# DELIGNE'S THEOREMS ON DEGENERATION OF SPECTRAL SEQUENCES

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Abstract. We present the proofs of Deligne's theorems on degeneration of the Leray spectral sequence, and the algebraic Hodge-de Rham spectral sequence.

## Contents

## Introduction

In this short article, we will present the proofs for the following two results due to Deligne [1]:

(i) (Thm. 1.13). Let $f : X \to S$ be a proper submersion of complex manifolds, such that $X$ satisfies some "appropriate Kähler condition." Then the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \underline{\mathbb{C}}) \implies H^{p+q}(X, \mathbb{C})$$

degenerates at $E_2$.

(ii) (Thm. 2.8). Let $X$ be a smooth, proper scheme over $\mathbb{C}$. Then the algebraic Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \implies \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}})$$

degenerates at $E_1$.

We comment on several points where this exposition differs from Deligne's original presentation [1]. In the proof of Prop. 1.4, although the result is an equivalence in the derived category $D(\mathcal{A})$, we choose to work with the homotopy category $K(\mathcal{A})$ to explicitly construct the quasi-isomorphisms before passing to $D(\mathcal{A})$. This preference is also reflected in the preceding sections. We also develop the "Lefscehtz decomposition" for degree-one and degree-two morphisms simultaneously (Lem. 1.6, Cor. 1.7), stressing the fact that formality is purely a consequence of "symmetry" (Thm. 1.9, 1.10). Here the fact that the "Lefschetz operator" is degree-two has no objective role. In the presentation of result (ii), we also write more extensively on the numerous identifications, as an attempt to make everything as explicit as possible. However, it is imperative to point out that all these differences are minor, and the author does not claim any originality beyond the level of exposition.

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## 1. DEGENERATION OF THE LERAY SPECTRAL SEQUENCE

1.1. Recall the following general result in the computation of hypercohomology groups.

**Lemma 1.1.** *Let $\mathcal{A}$ be an abelian category with enough injectives, and $T : \mathcal{A} \to \mathcal{B}$ be a left-exact functor of abelian categories. Let $X^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ be a bounded-below complex with objects in $\mathcal{A}$. Then there is a spectral sequence $E_r$ with*

$$E_2^{p,q} = R^pT(H^q(X^\bullet)) \implies R^{p+q}T(X^\bullet) \tag{1.1}$$

*which is functorial in $X^\bullet$ starting from $E_2$, and commutes with finite direct sums.*

*Proof.* The spectral sequence (1.1) is defined by the following steps:

(i) Take a Cartan-Eilenberg resolution $I^{\bullet,\bullet}$ of $X^\bullet$. Apply the functor $T$ and get a double complex $T(I^{\bullet,\bullet})$.

(ii) Consider the total complex $sI^\bullet$ of $I^{\bullet,\bullet}$, and a decreasing filtration on $T(sI^\bullet)$ by the second index

$$F^qT(sI^n) = \bigoplus_{\substack{p+r=n \\ r\geq q}} T(I^{p,r})$$

(iii) The spectral sequence (1.1) is the spectral sequence associated to the filtered complex $F^\bullet T(sI^\bullet)$.

The desired properties of $E^{p,q}$ all follow from properties of the Cartan-Eilenberg resolution. For details, see [2], III.7.7-15. □

The hypercohomology groups of a bounded-below complex generalize the ordinary cohomology groups on an object. In particular,

**Lemma 1.2.** *If $X^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ be concentrated at ith place, i.e. $X^\bullet = X[-i]$ for some $X \in \mathcal{A}$, then the spectral sequence (1.1) degenerates at $E_2$, with*

$$E_2^{p,q} = E_\infty^{p,q} = \begin{cases} R^pT(X) & : q = i \\ 0 & : q \neq i \end{cases} \tag{1.2}$$

*Proof.* It follows from (1.1) that

$$E_2^{p,q} = R^pT(H^q(X^\bullet)) = \begin{cases} R^pT(X) & : q = i \\ 0 & : q \neq i \end{cases}$$

and the degeneration at $E_2$ follows from the fact that $E_2^{p,q}$ is concentrated on a horizontal line in the first quadrant. □

