BONDAL-ORLOV RECONSTRUCTION THEOREM

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1. Some technical ingredients

Let X be a smooth projective variety over an algebraically closed field k. Let ω_X denote the canonical sheaf of X.

1.1. **Cohomology at "endpoints".** One advantage of working with the bounded derived category $\mathrm{D}^{\mathrm{b}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ is that any chain $\mathscr{E}^\bullet$ maps to its highest cohomology sheaf and receives a map from its lowest cohomology sheaf.[1] More precisely,

Lemma 1.1. *Let $\mathscr{E}^\bullet$ be an object in $D^b(X)$.*

(i) Set m_1 to be the maximal integer such that $\mathscr{H}^{m_1}(\mathscr{E}^\bullet) \neq 0$. Then there is a morphism

$$\mathscr{E}^\bullet \to \mathscr{H}^{m_1}(\mathscr{E}^\bullet)[-m_1]$$

which induces isomorphism on m_1th cohomology.

(ii) Set m_2 to be the minimum integer such that $\mathscr{H}^{m_2}(\mathscr{E}^\bullet) \neq 0$. Then there is a morphism

$$\mathscr{H}^{m_2}(\mathscr{E}^\bullet)[-m_2] \to \mathscr{E}^\bullet$$

which induces isomorphism on m_2th cohomology.

Proof. We only prove (i). There is a quasi-isomorphism

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathscr{E}^{m_1-1} & \longrightarrow & \ker(d^{m_1}) & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \mathscr{E}^{m_1-1} & \longrightarrow & \mathscr{E}^{m_1} & \xrightarrow{d^{m_1}} & \mathscr{E}^{m_1+1} & \longrightarrow & \cdots \end{array}$$

Hence we may replace $\mathscr{E}^\bullet$ by the upper chain. This chain clearly admits a morphism to $\mathscr{H}^{m_1}(\mathscr{E}^\bullet)[-m_1]$ which induces isomorphism on m_1th cohomology. □

This seemingly innocuous lemma turns out to be very useful. Here are some of its consequences:

Corollary 1.2. *Let $\mathscr{E}^\bullet$, m_1, m_2 be as above. Then for any object $\mathscr{F}$ of* $\mathrm{Coh}(X)$*, there are bijections:*

$$\mathrm{Hom}(\mathscr{H}^{m_1}(\mathscr{E}^\bullet), \mathscr{F}) \cong \mathrm{Hom}(\mathscr{E}^\bullet, \mathscr{F}[-m_1])$$
$$\mathrm{Hom}(\mathscr{F}, \mathscr{H}^{m_2}(\mathscr{E}^\bullet)) \cong \mathrm{Hom}(\mathscr{F}[-m_2], \mathscr{E}^\bullet).$$

Proof. Rightward maps are given by composition with the maps in Lemma 1.1. Leftward maps are given by taking cohomology. □

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[1] This technique already showed up in Alex Perry's talk, but let's review it.

Corollary 1.3. *Let $\mathscr{E}^\bullet$, m_2 be as above. Then there is a distinguished triangle:*

$$\mathscr{H}^{m_2}(\mathscr{E}^\bullet)[-m_2] \longrightarrow \mathscr{E}^\bullet \longrightarrow \mathscr{E}_1^\bullet \longrightarrow \mathscr{H}^{m_2}(\mathscr{E}^\bullet)[1-m_2]$$

where $\mathscr{E}_1^\bullet$ is an object in $\mathrm{D^b}(X)$ with length strictly less than that of $\mathscr{E}^\bullet$.

Recall that the *length* of $\mathscr{E}^\bullet$ is defined to be $m_1 - m_2$.

Proof. Construct $\mathscr{E}_1^\bullet$ as the cone of $\mathscr{H}^{m_2}(\mathscr{E}^\bullet)[-m_2] \to \mathscr{E}^\bullet$ and we obtain the above distinguished triangle. Taking cohomology yields

$$\mathscr{H}^{m_2}(\mathscr{E}_1^\bullet) = 0, \quad \mathscr{H}^{m}(\mathscr{E}_1^\bullet) = \mathscr{H}^{m}(\mathscr{E}^\bullet) \text{ for all } m > m_2.$$

Hence the length of $\mathscr{E}_1^\bullet$ is strictly less. □

1.2. **The Grothendieck spectral sequence.** Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be abelian categories. Use $\mathrm{K}(\mathscr{A})$ (resp. $\mathrm{K^b}(\mathscr{A})$) to denote the homotopy category of chain complexes (resp. bounded) in $\mathscr{A}$.

