of the square shows that $\mathcal{M}^0$ has the structure of a multiplicative line bundle:

$$\mathrm{mult}^* \mathcal{M}^0 \xrightarrow{\sim} \mathcal{M}^0 \boxtimes \mathcal{M}^0.$$

- We need to show that the above isomorphism respects the connections. However, the difference between $\mathrm{mult}^\bullet \nabla$ and $\nabla \boxtimes \nabla$ is a 1-form vanishing over $\{0\} \times \mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(X) \times \{0\}$; all such 1-forms must vanish by a general property of abelian varieties.[10]

□

Finally, we define $\mathcal{M}^\lambda$ over $\mathrm{Pic}^\lambda(X)$ using the isomorphism:

$$\mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Pic}^\lambda(X), \quad \mathcal{L} \rightsquigarrow \mathcal{L}(\lambda x)$$

[9] General stuff allows you to pass to analytic vector bundles over Y with analytic flat connection; then apply GAGA.

[10] Hint: all global 1-forms on an abelian variety are translation-invariant.

and the multiplicativity of $\mathcal{M}$ follows from the commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}^0(X)\times\mathrm{Pic}^0(X) \arrow[r] \arrow[d] & \mathrm{Pic}^0(X) \arrow[d] \\
\mathrm{Pic}^\lambda(X)\times\mathrm{Pic}^\mu(X) \arrow[r] & \mathrm{Pic}^{\lambda+\mu}(X).
\end{tikzcd}$$

2.2. **The characteristic p version.** This is contained in Yuval's notes on geometric class field theory: http://www.math.harvard.edu/~yifei/amigos/class_field_theory.pdf.

3. Outline of the Beilinson-Drinfeld approach

3.1. GL_1**-case again.** Let us revisit the proof for the GL_1-case (see §2.1). Recall that the composition: $X \xrightarrow{J^1} \mathrm{Pic}^1(X) \hookrightarrow \mathrm{Pic}(X)$ induces equivalence of categories:

$$\left\{\begin{matrix}\text{multiplicative local}\\ \text{systems on } \mathrm{Pic}(X)\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{rank-1 local systems}\\ \text{on } \mathrm{Pic}^1(X)\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{rank-1 local systems}\\ \text{on } X\end{matrix}\right\}. \tag{3.1}$$

The Hecke eigensheaf $\mathcal{M}$ is constructed as the image of the rank-1 local system σ on X in the first category.

Restricting our attention to local systems whose underlying line bundle is *trivial*, we obtain from (3.1) isomorphisms of vector spaces:

$$\left\{\begin{matrix}\text{translation-invariant}\\ \text{1-forms on } \mathrm{Pic}(X)\end{matrix}\right\} \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Pic}^1(X), \Omega^1_X) \xrightarrow{\sim} \mathrm{H}^0(X, \omega_X).$$

In other words, fixing $\sigma = (\mathcal{O}_X, \nabla)$, we may write $\nabla = d + \eta$ for some $\eta \in \mathrm{H}^0(X, \omega_X)$. Then we can regard $\mathcal{M} = (\mathcal{O}_{\mathrm{Pic}(X)}, d + \tilde{\eta})$ as the local system such that $\tilde{\eta}$ restricts to η along the Abel-Jacobi map.

3.1.1. We can reconceptualize the association $\eta \rightsquigarrow \mathcal{M}$ as follows. Let H be any commutative group scheme. The H-action on itself induces a map $T^*H \to \mathfrak{h}^*$, hence a morphism of k-algebras $\mathcal{O}_{\mathfrak{h}^*} \to \Gamma(T^*H, \mathcal{O})$. This morphism fits into a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{\mathfrak{h}^*} \arrow[rr] \arrow[d, "\simeq"] & & \Gamma(T^*H, \mathcal{O}) \arrow[d, "\simeq"] \\
\mathrm{gr}\, \mathrm{U}(\mathfrak{h}) \arrow[r, dotted] & \mathrm{gr}\, \Gamma(H, \mathcal{D}_H) \arrow[r, hook] & \Gamma(H, \mathrm{gr}\, \mathcal{D}_H).
\end{tikzcd} \tag{3.2}$$

where the dotted arrow arises from a morphism of filtered k-algebras: $\mathrm{U}(\mathfrak{h}) \to \Gamma(H, \mathcal{D}_H)$.