1.2. For applications later, we will compute the edge homomorphism $E_\infty^{0,i} \to E_2^{0,i}$ for the spectral sequence (1.1). Note that the construction in Lemma 1.1 implies

$$E_2^{0,i} = T(H^i(X^\bullet)) = T(H^i(sI^\bullet)), \quad E_\infty^{0,i} = \mathrm{Gr}^0 R^iT(X^\bullet) = \mathrm{Gr}^0 H^iT(sI^\bullet)$$

There is an induced homomorphism

$$H^iT(sI^\bullet) \to T(H^i(sI^\bullet)) \tag{1.3}$$

which factors through the edge homomorphism $E_\infty^{0,i} \to E_2^{0,i}$. Indeed, the homomorphism (1.3) is given by the following construction:

(i) Let $Z^i(sI^\bullet)$ denote the kernel of $d$ on $sI^i$. Then there is an exact sequence

$$0 \to Z^i(sI^\bullet) \to sI^i \xrightarrow{d} sI^{i+1}$$

Since $T$ is left-exact, we obtain an exact sequence

$$0 \to T(Z^i(sI^\bullet)) \to T(sI^i) \xrightarrow{T(d)} T(sI^{i+1}) \tag{1.4}$$

(ii) The exact sequence (1.4) gives rise to an isomorphism

$$Z^iT(sI^\bullet) \xrightarrow{\sim} T(Z^i(sI^\bullet)) \tag{1.5}$$

where $Z^iT(sI^\bullet)$ denotes the kernel of $T(d)$ on $sI^\bullet$.

(iii) Post-compose (1.5) with $T(Z^i(sI^\bullet)) \to T(H^i(sI^\bullet))$, given by the functoriality of $T$, we obtain a map $Z^iT(sI^\bullet) \to T(H^i(sI^\bullet))$ which induces (1.3).

**Example 1.3.** Let $T = \mathrm{Hom}_{\mathcal{A}}(Y, -)$ for a fixed object $Y \in \mathcal{A}$. We will follow the above construction to describe the homomorphism (1.3) for $\mathrm{Hom}_{\mathcal{A}}(Y, -)$:

$$H^i\mathrm{Hom}_{\mathcal{A}}(Y, sI^\bullet) \to \mathrm{Hom}_{\mathcal{A}}(Y, H^i(sI^\bullet)) \tag{1.6}$$

Indeed, given any element $t \in H^i\mathrm{Hom}_{\mathcal{A}}(Y, sI^\bullet)$, corresponding to the above three steps, we have

(i) $t$ is represented by some homomorphism $t' : Y \to sI^i$ such that $d \circ t' = 0$, i.e. $t' \in T(sI^i)$ and is mapped to zero under $T(d)$.
(ii) $t'$ induces a homomorphism $t'' : Y \to Z^i(sI^\bullet)$, i.e. $t'' \in T(Z^i(sI^\bullet))$.
(iii) The induced map $t''' : Y \to H^i(sI^\bullet)$ is well-defined, and independent of the choice of $t'$.

It follows from this construction that the homomorphism (1.6) admits the following factorization

$$\begin{tikzcd}
H^i\mathrm{Hom}_{\mathcal{A}}(Y, sI^\bullet) \arrow[r, equal] \arrow[dr, "(1.6)"'] & \mathrm{Hom}_{K(\mathcal{A})}(Y, sI[i]^\bullet) \arrow[d, "H^0"] \\
 & \mathrm{Hom}_{\mathcal{A}}(Y, H^i(sI^\bullet))
\end{tikzcd}$$

where $K(\mathcal{A})$ denotes the homotopy category of complexes of $\mathcal{A}$.

1.3. **Criterion of formality.** The following result is due to Deligne ([1], Prop. 1.2).

**Proposition 1.4.** *Let $\mathcal{A}$ be an abelian category with enough injectives, and $X^\bullet \in \mathrm{Kom}^b(\mathcal{A})$ be a bounded complex. Then the followings are equivalent:*

*(i) For any left-exact functor $T : \mathcal{A} \to \mathcal{B}$ to another abelian category $\mathcal{B}$, the spectral sequence (1.1) associated to $T$ and $X^\bullet$ degenerates at $E_2$.*
*(ii) For any $i$, and the left-exact functor $T^i = \mathrm{Hom}_{\mathcal{A}}(H^i(X^\bullet), -) : \mathcal{A} \to \mathbf{Ab}$, the spectral sequence (1.1) associated to $T$ and $X^\bullet$ degenerates at $E_2$.*
*(iii) There exists an isomorphism in $D^b(\mathcal{A})$:*

$$X^\bullet \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} H^i(X^\bullet)[-i] \tag{1.7}$$

A complex $X^\bullet$ (not necessarily bounded) is *formal* if it satisfies the condition in (iii).