Theorem 1.4 (See [3], Proposition 2.66)**.** *Let $F_1 : \mathrm{K^b}(\mathscr{A}) \to \mathrm{K^b}(\mathscr{B})$ and $F_2 : \mathrm{K^b}(\mathscr{B}) \to \mathrm{K}(\mathscr{C})$ be two exact functors. Suppose that $\mathscr{A}$ and $\mathscr{B}$ contain enough injectives and that the image under F_1 of a complex $\mathscr{I}^\bullet$ with each $\mathscr{I}^i$ injective is contained in an F_2-adapted triangulated subcategory $\mathscr{K}_{F_2}$.*

Then for any any complex $\mathscr{F}^\bullet \in \mathrm{D^b}(\mathscr{A})$ there exists a spectral sequence

$$E_2^{p,q} = R^pF_2(R^qF_1(\mathscr{F}^\bullet)) \implies E^n = R^n(F_2 \circ F_1)(\mathscr{F}^\bullet)$$

Talking about derived functors actually creates some trouble for us: the category $\mathrm{Coh}(X)$ does not contain enough injectives in general. However, by passing to $\mathrm{D^b_{Coh}}(\mathrm{QCoh}(X))$, the derived category of complexes of quasi-coherent sheaves with coherent cohomology, we can still make use of much of the derived functors machinery.

Here are the relevant results that will be used in these notes:

Proposition 1.5. *For all $\mathscr{E}^\bullet, \mathscr{F}^\bullet$ in $\mathrm{D^b}(X)$,*

(i) there is an isomorphism

$$\mathrm{Ext}^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \cong \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i])$$

(ii) there are spectral sequences

$$E_2^{p,q} = \mathrm{Ext}^p(\mathscr{E}^\bullet, \mathscr{H}^q(\mathscr{F}^\bullet)) \implies \mathrm{Ext}^{p+q}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \tag{1.1}$$

$$E_2^{p,q} = \mathrm{Ext}^p(\mathscr{H}^{-q}(\mathscr{E}^\bullet), \mathscr{F}^\bullet) \implies \mathrm{Ext}^{p+q}(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \tag{1.2}$$

where the Ext*'s are computed in* $\mathrm{QCoh}(X)$.

Here is how one proves Proposition 1.5.[a] The essential ingredient is the following

Proposition 1.6 ([3], Proposition 3.5)**.** *The natural functor $\mathrm{D^b}(X) \to \mathrm{D^b_{Coh}}(\mathrm{QCoh}(X))$ is an equivalence of categories.*

The proof of this fact is not hard; however, the assumption on boundedness does play a role. Now, part (i) follows from

$$\begin{aligned}\mathrm{Ext}^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet) &\cong \mathrm{Hom}_{\mathrm{D^b}(\mathrm{QCoh}(X))}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i]) \\ &\cong \mathrm{Hom}_{\mathrm{D^b_{Coh}}(\mathrm{QCoh}(X))}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i]) \\ &\cong \mathrm{Hom}_{\mathrm{D^b}(X)}(\mathscr{E}^\bullet, \mathscr{F}^\bullet[i])\end{aligned}$$

Part (ii) follows from Grothendieck's spectral sequences, first with $F_2 = \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathscr{E}^\bullet, -)$, $F_1 = \mathrm{id}$, and second with $F_2 = \mathrm{Hom}_{\mathrm{QCoh}(X)}(-, \mathscr{F}^\bullet)$, $F_1 = \mathrm{id}$.

[a] It's safe to skip this and just assume the proposition.

Corollary 1.7. *For any $\mathscr{E}^\bullet, \mathscr{F}^\bullet$ in $\mathrm{D^b}(X)$ and $i \in \mathbb{Z}$, the k-vector space $\mathrm{Ext}^i(\mathscr{E}^\bullet, \mathscr{F}^\bullet)$ is finite-dimensional.*

In particular, $\mathrm{D^b}(X)$ is a k-linear category with finite-dimensional hom-sets.[2]

[2] This means that the hom-sets of $\mathscr{A}$ are k-vector spaces, and compositions are k-bilinear maps.

Proof. We know that $\mathrm{Ext}^i(\mathscr{E},\mathscr{F})$ is finite dimensional for $\mathscr{E},\mathscr{F}$ concentrated in degree zero (i.e., they are just sheaves). Then the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p(\mathscr{E},\mathscr{H}^q(\mathscr{F}^\bullet)) \implies \mathrm{Ext}^{p+q}(\mathscr{E},\mathscr{F}^\bullet)$$

shows that $\mathrm{Ext}^i(\mathscr{E},\mathscr{F}^\bullet)$ is finite dimensional. Finally, (1.2) shows that $\mathrm{Ext}^i(\mathscr{E}^\bullet,\mathscr{F}^\bullet)$ is finite-dimensional in general. □

1.3. **Serre duality.** Serre duality can be formulated over $\mathrm{D^b}(X)$ as the existence of a "Serre functor."

Definition 1.8. Let $\mathscr{A}$ be a k-linear category. A *Serre functor* is a k-linear equivalence $S : \mathscr{A} \to \mathscr{A}$ such that for any two objects a, a' of $\mathscr{A}$, there exists an isomorphism

$$\mathrm{Hom}_{\mathscr{A}}(a,a') \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}}(a',S(a))^\vee$$

which is natural in a and a'.