Remark 3.1. We would like to think of $\mathcal{O}_{\mathfrak{h}^*}$ as defining a "completely integrable system" in $\Gamma(T^*H, \mathcal{O})$, and the map $\mathrm{U}(\mathfrak{h}) \to \Gamma(H, \mathcal{D}_H)$ as the quantization of such.

For $H = \mathrm{Pic}(X)$, if we are given $\eta \in \mathrm{H}^0(X, \omega_X)$, corresponding to a character of $\mathcal{O}_{\mathrm{H}^0(X,\omega_X)}$, the construction of $\mathcal{M}$ proceeds by:

- identifying $\mathcal{O}_{\mathrm{H}^0(X,\omega_X)} \xrightarrow{\sim} \mathrm{U}(\mathfrak{h})$ (via the Abel-Jacobi map, say);
- define $\mathcal{M} := \mathcal{D}_H \underset{\mathrm{U}(\mathfrak{h})}{\otimes} k$ using the above character of $\mathrm{U}(\mathfrak{h})$.

Remark 3.2. Indeed, over any commutative group scheme H, the $\mathcal{D}_H$-module $\mathcal{D}_H \underset{\mathrm{U}(\mathfrak{h})}{\otimes} k$ associated to a character $\tilde{\eta} : U(\mathfrak{h}) \to k$ is $(\mathcal{O}_H, d + \tilde{\eta})$ (where $\tilde{\eta}$ is regarded as an invariant 1-form, via $\mathrm{U}(\mathfrak{h}) \cong \mathcal{O}_{\mathfrak{h}^*}$.)

3.2. **Towards a semisimple group G.** Of course, Bun_G is *not* a commutative group stack, but the above picture generalizes:

	GL_1-case	semisimple case
classical	$T^*H \to \mathfrak{h}^*$	Hitchin fibration $T^*\mathrm{Bun}_G \to \mathrm{Hitch}_G(X)$
	$\mathrm{H}^0(X, \omega_X)$	classical oper $\mathrm{Op}^{\mathrm{cl}}_{\check{G}}(X)$
quantum	$\mathrm{U}(\mathfrak{h}) \to \Gamma(H, \mathcal{D}_H)$	Kac-Moody localization $\mathrm{H}_\nabla(X, \mathfrak{z}_{\mathrm{crit}}) \to \Gamma(\mathrm{Bun}_G, \mathcal{D}(\mathcal{L}_{G,\det}^{-\frac{1}{2}}))$
	$\mathrm{U}(\mathfrak{h}) \xrightarrow{\sim} \mathrm{H}^0(X, \omega_X)$	Feigin-Frenkel isomorphism $\mathrm{H}_\nabla(X, \mathfrak{z}_{\mathrm{crit}}) \xrightarrow{\sim} \mathcal{O}_{\mathrm{Op}_{\check{G}}(X)}$

I will now briefly explain all the items in the right column.

3.2.1. Note that for G a semisimple group, Bun_G is good for lazybones (Theorem 1.12). In particular, $T^*\mathrm{Bun}_G$ is a classical stack, so it is given by
$$T^*\mathrm{Bun}_G \xrightarrow{\sim} \underline{\mathrm{Spec}}_{\mathrm{Bun}_G}(\mathrm{Sym}\,\mathrm{H}^0(\mathcal{T}_{\mathrm{Bun}_G})).$$
Since $\mathcal{T}_{\mathrm{Bun}_G} \xrightarrow{\sim} \mathrm{R}\Gamma(X, \mathfrak{g}_{\mathcal{P}})[1]$, we have:
$$\mathrm{H}^0(\mathcal{T}_{\mathrm{Bun}_G}) \xrightarrow{\sim} \mathrm{H}^1(X, \mathfrak{g}_{\mathcal{P}}) \xrightarrow{\sim} \mathrm{H}^0(X, \mathfrak{g}^*_{\mathcal{P}} \otimes \omega_X)^*.$$
In other words, $T^*\mathrm{Bun}_G$ is the moduli space of pairs $(\mathcal{P}_G, \theta)$, where $\mathcal{P}_G$ is a G-bundle over X and $\theta \in \mathrm{H}^0(X, \mathfrak{g}^*_{\mathcal{P}} \otimes \omega_X)$. This latter space is also called the *Higgs* moduli space Higgs_G.