*Proof.* The implication (i) $\Longrightarrow$ (ii) is trivial. To prove (iii) $\Longrightarrow$ (i), note that the spectral sequence (1.1) applied to each $H^i(X^\bullet)[-i]$ and any left exact functor $T : \mathcal{A} \to \mathcal{B}$ degenerates at $E_2$ by Lemma 1.2. Furthermore, since the spectral sequence (1.1) is functorial in $X^\bullet$ and commutes with finite direct sums, the hypothesis of (iii) implies (i).

We now prove (ii) $\Longrightarrow$ (iii). The spectral sequence (1.1) associated to $T^i$ and $X^\bullet$ reads

$$E_2^{p,q} = \mathrm{Ext}^p(H^i(X^\bullet), H^q(X^\bullet)) \implies \mathrm{Ext}^{p+q}(H^i(X^\bullet), X^\bullet) \tag{1.8}$$

Let $I^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $X^\bullet$ and $sI^\bullet$ be its total complex. The edge homomorphism $E_\infty^{0i} \to E_2^{0i}$ induces a homomorphism (as in (1.3)):

$$H^i\mathrm{Hom}_{\mathcal{A}}(H^i(X^\bullet), sI^\bullet) \to \mathrm{Hom}_{\mathcal{A}}(H^i(X^\bullet), H^i(sI^\bullet)) \tag{1.9}$$

which factors through $H^0$ by Example 1.3:

$$\begin{tikzcd}
H^i\mathrm{Hom}_{\mathcal{A}}(H^i(X^\bullet), sI^\bullet) \arrow[r, equal] \arrow[dr, "(1.9)"'] & \mathrm{Hom}_{K(\mathcal{A})}(H^i(X^\bullet), sI[i]^\bullet) \arrow[d, "H^0"] \\
 & \mathrm{Hom}_{\mathcal{A}}(H^i(X^\bullet), H^i(sI^\bullet))
\end{tikzcd}$$

The hypothesis (ii) implies that the homomorphism (1.9) is surjective. Hence we obtain a surjective homomorphism

$$\mathrm{Hom}_{K(\mathcal{A})}(H^i(X^\bullet), sI[i]^\bullet) \xrightarrow{H^0} \mathrm{Hom}_{\mathcal{A}}(H^i(X^\bullet), H^i(X^\bullet)) \tag{1.10}$$

or equivalently, a surjective homomorphism

$$\mathrm{Hom}_{K(\mathcal{A})}(H^i(X^\bullet)[-i], sI^\bullet) \xrightarrow{H^i} \mathrm{Hom}_{\mathcal{A}}(H^i(X^\bullet), H^i(X^\bullet)) \tag{1.11}$$

By taking a pre-image of the identity map on $H^i(X^\bullet)$ for each $i$, we obtain a quasi-isomorphism

$$\bigoplus_{i\in\mathbb{Z}} H^i(X^\bullet)[-i] \to sI^\bullet$$

Together with the quasi-ismorphism $X^\bullet \to sI^\bullet$, we find the isomorphism in $D^b(\mathcal{A})$ of (iii). □

**Remark 1.5.** The proof above is essentially the same as the one given in [1], except that we work with the homotopy category $K(\mathcal{A})$ to make the constructions more explicit.

1.4. We now prove abstract versions of the Lefschetz decomposition.

**Lemma 1.6** (Degree-one version). *Let $\mathcal{A}$ be an abelian category, and let*

$$V^0 \xrightarrow{L} V^1 \xrightarrow{L} \cdots \xrightarrow{L} V^k \xrightarrow{L} \cdots \xrightarrow{L} V^n \tag{1.12}$$

*be a chain of morphisms, all denoted by L for notational convenience. If for all $0 \le k \le n/2$, the composition $L^{n-2k} : V^k \to V^{n-k}$ is an isomorphism, then*

$$V^k = \bigoplus_{0\le r\le k} L^r V^{k-r}_{\text{prim}} \;\; \textit{for} \;\; 0 \le k \le \frac{n}{2} \tag{1.13}$$

*where $V^k_{\text{prim}} = \operatorname{Ker}(L^{n-2k+1}$ on $V^k)$.*

This lemma expresses the fact that a Lefschetz decomposition exists for every chain of morphisms that is "symmetric about the middle."