It's somewhat surprising that Serre functors are always compatible with k-linear equivalences.

Lemma 1.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be k-linear categories with* finite-*dimensional hom-sets. Suppose $S_{\mathscr{A}}$ (resp. $S_{\mathscr{B}}$) is a Serre functor on $\mathscr{A}$ (resp. $\mathscr{B}$). Then any k-linear equivalence $F : \mathscr{A} \to \mathscr{B}$ satisfies*

$$S_{\mathscr{B}} \circ F \cong F \circ S_{\mathscr{A}}.$$

In particular, a Serre functor (if exists) is unique up to isomorphism.

Proof. There is a chain of natural isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{B}}(F(a),F(S_{\mathscr{A}}(a'))) &\cong \mathrm{Hom}_{\mathscr{A}}(a,S_{\mathscr{A}}(a')) \\
&\cong \mathrm{Hom}_{\mathscr{A}}(a',a)^\vee \\
&\cong \mathrm{Hom}_{\mathscr{B}}(F(a'),F(a))^\vee \\
&\cong \mathrm{Hom}_{\mathscr{B}}(F(a),S_{\mathscr{B}}(F(a')))
\end{aligned}$$

In the last isomorphism, we used finite-dimensionality. Now, as F is essentially surjective, the conclusion follows from the Yoneda lemma. □

Theorem 1.10 (Serre duality)**.** *The functor $S_X : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ defined by*

$$S_X(\mathscr{E}^\bullet) = (\mathscr{E}^\bullet \otimes \omega_X)[n]$$

is a Serre functor.

Proof. Let me just refer you to [3, Theorem 3.12]. □

Using Serre duality, we can calculate the homological dimension of $\mathrm{Coh}(X)$.

Definition 1.11. Let $\mathscr{A}$ be an abelian category. The *homological dimension* of $\mathscr{A}$ is the largest integer n for which $\mathrm{Ext}^i(a,a') = 0$ for all $i > n$.

Corollary 1.12. *The homological dimension of* $\mathrm{Coh}(X)$ *equals the dimension of X.*

Proof. Let $n = \dim(X)$. Then for all $i > n$,

$$\mathrm{Ext}^i(\mathscr{F},\mathscr{G}) \cong \mathrm{Ext}^{n-i}(\mathscr{G},\mathscr{F}\otimes\omega_X)^\vee = 0$$

for all coherent sheaves $\mathscr{F},\mathscr{G}$ on X. Note also that

$$\mathrm{Ext}^n(\mathscr{O}_X,\omega_X) \cong \mathrm{Hom}(\mathscr{O}_X,\mathscr{O}_X)^\vee \neq 0.$$

Hence $\mathrm{Coh}(X)$ has homological dimension exactly n. □

1.4. **An application: curves.** What does $\mathrm{D^b}(X)$ look like when X is a curve? The answer turns out to be very simple:

Theorem 1.13. *Let X be a smooth projective curve. Then any object $\mathscr{E}^\bullet$ in $\mathrm{D^b}(X)$ is* formal, *i.e.*[3]

$$\mathscr{E}^\bullet \cong \bigoplus \mathscr{H}^i(\mathscr{E}^\bullet)[-i].$$

Proof. Let $\mathscr{E}^\bullet$ be an object in $\mathrm{D^b}(X)$. We induct on the length of $\mathscr{E}^\bullet$. By Corollary 1.3, there is a distinguished triangle

$$\mathscr{H}^m(\mathscr{E}^\bullet)[-m] \longrightarrow \mathscr{E}^\bullet \longrightarrow \mathscr{E}_1^\bullet \longrightarrow \mathscr{H}^m(\mathscr{E}^\bullet)[1-m]$$

where m is the least integer such that $\mathscr{H}^m(\mathscr{E}^\bullet)$ is nonzero, and $\mathscr{E}_1^\bullet$ has strictly smaller length than $\mathscr{E}^\bullet$. Induction hypothesis shows that $\mathscr{E}_1^\bullet$ is formal. Hence

$$\mathscr{H}\mathrm{om}(\mathscr{E}_1^\bullet, \mathscr{H}^m(\mathscr{E}^\bullet)[1-m]) \cong \bigoplus_{i>m} \mathrm{Ext}^{1+i-m}(\mathscr{H}^i(\mathscr{E}_1^\bullet), \mathscr{H}^m(\mathscr{E}^\bullet)) = 0$$

as $\mathrm{Coh}(X)$ has homological dimension 1. We then appeal to the following general

Lemma 1.14. *Given a distinguished triangle*

$$a \longrightarrow b \longrightarrow c \longrightarrow a[1],$$

if the map $c \to a[1]$ is zero, then the sequence $a \to b \to c$ splits.

> There is a diagram of distinguished triangles:
>
> $$\begin{array}{ccccccc} a & \longrightarrow & b & \longrightarrow & c & \xrightarrow{0} & a[1] \\ \downarrow & & \vdots & & \downarrow & & \downarrow \\ a & \longrightarrow & a \oplus c & \longrightarrow & c & \longrightarrow & a[1] \end{array}$$
>
> so the dotted arrow exists and must be an isomorphism, by the five-lemma.