Denote by $\mathfrak{g}/\!\!/G$ the GIT quotient $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}^*)^G)$. It is a classical theorem that $\mathrm{Sym}(\mathfrak{g}^*)^G$ is a finitely generated free algebra,[11] so $\mathfrak{g}/\!\!/G$ is a vector space. We define $\mathrm{Hitch}_G(X)$ by
$$\mathrm{Hitch}_G(X) := \mathrm{H}^0(X, \omega_X \overset{\mathbb{G}_m}{\times} \mathfrak{g}/\!\!/G)$$
where ω_X is regarded as a $\mathbb{G}_m$-torsor. There is a natural map:
$$\mathrm{Higgs}_G \to \mathrm{Hitch}_G(X), \quad (\mathcal{P}_G, \theta) \rightsquigarrow \overline{\theta},$$
which is called the *Hitchin fibration*.

Remark 3.3. Like what we have seen in the GL_1-case, the induced map $\mathcal{O}_{\mathrm{Hitch}_G(X)} \to \Gamma(\mathrm{Higgs}_G, \mathcal{O})$ realizes $\mathcal{O}_{\mathrm{Hitch}_G(X)}$ as a completely integrable system. This is a nontrivial theorem, due to N. Hitchin.

3.2.2. I will tell you what *classical* opers are. Fix a Borel $B \hookrightarrow G$. Let
$$\mathfrak{h}^{-1} := \mathfrak{b} \oplus \sum_{\check{\alpha} \in \Delta^-} \mathfrak{h}^{\check{\alpha}}$$
which is a B-invariant subspace of $\mathfrak{g}$. Denote by $\mathfrak{h}^{-1}_{\mathrm{n.d.}}$ the *non-degenerate* locus of $\mathfrak{h}^{-1}$, i.e., the subset of elements whose projection to each root space $\mathfrak{h}^{\check{\alpha}}$ is nonzero. Then one can show that $\mathfrak{h}^{-1}_{\mathrm{n.d.}}/\!\!/B$ is again a vector space, and we can define:
$$\mathrm{Op}^{\mathrm{cl}}_G(X) := \mathrm{H}^0(X, \omega_X \overset{\mathbb{G}_m}{\times} \mathfrak{h}_{\mathrm{n.d.}}/\!\!/B).$$
The important fact (which is just a matter of group theory) is:

Lemma 3.4. *There is a canonical isomorphism* $\mathrm{Hitch}_G(X) \xrightarrow{\sim} \mathrm{Op}^{\mathrm{cl}}_G(X)$.

[11] For instance, when $G = \mathrm{GL}_n$, a set of generators for $\mathrm{Sym}(\mathfrak{g}^*)^G$ can be given as coefficients of the characteristic polynomial of elements in $\mathfrak{g} = \mathfrak{gl}_n$. These range from the degree-1 polynomial $\mathrm{Tr} : \mathfrak{g} \to k$ to the degree-n polynomial $\det : \mathfrak{g} \to k$.

Remark 3.5. The above lemma is the nonabelian analogue of the isomorphism $\mathfrak{h}^* \xrightarrow{\sim} \mathrm{H}^0(X, \omega_X)$.

We will see a geometric object $\mathrm{Op}_G(X)$, called *opers*, that degenerate into $\mathrm{Op}_G^{\mathrm{cl}}(X)$ in a certain sense. Right now, let us just assume that it exists as a closed substack of $\mathrm{LocSys}_G(X)$.

3.2.3. Defining the object $\mathrm{H}_\nabla(X, \mathfrak{z}_{\mathrm{crit}})$ requires the theory of chiral algebras.

Remark 3.6. Informally, a *chiral algebra* is a $\mathcal{D}_X$-module equipped with a "meromorphic Lie bracket." Any chiral algebra has a *center* which is just a commutative $\mathcal{D}_X$-algebra. There is a functor:

$$- \otimes \mathcal{O}_X : \mathbf{Alg}_{/k} \to \mathcal{D}_X\text{-}\mathbf{Alg}^{\mathrm{comm}}$$

which admits a *left* adjoint, denoted by $\mathrm{H}_\nabla(X, -)$. It is called the *conformal block* of a commutative $\mathcal{D}_X$-algebra.