*Proof.* We prove this lemma for

$$\mathcal{A} = \text{category of } R\text{-modules, for some ring } R \text{ with identity}$$

The general version follows from the embedding theorem of Freyd-Mitchell. In this case, it suffices to show that each $\alpha \in V^k$, with $0 \le k \le \frac{n}{2}$, can be expressed in the form

$$\alpha = \sum_{0\le r\le k} L^r \alpha_r \;\; \text{where} \;\; \alpha_r \in V^{k-r}_{\text{prim}} \tag{1.14}$$

in a unique way. We proceed by induction on $k$. The case for $k = 0$ is trivial, since $V^0 = V^0_{\text{prim}}$ by degree considerations. Given $k \ge 1$, we assume the existence and uniqueness of the expression (1.14), for any $\alpha$ of degree less than $k$.

We first prove the existence of such an expression for $\alpha \in V^k$. Note that

$$L^{n-2k+1}(\alpha) \in V^{n-k+1} = L^{n-2k+2}(V^{k-1})$$

Hence there exists some $\beta \in V^{k-1}$ with $L^{n-2k+1}(\alpha) = L^{n-2k+2}(\beta)$. Therefore $L^{n-2k+1}(\alpha - L\beta) = 0$, and

$$\alpha = \alpha_0 + L\beta \;\; \text{where} \;\; \alpha_0 \in V^k_{\text{prim}}$$

Hence, the fact that $\beta$ can be expressed in the form of (1.14) implies that $\alpha$ can be expressed in this form as well.

We now prove the uniqueness of the expression (1.14) for $\alpha \in V^k$. It suffices to show that

$$\sum_{0\le r\le k} L^r \alpha_r = 0 \;\; \text{where} \;\; \alpha_r \in V^{k-r}_{\text{prim}} \implies \alpha_r = 0 \;\; \text{for all } 0 \le r \le k$$

Suppose $\alpha_0 = 0$. Then

$$0 = \sum_{1\le r\le k} L^r \alpha_r = L \sum_{1\le r\le k} L^{r-1} \alpha_r$$

Since $L^{r-1}\alpha_r \in V^{k-1}$ and $L^{n-2k+2}$ is injective on $V^{k-1}$, in particular, $L$ is injective on $V^{k-1}$. Hence

$$\sum_{1\le r\le k} L^{r-1} \alpha_r = 0$$

and induction hypothesis shows that $\alpha_r = 0$ for all $0 \le r \le k$. Now, suppose $\alpha_0 \ne 0$. Thus $\alpha_0 \in V^k_{\text{prim}}$, and $L^{n-2k+1}\alpha_0 = 0$. Therefore

$$\sum_{0\le r\le k} L^r \alpha_r = 0 \implies L^{n-2k+1} \sum_{1\le r\le k} L^r \alpha_r = 0 = L^{n-2k+2} \sum_{1\le r\le k} L^{r-1} \alpha_r$$

Since each $L^{r-1}\alpha_r \in V^{k-1}$, and $L^{n-2k+2}$ is in particular injective on $V^{k-1}$,

$$\sum_{1\leq r\leq k} L^{r-1}\alpha_r = 0$$

The induction hypothesis again shows that $\alpha_r = 0$ for all $1 \leq r \leq k$, and consequently $\alpha_0 = 0$ as well. The proof is complete. ☐

**Corollary 1.7** (Degree-two version)**.** *Let $\mathcal{A}$ be an abelian category, and let*

$$V = \bigoplus_{k\in\mathbb{Z}} V^k$$

*where $V^k \in \mathrm{Ob}(\mathcal{A})$ for all $k \in \mathbb{Z}$, such that there exists some $n \in \mathbb{N}$, with $V^k = 0$ unless $0 \leq k \leq 2n$. Suppose there exists a morphism $L : V \to V$ of degree 2, such that*

$$L^{n-k} : V^k \to V^{2n-k} \quad \textit{is an isomorphism for all } 0 \leq k \leq n \tag{1.15}$$

*Then, if we let $V^k_{\mathrm{prim}} = \mathrm{Ker}(L^{n-k+1}$ on $V^k)$, there holds*

$$V^k = \bigoplus_{0\leq r\leq \frac{k}{2}} L^r V^{k-2r}_{\mathrm{prim}} \quad \textit{for} \;\; 0 \leq k \leq n \tag{1.16}$$