Therefore $\mathscr{E}^\bullet \cong \mathscr{H}^m(\mathscr{E}^\bullet)[-m] \oplus \mathscr{E}_1^\bullet$ and the proof is complete. □

2. The reconstruction theorem

2.1. **What is it?** Bondal-Orlov reconstruction theorem gives a way to recover a smooth projective variety from its derived category. The precise statement is as follows:

> Let X and Y be smooth projective varieties over an algebraically closed field k.
>
> **Theorem 2.1** (Bondal, Orlov)**.** *Assume that either ω_X or ω_X^{-1} is ample. If there exists an exact equivalence $\mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(Y)$, then X and Y are isomorphic.*

Throughout the rest of the notes, X and Y will be smooth projective varieties over k. Their Serre functors are denoted by S_X and S_Y. If $x \in X$ is a (closed) point, let $k(x)$ denote the skyscraper sheaf at x with values in k.

[3]It turns out that for any abelian category $\mathscr{A}$, a complex $\mathscr{E}^\bullet$ in $\mathrm{D^b}(\mathscr{A})$ is formal if and only if for any left-exact functor $F : \mathscr{A} \to \mathscr{B}$, the Grothendieck spectral sequence

$$E_2^{p,q} = R^pF(H^q(\mathscr{E}^\bullet)) \implies R^{p+q}F(\mathscr{E}^\bullet)$$

degenerates at E_2. Given a morphism $f : X \to S$ of ringed spaces, let $\mathscr{E}^\bullet = Rf_*\mathscr{F}$ for some $\mathscr{O}_X$-module $\mathscr{F}$ on X, and $F = \Gamma(S, -)$. The above spectral sequence specializes to the Leray spectral sequence:

$$E_2^{p,q} = H^p(Y, R^qf_*\mathscr{F}) \implies H^{p+q}(X, \mathscr{F}).$$

The combination of these two facts actually has applications in Hodge theory. (See [2]).

2.2. **Point-like objects in** $\mathrm{D^b}(X)$**.** Recall that $\mathrm{D^b}(X)$ is a k-linear category with finite-dimensional hom-sets (Corollary 1.7).

Definition 2.2. An object $\mathscr{P}^\bullet \in \mathrm{D^b}(X)$ is *point-like* if

(i) $S_X(\mathscr{P}^\bullet) \cong \mathscr{P}^\bullet[d]$ for some d,
(ii) $\mathrm{Hom}(\mathscr{P}^\bullet, \mathscr{P}^\bullet[i]) = 0$ for $i < 0$, and
(iii) $\mathrm{Hom}(\mathscr{P}^\bullet, \mathscr{P}^\bullet) = k$ (as k-algebras).

In particular, every nonzero element in $\mathrm{Hom}(\mathscr{P}^\bullet, \mathscr{P}^\bullet)$ is invertible.

If $\omega_X \cong \mathscr{O}_X$ (e.g. an elliptic curve), then there can be many extra point-like objects. Indeed, any sheaf $\mathscr{F}$ satisfying (iii) will be point-like, since $S_X(\mathscr{F}) \cong \mathscr{F}[n]$, and (ii) is trivially satisfied for sheaves since $\mathrm{Hom}(\mathscr{F}, \mathscr{F}[i]) \cong \mathrm{Ext}^i(\mathscr{F}, \mathscr{F}) = 0$ when $i < 0$. In particular, $\mathscr{O}_X$ itself will be point-like.

Proposition 2.3. *(i) Every sheaf of the form $k(x)[m]$, for some closed point $x \in X$ and $m \in \mathbb{Z}$, is point-like in $\mathrm{D^b}(X)$.*
(ii) If either ω_X or ω_X^{-1} is ample, then any point-like object in $\mathrm{D^b}(X)$ is isomorphic to some $k(x)[m]$.

Proof. The proof of (i) is omitted.

Let $n = \dim(X)$. Condition (iii) guarantees that any point-like object $\mathscr{P}^\bullet$ is nontrivial. Suppose $\mathscr{H}^i(\mathscr{P}^\bullet) \neq 0$ for some i. By condition (i), there holds

$$\mathscr{P}^\bullet \otimes \omega_X[n] \cong S_X(\mathscr{P}^\bullet) \cong \mathscr{P}^\bullet[d]$$

Since tensoring with ω_X does not change cohomology, we must have $n = d$ and $\mathscr{H}^i(\mathscr{P}^\bullet) \otimes \omega_X \cong \mathscr{H}^i(\mathscr{P}^\bullet)$. Write $\mathscr{F} = \mathscr{H}^i(\mathscr{P}^\bullet)$. Under the assumption that ω_X is ample or anti-ample, the Hilbert polynomial

$$P(k) = \chi(\mathscr{F} \otimes \omega_X^k)$$

is constant. This shows that $\mathscr{F}$ has zero-dimensional support. Since this holds for all i, $\mathscr{P}^\bullet$ must have zero-dimensional support.[4] We have then reduced the problem to the following special case:

Lemma 2.4. *Suppose a point-like object $\mathscr{P}^\bullet$ has zero-dimensional support. Then it is isomorphic to some $k(x)[m]$.*

Proof. If $\mathrm{supp}(\mathscr{P}^\bullet)$ has more than one point, then $\mathscr{P}^\bullet$ can be written as a direct sum.[a] This contradicts condition (iii), since the projections of $\mathscr{P}^\bullet$ to its summands aren't invertible.