To each G-invariant symmetric bilinear form κ on $\mathfrak{g}$, one can associate a chiral algebra $\mathcal{A}(\mathfrak{g}, \kappa)$, and therefore its center $\mathfrak{z}_\kappa$. The important observation is as follows:

– Consider the *determinant* line bundle $\mathcal{L}_{G,\det}$ over Bun_G, defined by the formula:

$$\mathcal{L}_{G,\det}\big|_S := \det \mathrm{R}\,\Gamma(X, \mathfrak{g}_{\mathcal{P}_G}), \quad S \xrightarrow{\mathcal{P}_G} \mathrm{Bun}_G\,.$$

Then there is a morphism of commutative $\mathcal{D}_X$-algebras:

$$\mathfrak{z}_{c\cdot\mathrm{Kil}} \to \Gamma(\mathrm{Bun}_G, \mathcal{D}(\mathcal{L}_{G,\det}^c)) \otimes \mathcal{O}_X\,. \tag{3.3}$$

The construction of (3.3) takes quite a bit of work. Assuming that we have it at our disposal, the adjoint property of $\mathrm{H}_\nabla(X, -)$ will give us a morphism of k-algebras:

$$\mathrm{H}_\nabla(X, \mathfrak{z}_{c\cdot\mathrm{Kil}}) \to \Gamma(\mathrm{Bun}_G, \mathcal{D}(\mathcal{L}_{G,\det}^c)). \tag{3.4}$$

3.2.4. We now come to a piece of dark magic:

Theorem 3.7 (Feigin-Frenkel)**.** *There is an isomorphism of k-algebras:*

$$\mathrm{H}_\nabla(X, \mathfrak{z}_{-\frac{1}{2}\cdot\mathrm{Kil}}) \xrightarrow{\sim} \mathcal{O}_{\mathrm{Op}_{\check{G}}(X)}.$$

In fact, for any form κ other than $-\frac{1}{2}\cdot\mathrm{Kil}$, the algebra $\mathrm{H}_\nabla(X, \mathfrak{z}_\kappa)$ will be degenerate (i.e., isomorphic to k). We call $\mathrm{crit} := -\frac{1}{2}\cdot\mathrm{Kil}$ the *critical level.* Setting $c = -\frac{1}{2}$ in (3.4), we obtain the morphism

$$\mathrm{H}_\nabla(X, \mathfrak{z}_{\mathrm{crit}}) \to \Gamma(\mathrm{Bun}_G, \mathcal{D}(\mathcal{L}_{G,\det}^{-\frac{1}{2}}))$$

of *Kac-Moody localization* (at the critical level).

3.2.5. Putting together all our ingredients, the analogue of diagram (3.2) reads as follows:

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Hitch}_G(X)} \arrow[rr] \arrow[d, "\simeq"] & & \Gamma(T^*\mathrm{Bun}_G, \mathcal{O}) \arrow[d, "\simeq"] \\
\mathrm{gr}\,\mathrm{H}_\nabla(X, \mathfrak{z}_{\mathrm{crit}}) \arrow[r] & \mathrm{gr}\,\Gamma(\mathrm{Bun}_G, \mathcal{D}(\mathcal{L}_{G,\det}^{-\frac{1}{2}})) \arrow[r, hook] & \Gamma(\mathrm{Bun}_G, \mathrm{gr}\,\mathcal{D}(\mathcal{L}_{G,\det}^{-\frac{1}{2}})).
\end{tikzcd}$$

Given an oper $\sigma \in \mathrm{Op}_{\check{G}}(X) \hookrightarrow \mathrm{LocSys}_{\check{G}}(X)$, we obtain a $\mathcal{D}(\mathcal{L}_{G,\det}^{-\frac{1}{2}})$-module over Bun_G by:

$$\tilde{\mathcal{M}}^\sigma := \Gamma(\mathrm{Bun}_G, \mathcal{D}(\mathcal{L}_{G,\det}^{-\frac{1}{2}})) \underset{\mathrm{H}_\nabla(X, \mathfrak{z}_{\mathrm{crit}})}{\otimes} k,$$

using the character of $\mathrm{H}_\nabla(X, \mathfrak{z}_{\mathrm{crit}})$ corresponding to σ. Then $\tilde{\mathcal{M}}^\sigma$ will satisfy the Hecke eigenproperty (see §1.6) by the fact that the Feigin-Frenkel isomorphism is "compatible" with geometric Satake equivalence. Compatibility with factorization structure will essentially follow from the theory of chiral algebras.

3.2.6. Finally, in order to construct an honest (i.e., untwisted) $\mathcal{D}$-module over Bun_G, we appeal to the existence of the *Pfaffian*:

Lemma 3.8. *The line bundle* $\mathcal{L}_{G,\mathrm{det}}$ *admits a square root.*

References

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