*Proof.* We get two chains of morphisms

$$V^0 \xrightarrow{L} V^2 \xrightarrow{L} V^4 \xrightarrow{L} \cdots \xrightarrow{L} V^{2n} \tag{1.17}$$

$$V^1 \xrightarrow{L} V^3 \xrightarrow{L} \cdots \xrightarrow{L} V^{2n-1} \tag{1.18}$$

satisfying the conditions of Lemma 1.6 with appropriate indices. The result follows from applying Lemma 1.6 and an adjustment of indices. ☐

**Remark 1.8.** One does not have a verbatim generalization of the above results to $L : V \to V$ of degree $s \geq 3$, for the simple reason that the decompositions similar to (1.17) and (1.18) will not have the required symmetry in order to apply Lemma 1.6. In the case $s \geq 3$, one has to impose the condition that $V^k = 0$ if $k$ is not a multiple of $s$, to obtain a similar result as Corollary 1.7.

### 1.5. **Criterion of degeneration.** The following results appear as Theorem I.5 and Remark I.9 in [1].

**Theorem 1.9.** *Let $\mathcal{A}$ be an abelian category, and $X^\bullet \in D^b(\mathcal{A})$ be a complex bounded in degrees from 0 to $n$. Let $L : X^\bullet \to X[1]^\bullet$ be a morphism in $D^b(\mathcal{A})$ such that the composition of induced morphisms $L^{n-2k} : H^k(X^\bullet) \to H^{n-k}(X^\bullet)$ is an isomorphism. Then $X^\bullet$ is formal.*

*Proof.* In light of Proposition 1.4, we only need to check that for any left-exact functor $T : \mathcal{A} \to \mathcal{B}$, the spectral sequence (1.1) degenerates at $E_2$, i.e. the differentials $d_r$ on $E_r$ vanishes for $r \geq 2$. Note that the fact that (1.1) is defined using the second filtration, and is functorial starting from $E_2$, we obtain a morphism

$$L : E_r^{p,q} \to E_r^{p,q+1} \;\; \text{for } r \geq 2$$

which commutes with $d_r$. To show that $d_r = 0$ for $r \geq 2$, we assume that $d_2 = \cdots = d_{r-1} = 0$; the assumption is vacuous for $r = 2$, so we will obtain both the base case and the induction step at once. Based on this assumption, we see that

$$E_r^{p,q} = E_2^{p,q} = R^pT(H^q(X^\bullet))$$

By Lemma 1.6, $L^{n-2k} : H^k(X^\bullet) \to H^{n-k}(X^\bullet)$ being an isomorphism implies that

$$H^k(X^\bullet) = \bigoplus_{0\leq r\leq k} L^r H^{k-r}_{\mathrm{prim}}(X^\bullet) \;\; \text{for} \;\; 0 \leq k \leq \frac{n}{2}$$

where $H^k_{\text{prim}}(X^\bullet) = \text{Ker}(L^{n-2k+1} \text{ on } H^k(X^\bullet))$. Hence it suffices to show that $d_r = 0$ on $R^pT(H^q_{\text{prim}}(X^\bullet))$. Consider the following commutative diagram:

$$\begin{tikzcd}
R^pT(H^q_{\text{prim}}(X^\bullet)) \arrow[r, "d_r"] \arrow[d, "L^{n-2q+1}"'] & R^{p+r}T(H^{q-r+1}(X^\bullet)) \arrow[d, "L^{n-2q+1}"] \arrow[dd, bend left, "\cong"] \\
R^pT(H^{n-q+1}(X^\bullet)) \arrow[r, "d_r"] & R^{p+r}T(H^{n-q-r+2}(X^\bullet)) \arrow[d, "L^{2r-3}"] \\
 & R^{p+r}T(H^{n-q+r-1}(X^\bullet))
\end{tikzcd} \tag{1.19}$$

Note that the left-column arrow vanishes, by definition of $H^q_{\text{prim}}(X^\bullet)$, while the $L^{n-2q+1}$ on the right column is injective. Therefore the upper-horizontal morphism $d_r$ vanishes. This completes the proof. ☐

We have the corresponding result for a degree-two morphism.