Now, we can assume that the $x \in X$ is the support of $\mathscr{P}^\bullet$. Let m_1 (resp. m_2) be the maximal (resp. minimal) integer such that $\mathscr{H}^{m_1}(\mathscr{P}^\bullet) \neq 0$ (resp. $\mathscr{H}^{m_2}(\mathscr{P}^\bullet) \neq 0$). Let $\mathscr{H}^{m_1}(\mathscr{P}^\bullet) \to \mathscr{H}^{m_2}(\mathscr{P}^\bullet)$ be a nontrivial map, which exists by the following algebraic

Lemma 2.5. *Let M be a finite module over a Noetherian local ring $(A, \mathfrak{m})$ with support $\{\mathfrak{m}\}$. Then there exists an injection $A/\mathfrak{m} \to M$ and a surjection $M \to A/\mathfrak{m}$.*

Indeed, the condition on support means that $\mathfrak{m} = \mathrm{Ann}(M)$. Hence an injection $A/\mathfrak{m} \to M$ can be induced from any nontrivial map $A \to M$. A surjection $M \to A/\mathfrak{m}$ exists since $M = M/\mathfrak{m}M$ is a finite-dimensional $A/\mathfrak{m}$-vector space.

Back to the proof of Lemma 2.4. We obtain (by Lemma 1.1) a nontrivial map by the composition

$$\mathscr{P}^\bullet[m_1] \longrightarrow \mathscr{H}^{m_1}(\mathscr{P}^\bullet) \longrightarrow \mathscr{H}^{m_2}(\mathscr{P}^\bullet) \longrightarrow \mathscr{P}^\bullet[m_2]$$

But $m_1 \geq m_2$. So condition (ii) shows that $m_1 = m_2 = m$. In other words, $\mathscr{P}^\bullet = \mathscr{P}[m]$ for some sheaf $\mathscr{P}$. Now condition (iii) implies that $\mathscr{P} = k(x)$. ∎

[a] This was proved in Alex Perry's talk. You can also find the proof in [3, Lemma 3.9].

The proof of Proposition 2.3 is now complete. ∎

[4] The *support* of a complex of sheaves is the union of the support of its cohomology sheaves.

2.3. **Invertible objects in** $\mathrm{D^b}(X)$.

Definition 2.6. An object $\mathscr{L}^\bullet \in \mathrm{D^b}(X)$ is *invertible* if for any point-like object $\mathscr{P}^\bullet$, there exists an integer $n_{\mathscr{P}^\bullet}$ such that

$$\mathrm{Hom}(\mathscr{L}^\bullet, \mathscr{P}^\bullet[i]) = \begin{cases} k & \text{if } i = n_{\mathscr{P}^\bullet} \\ 0 & \text{if otherwise} \end{cases}$$

Proposition 2.7. *(i) Every invertible object in* $\mathrm{D^b}(X)$ *is of the form* $\mathscr{L}[m]$ *for some line bundle* $\mathscr{L}$*.*
(ii) If either ω_X *or* ω_X^{-1} *is ample, then every object of the form* $\mathscr{L}[m]$ *is invertible.*

Proof of (i). Consider an invertible object $\mathscr{L}^\bullet$ in $\mathrm{D^b}(X)$. Let m_1 be the largest integer for which $\mathscr{H}^{m_1}(\mathscr{L}^\bullet)$ is nonzero. Consider any skyscraper sheaf $k(x)$ with $x \in \mathrm{supp}(\mathscr{H}^{m_1}(\mathscr{L}^\bullet))$. Then Corollary 1.2 shows that

$$\mathrm{Hom}(\mathscr{L}^\bullet, k(x)[-m_1]) \cong \mathrm{Hom}(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x)) \neq 0 \tag{2.1}$$

Hence the constant $n_{k(x)} = -m_1$. It follows that $\mathrm{Hom}(\mathscr{L}^\bullet, k(x)[-m]) = 0$ whenever $m \neq -m_1$. In particular,

$$\mathrm{Ext}^{1-m_1}(\mathscr{L}^\bullet, k(x)) = 0.$$

Using the spectral sequence (1.2):

$$E_2^{p,q} = \mathrm{Ext}^p(\mathscr{H}^{-q}(\mathscr{L}^\bullet), k(x)) \implies \mathrm{Ext}^{p+q}(\mathscr{L}^\bullet, k(x))$$

together with $E_2^{1,-m_1} = E_\infty^{1,-m_1}$ (by looking at d^2), we see that

$$E_2^{1,-m_1} = \mathrm{Ext}^1(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x))$$

is a graded piece in $\mathrm{Ext}^{1-m_1}(\mathscr{L}^\bullet, k(x))$, which must then be zero. This shall give us the local freeness of $\mathscr{H}^{m_1}(\mathscr{L}^\bullet)$ via the algebraic

Lemma 2.8. *Let* M *be a finite module over a Noetherian local ring* $(A, \mathfrak{m})$*. If* $\mathrm{Ext}^1_A(M, A/\mathfrak{m}) = 0$*, then* M *is free.*

> Let $k = A/\mathfrak{m}$. Use Nakayama lemma to lift a k-basis of $M/\mathfrak{m}M$ to generators of M. We then obtain an exact sequence of A-modules:
>
> $$0 \longrightarrow N \xrightarrow{\varphi} A^{\oplus n} \longrightarrow M \longrightarrow 0$$
>
> where N is a finite A-module, and φ induces the zero map $\tilde{\varphi} : N/\mathfrak{m}N \to k^{\oplus n}$. The vanishing of $\mathrm{Ext}^1_A(M, A/\mathfrak{m})$, on the other hand, shows that φ^* in the following diagram
>
> $$\begin{array}{ccc} \mathrm{Hom}_A(A^{\oplus n}, k) & \xrightarrow{\varphi^*} & \mathrm{Hom}_A(N, k) \\ \| & & \| \\ \mathrm{Hom}_k(k^{\oplus n}, k) & \xrightarrow{\tilde{\varphi}^*} & \mathrm{Hom}_k(N/\mathfrak{m}N, k) \end{array}$$
>
> is surjective. This cannot happen unless $N/\mathfrak{m}N = 0$. Thus $N = 0$ by Nakayama, and M is free.

However, we cannot conclude the local freeness of $\mathscr{H}^{m_1}(\mathscr{L}^\bullet)$ yet—we have vanishing of the global Ext, but the lemma requires vanishing of the local Ext. To apply the lemma, we appeal to the local-to-global spectral sequence

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x))) \implies \mathrm{Ext}^{p+q}(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x))$$

Again, $E_2^{0,1} = E_\infty^{0,1}$, which is a graded piece of $\mathrm{Ext}^1(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x))$. Hence

$$H^0(X, \mathscr{E}xt^1(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x))) = 0.$$

Since $\mathscr{E}xt^1(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x))$ is concentrated at $x \in X$,[5] it must vanish. The algebraic lemma then implies that $\mathscr{H}^{m_1}(\mathscr{L}^\bullet)$ is locally free. In particular, its support is all of X.

Claim. $\mathscr{H}^{m_1}(\mathscr{L}^\bullet)$ *is a line bundle*

[5] For finite modules over Noetherian schemes, $\mathscr{E}xt$ commutes with localization.

For any $y \in X$, (2.1) applies, and we get

$$k \cong \mathrm{Hom}(\mathscr{L}^\bullet, k(y)[-m_1]) \cong \mathrm{Hom}(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(y))$$

so $\mathscr{H}^{m_1}(\mathscr{L}^\bullet)$ has rank 1.

Claim. *$\mathscr{H}^m(\mathscr{L}^\bullet) = 0$ for $m < m_1$.*

By (descending) induction, we may assume $\mathscr{H}^{m+1}(\mathscr{L}^\bullet) = \mathscr{H}^{m+2}(\mathscr{L}^\bullet) = \cdots = \mathscr{H}^{m_1-1}(\mathscr{L}^\bullet) = 0$. It suffices to show the vanishing of $\mathrm{Hom}(\mathscr{H}^m(\mathscr{L}^\bullet), k(x))$ for all $x \in X$. Note that this is the $E_2^{0,-m}$-term in the spectral sequence (1.2):

$$E_2^{p,q} = \mathrm{Ext}^p(\mathscr{H}^{-q}(\mathscr{L}^\bullet), k(x)) \implies \mathrm{Ext}^{p+q}(\mathscr{L}^\bullet, k(x)).$$

Note that d^2 on this term vanishes because its codomain is

$$E_2^{2,-m-1} = \mathrm{Ext}^2(\mathscr{H}^{m+1}(\mathscr{L}^\bullet), k(x))$$

which is zero:

(i) if $m = m_1 - 1$, it is zero because $\mathrm{Ext}^2(\mathscr{H}^{m_1}(\mathscr{L}^\bullet), k(x)) \cong H^2(X, \mathscr{H}^{m_1}(\mathscr{L}^\bullet)^\vee \otimes k(x)) = 0$ using local freeness of $\mathscr{H}^{m_1}(\mathscr{L}^\bullet)$,

(ii) if $m < m_1 - 1$, it is zero by induction hypothesis.