**Theorem 1.10.** *Let $\mathcal{A}$ be an abelian category, and $X^\bullet \in D^b(\mathcal{A})$ be a complex bounded in degrees from $0$ to $2n$. Let $L : X^\bullet \to X[2]^\bullet$ be a morphism in $D^b(\mathcal{A})$ such that the composition of induced morphisms $L^{n-k} : H^k(X^\bullet) \to H^{2n-k}(X^\bullet)$ is an isomorphism. Then $X^\bullet$ is formal.*

*Proof.* The proof is identical to that of Theorem 1.9, except that we have a morphism

$$L : E_r^{p,q} \to E_r^{p,q+2} \ \text{ for } r \geq 2$$

which commutes with $d_r$, and we use Corollary 1.7 to obtain the decomposition

$$H^k(X^\bullet) = \bigoplus_{0 \leq r \leq \frac{k}{2}} L^r H^{k-2r}_{\text{prim}}(X^\bullet) \ \text{ for } \ 0 \leq k \leq n$$

where $H^k_{\text{prim}} = \text{Ker}(L^{n-k+1} \text{ on } H^k(X^\bullet))$. Finally, in place of diagram (1.19), we consider

$$\begin{tikzcd}
R^pT(H^q_{\text{prim}}(X^\bullet)) \arrow[r, "d_r"] \arrow[d, "L^{n-q+1}"'] & R^{p+r}T(H^{q-r+1}(X^\bullet)) \arrow[d, "L^{n-q+1}"] \arrow[dd, bend left, "\cong"] \\
R^pT(H^{2n-q+2}(X^\bullet)) \arrow[r, "d_r"] & R^{p+r}T(H^{2n-q-r+3}(X^\bullet)) \arrow[d, "L^{r-2}"] \\
 & R^{p+r}T(H^{2n-q+r-1}(X^\bullet))
\end{tikzcd} \tag{1.20}$$

and the argument runs through as before. ☐

**Remark 1.11.** As in Remark 1.8, the above results generalize to the following situation: let $X^\bullet \in D^b(\mathcal{A})$ be bounded in degrees from 0 to $ns$, such that $H^k(X^\bullet) = 0$ if $k$ is not a multiple of $s$. Then if there is a morphism $L : X^\bullet \to X[s]^\bullet$ which induces an isomorphism $L^{n-2k} : H^{ks}(X^\bullet) \to H^{(n-k)s}(X^\bullet)$, then $X^\bullet$ is formal.

1.6. We note an application of Theorem 1.9 and 1.10 to sheaves of rings on a topological space $X$, and the Leray spectral sequence associated to a continuous map $f : X \to Y$. We work with degree-one and degree-two cases simultaneously.

Let $\mathcal{A}$ be a sheaf of rings on $X$, and fix $\omega \in H^1(X, \mathcal{A})$ (resp. $\omega \in H^2(X, \mathcal{A})$). Let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-modules. Then $\omega$ defines a morphism $L : \mathcal{F} \to \mathcal{F}[1]$ (resp. $L : \mathcal{F} \to \mathcal{F}[2]$) in $D^b(\mathbf{Mod}_{\mathcal{A}})$, which gives rise to

$$L : Rf_*\mathcal{F} \to Rf_*\mathcal{F}[1] \ \ (\text{resp. } L : Rf_*\mathcal{F} \to Rf_*\mathcal{F}[2])$$

by functoriality. $\mathcal{F}$ is said to *satisfy the Lefschetz condition relative to* $\omega$ if

(i) $Rf_*\mathcal{F}$ is in $D^b(Y)$, and bounded in degrees from 0 to some $n$ (resp. $2n$).

(ii) The composition of induced morphisms

$$L^{n-2k} : R^k f_* \mathcal{F} \to R^{n-k} f_* \mathcal{F} \ \ (\text{resp. } L^{n-k} : R^k f_* \mathcal{F} \to R^{2n-k} f_* \mathcal{F})$$

is an isomorphism for all $0 \leq k \leq \frac{n}{2}$ (resp. $0 \leq k \leq n$).