Altogether, $E_2^{0,-m} = E_\infty^{0,-m}$, a graded piece of $\mathrm{Ext}^{-m}(\mathscr{L}^\bullet, k(x)) \cong \mathrm{Hom}(\mathscr{L}^\bullet, k(x)[-m])$, which vanishes by hypothesis.

It follows that $\mathscr{L} \cong \mathscr{H}^{m_1}(\mathscr{L}^\bullet)[-m_1]$. □

Proof of (ii). Consider $\mathscr{L}[m]$ for some line bundle $\mathscr{L}$. By Proposition 2.3, any point-like object in $\mathrm{D^b}(X)$ is isomorphic to some $k(x)[m']$. Checking that $\mathscr{L}[m]$ is an invertible object boils down to a computation:

$$\begin{aligned}\mathrm{Hom}(\mathscr{L}[m], k(x)[i+m']) &\cong \mathrm{Hom}(\mathscr{L}, k(x)[i+m'-m]) \\ &\cong H^{i+m'-m}(X, \mathscr{L}^\vee \otimes k(x)) \\ &= 0\end{aligned}$$

unless $i = m - m'$, and it is isomorphic k when $i = m - m'$. □

2.4. **The reconstruction.** Let $F : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ be an exact equivalence. Assume that either ω_X or ω_X^{-1} is ample. Propositions 2.3 and 2.7 establish the following[6]:

$$\begin{array}{ccc}\{\text{point-like objects in } \mathrm{D^b}(X)\} & \xleftrightarrow{F} & \{\text{point-like objects in } \mathrm{D^b}(Y)\} \\ \| & & \uparrow \\ \{k(x)[m]\} & & \{k(y)[m]\}\end{array}$$

and

$$\begin{array}{ccc}\{\text{invertible objects in } \mathrm{D^b}(X)\} & \xleftrightarrow{F} & \{\text{invertible objects in } \mathrm{D^b}(Y)\} \\ \| & & \downarrow \\ \{\mathscr{L}[m]\} & & \{\mathscr{M}[m]\}\end{array}$$

Proof of Theorem 2.1. We will proceed by reconstructing Y step by step from its derived category.

Reduction. *We may assume $F(\mathscr{O}_X) \cong \mathscr{O}_Y$.*

Indeed, $F(\mathscr{O}_X) \cong \mathscr{M}[m]$ for some line bundle $\mathscr{M}$ on Y. Compose F by the exact equivalences $[-m]$ and $\mathscr{M}^\vee \otimes -$, we obtain a new exact equivalence sending $\mathscr{O}_X$ to $\mathscr{O}_Y$.

Claim. *Point-like objects in $\mathrm{D^b}(Y)$ are of the form $k(y)[m]$.*

[6]These pretty diagrams (just like everything else in these notes!) are taken from [3].

Let $\mathscr{P}^\bullet$ be a point-like object in $\mathrm{D}^{\mathrm{b}}(Y)$ which is not of the form $k(y)[m]$ for any $y \in Y$ and $m \in \mathbb{Z}$. We first show that it is orthogonal to every $k(y)[m]$. Indeed, say $\mathscr{P}^\bullet \cong F(k(x_0)[m_0])$ and $k(y) \cong F(k(x_y)[m_y])$. Then $x_0 \neq x_y$. So

$$\begin{aligned}\mathrm{Hom}(\mathscr{P}^\bullet, k(y)[m]) &\cong \mathrm{Hom}(k(x_0)[m_0], k(x_y)[m_y+m]) \\ &\cong 0\end{aligned}$$

since the two complexes have distinct supports.

Now, we will show that $\mathscr{P}^\bullet$ by exhibiting some $k(y)[m]$ for which

$$\mathrm{Hom}(\mathscr{P}^\bullet, k(y)[m]) = \mathrm{Ext}^m(\mathscr{P}^\bullet, k(y))$$

is non-vanishing. Indeed, let m_1 be the largest number such that $\mathscr{H}^{m_1}(\mathscr{P}^\bullet)$ is nonzero, and let y be in the support of $\mathscr{H}^{m_1}(\mathscr{P}^\bullet)$. Consider the spectral sequence (1.2):

$$E_2^{p,q} = \mathrm{Ext}^p(\mathscr{H}^{-q}(\mathscr{P}^\bullet), k(y)) \implies \mathrm{Ext}^{p+q}(\mathscr{P}^\bullet, k(y)).$$

Since $E_2^{0,-m_1} = H^0(\mathscr{H}^{m_1}(\mathscr{P}^\bullet), k(y))$ receives differentials from negative Ext-groups and sends elements to $\mathrm{Ext}^2(\mathscr{H}^{m_1+1}(\mathscr{P}^\bullet), k(y)) = 0$, we have $E_2^{0,-m_1} = E_\infty^{0,-m_1}$. Hence $\mathrm{Ext}^{-m_1}(\mathscr{P}^\bullet, k(y))$ is nonzero.[a]

[a] This part of the argument actually shows that the skyscraper sheaves form a "spanning class" of $\mathrm{D}^{\mathrm{b}}(Y)$. See [3, §3.2].