**Proposition 1.12.** *If $\mathcal{F}$ satisfies the Lefschetz condition relative to $\omega$, then $Rf_*\mathcal{F}$ is formal, and the Leray spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \implies H^{p+q}(X, \mathcal{F}) \tag{1.21}$$

*degenerates at $E_2$.*

*Proof.* The cases for $\omega \in H^1(X, \mathcal{A})$, respectively $H^2(X, \mathcal{A})$ follow from Theorem 1.9, respectively 1.10. Note that the Leray spectral sequence is by construction the spectral sequence (1.1) associated to the object $Rf_*\mathcal{F} \in D^b(Y)$ and the left-exact functor $\Gamma(Y, -)$. □

Suppose $f : X \to S$ is a proper submersion of complex manifolds with codimension $n$, and let $A = \mathbb{C}$, $\mathbb{R}$, or $\mathbb{Q}$. Suppose there is a cohomology class $\omega \in H^2(X, A)$ whose restriction to each fiber $X_t = f^{-1}(t)$ is a Kähler class $\omega_t \in H^2(X_t, A)$. Let $\mathcal{F} = \underline{A}$, the constant sheaf with coefficients in $A$. Then the induced morphism

$$L : Rf_*\underline{A} \to Rf_*\underline{A}[2]$$

is given stalk-wise by

$$\begin{tikzcd}
(R^k f_* \underline{A})_t \arrow[r] \arrow[d, equal] & (R^{k+2} f_* \underline{A})_t \arrow[d, equal] \\
H^k(X_t, \underline{A}) \arrow[r, "\cup \omega_t"] & H^{k+2}(X_t, \underline{A})
\end{tikzcd} \tag{1.22}$$

Since $H^k(X_t, \underline{A}) = 0$ for $k > 2n$, $Rf_*\underline{A}$ is bounded in degrees from 0 to $2n$. Furthermore, it follows from the hard Lefschetz theorem and diagram (1.22) that

$$L^{n-k} : R^k f_* \underline{A} \to R^{2n-k} f_* \underline{A}$$

is an isomorphism for all $0 \leq k \leq n$. Hence $\underline{A}$ satisfies the Lefschetz condition relative to $\omega$, and Proposition 1.12 implies

**Theorem 1.13.** *Suppose $f : X \to S$ is a proper submersion of complex manifolds with codimension $n$, and let $A = \mathbb{C}$, $\mathbb{R}$, or $\mathbb{Q}$. Suppose there is a cohomology class $\omega \in H^2(X, A)$ whose restriction to each fiber $X_t = f^{-1}(t)$ is a Kähler class $\omega_t \in H^2(X_t, A)$. Then $Rf_*\underline{A} \in D^b(S)$ is formal, and the Leray spectral sequence*

$$E_2^{p,q} = H^p(Y, R^q f_* \underline{A}) \implies H^{p+q}(X, A)$$

*degenerates at $E_2$.* □

Theorem 1.13 has a wide range of applications. For example, it can be used to prove the global invariant cycle theorem. We direct the reader to [7], §4.3 for details.

## 2. Degeneration of the Hodge-de Rham spectral sequence

2.1. Besides the spectral sequence (1.1), one may consider yet another spectral sequence defined via the filtration by the first index. We summarize the result as

**Lemma 2.1.** *Let $\mathcal{A}$ be an abelian category with enough injectives, and $T : \mathcal{A} \to \mathcal{B}$ be a left-exact functor of abelian categories. Let $X^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ be a bounded-below complex with objects in $\mathcal{A}$. Then there is a spectral sequence $E_r$ with*

$$E_1^{p,q} = R^q T(X^p) \implies R^{p+q} T(X^\bullet) \tag{2.1}$$

*which is functorial in $X^\bullet$ starting from $E_1$, and commutes with finite direct sums.*

*Proof.* The construction of (2.1) is identical to that of (1.1) (cf. proof of Lemma 1.1), except that in step (ii), we take the filtration on $T(sI^\bullet)$ given by the first index:

$$F^p T(sI^n) = \bigoplus_{\substack{r+q=n \\ r \geq p}} T(I^{r,q})$$

The functorial properties still follow from those of the Cartan-Eilenberg resolution. □

Let $f : X \to S$ be a proper, smooth morphism of schemes. Consider the complex $(\Omega^\bullet_{X/S}, d)$ of relative algebraic differential forms with exterior differential $d$. Then $\Omega^\bullet_{X/S} \in \mathrm{Kom}^+(\mathbf{Coh}(X))$. Let

$$T = f_* : \mathbf{Coh}(X) \to \mathbf{Coh}(S)$$

Then the associated spectral sequence (2.1):

$$E_1^{p,q} = R^q f_* \Omega^p_{X/S} \implies R^{p+q} f_* \Omega^\bullet_{X/S} \tag{2.2}$$

is called the *algebraic Hodge-de Rham spectral sequence*. Note that if $S = \mathrm{Spec}(\mathbb{C})$, then (2.2) is equivalent to

$$E_1^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \implies \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \tag{2.3}$$