Claim. *(i) For each $x \in X$, there is some $y \in Y$ with $F(k(x)) \cong k(y)$.*
(ii) $\dim(X) = \dim(Y)$.
(iii) For each integer k, there holds $F(\omega_X^k) \cong \omega_Y^k$.

Say $F(k(x)) \cong k(y)[m]$. We want to show $m = 0$. Indeed,

$$H^m(Y, k(y)) \cong \mathrm{Hom}(\mathscr{O}_Y, k(y)[m]) \cong \mathrm{Hom}(\mathscr{O}_X, k(x)) \cong k$$

so m has to be zero.

Let x, y be as in part (i). The second part follows from the commutation of F with Serre functors (Lemma 1.9):

$$k(y)[\dim X] \cong F(k(x)[\dim X]) \cong F(S_X(k(x))) \cong S_Y(F(k(x))) \cong k(y)[\dim Y]$$

But this could only happen when $\dim X = \dim Y$.

Now, let $n = \dim X = \dim Y$. For every integer k,

$$F(\omega_X^k) \cong F(S_X^k(\mathscr{O}_X)[-kn]) \cong S_Y^k(F(\mathscr{O}_X))[-kn] \cong S_Y^k(\mathscr{O}_Y)[-kn] \cong \omega_Y^k.$$

This established part (iii).

Claim. *If ω_X is ample (or anti-ample), so is ω_Y.*

Suppose ω_X^k is very ample. We need to show that the same holds for ω_Y^k. Recall that a line bundle $\mathscr{L}$ on X is very ample if

(i) it *separates points*, i.e.

$$H^0(X, \mathscr{L}) \to H^0(X, k(x_1) \oplus k(x_2))$$

is surjective for all $x_1 \neq x_2$ in X.

(ii) it *separates tangent vectors*, i.e.

$$H^0(X, \mathscr{L}) \to H^0(X, \mathscr{O}_{Z_x})$$

is surjective when $\mathscr{O}_{Z_x}$ is a nontrivial extension of $k(x)$ by $k(x)$, for all $x \in X$.

Both properties are captured by the derived category. For (i), use the commutative diagram

$$\begin{array}{ccc} H^0(X, \mathscr{L}) & \longrightarrow & H^0(X, k(x_1) \oplus k(x_2)) \\ \| & & \| \\ \mathrm{Hom}(\mathscr{O}_X, \mathscr{L}) & \longrightarrow & \mathrm{Hom}(\mathscr{O}_X, k(x_1) \oplus k(x_2)) \end{array}$$

and the fact that the lower row is isomorphic to $\mathrm{Hom}(\mathscr{O}_Y, F(\mathscr{L})) \to \mathrm{Hom}(\mathscr{O}_Y, k(y_1) \oplus k(y_2))$. For (ii), use the fact that $\mathscr{O}_{Z_x}$ corresponds to a nonzero extension class $e \in \mathrm{Hom}(k(x), k(x)[1])$.

We have established an isomorphism of graded vector spaces:

$$\bigoplus_{k\in\mathbb{Z}} H^0(X, \omega_X^k) \xrightarrow{\sim} \bigoplus_{k\in\mathbb{Z}} H^0(Y, \omega_Y^k)$$

This isomorphism preserves the ring structure, since for sections

$$s_i \in H^0(X, \omega_X^{k_i}) \cong \mathrm{Hom}(\mathscr{O}_X, \omega_X^{k_i}), \quad (i = 1, 2)$$

we have $s_1 \cdot s_2 = S_X^{k_1}(s_2)[-k_1 n] \circ s_1$. To see why this equality holds, simply note that $S_X^{k_1}(s_2)[-k_1 n] = \mathrm{id}_{\omega_X^{k_1}} \otimes s_2$, and the following diagram commutes:

$$\begin{array}{ccc} \mathscr{O}_X & \xrightarrow{s_1} & \omega_X^{k_1} \\ & \searrow^{s_1 \cdot s_2} & \downarrow{\scriptstyle \mathrm{id}_{\omega_X^{k_1}} \otimes s_2} \\ & & \omega_X^{k_1+k_2} \end{array}$$

Now, by ampleness (or anti-ampleness) of both ω_X and ω_Y,

$$X \cong \mathrm{Proj}\left(\bigoplus H^0(X, \omega_X^k)\right), \quad Y \cong \mathrm{Proj}\left(\bigoplus H^0(Y, \omega_Y^k)\right)$$

and we have $X \cong Y$. □

References

[1] Bondal, Alexei, and Dmitri Orlov. "Reconstruction of a variety from the derived category and groups of autoequivalences." *Compositio Mathematica* 125.03 (2001): 327-344.

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[3] Huybrechts, Daniel. *Fourier-Mukai Transforms in Algebraic Geometry.* USA: Oxford University Press, 2006.