For a complex manifold $M$, the *holomorphic Hodge-de Rham spectral sequence* refers to the spectral sequence (2.1) associated to the complex $(\Omega^\bullet_M, d)$ of holomorphic differential forms, and the left-exact functor

$$T = \Gamma(M, -) : \mathbf{Coh}(M) \to \mathbf{Vect}_{\mathbb{C}}$$

This spectral sequence ${}^hE_r^{p,q}$ may be written as

$${}^hE_1^{p,q} = H^q(M, \Omega^p_M) \implies \mathbb{H}^{p+q}(M, \Omega^\bullet_M) \tag{2.4}$$

These two notions are related by the following

**Lemma 2.2** (cf. [4]). *Let $X$ be a proper, smooth scheme of finite type over $\mathbb{C}$, and let $X^h$ be its analytification. Then the GAGA natural isomorphisms $H^q(X, \Omega^p_{X/\mathbb{C}}) \cong H^q(X^h, \Omega^p_{X^h})$ induce isomorphisms:*

$$(E_r^{p,q}, d_r) \cong ({}^hE_r^{p,q}, d_r) \text{ for all } r \geq 1 \tag{2.5}$$

*where $E_r$ is the spectral sequence (2.3), and ${}^hE_r$ is the spectral sequence (2.4) with $M = X^h$.*

In particular, we have an identification for $E_\infty$ and ${}^hE_\infty$. An immediate consequence is that the hypercohomology groups agree:

**Lemma 2.3.** *Let $X$ be a proper, smooth scheme of finite type over $\mathbb{C}$, and let $X^h$ be its analytification. The pullback map on hypercohomology $\mathbb{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}) \to \mathbb{H}^n(X^h, \Omega^\bullet_{X^h})$ induces*

$$\begin{tikzcd} F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \arrow[r] \arrow[d, equal] & E_\infty^{p,q} \arrow[d, equal] \\ F^p\mathbb{H}^{p+q}(X^h, \Omega^\bullet_{X^h}) \arrow[r] & {}^hE_\infty^{p,q} \end{tikzcd} \tag{2.6}$$

*and in particular, the pullback $\mathbb{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}) \to \mathbb{H}^n(X^h, \Omega^\bullet_{X^h})$ is an isomorphism.*

*Proof.* Indeed, the relevant morphisms on cohomology are induced by the map of complexes $f^{-1}\Omega^\bullet_{X/\mathbb{C}} \to \Omega^\bullet_{X^h}$. It follows from the functoriality of the Cartan-Eilenberg resolution that the diagram

$$\begin{tikzcd} F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \arrow[r] \arrow[d] & E_\infty^{p,q} \arrow[d] \\ F^p\mathbb{H}^{p+q}(X^h, \Omega^\bullet_{X^h}) \arrow[r] & {}^hE_\infty^{p,q} \end{tikzcd}$$

commutes. The arrow on the right column is an isomorphism by Lemma 2.2. Using a decreasing induction on $p$, and the five-lemma for

$$\begin{tikzcd} 0 \arrow[r] & F^{p+1}\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \arrow[r] \arrow[d, equal] & F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \arrow[r] \arrow[d] & E_\infty^{p,q} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & F^{p+1}\mathbb{H}(X^h, \Omega^\bullet_{X^h}) \arrow[r] & F^p\mathbb{H}(X^h, \Omega^\bullet_{X^h}) \arrow[r] & {}^hE_\infty^{p,q} \arrow[r] & 0 \end{tikzcd}$$

proves (2.6). The result $\mathbb{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{H}^n(X^h, \Omega^\bullet_{X^h})$ follows by taking $p = 0$. □

It is worth nothing that similar results hold in more general context. For example, when $X$ is smooth and affine, but not necessarily proper (cf. [4]).

2.2. **Projective case.** When $X$ is a smooth, projective scheme over $\mathbb{C}$, the associated analytic space $X^h$ is a closed submanifold of a complex projective space, hence endowed with a Kähler metric. Let $H^{p,q}(X^h)$ be the subspace of $H^{p+q}(X^h,\mathbb{C})$ consisting of cohomology classes with a $(p,q)$-form representative. The theory of harmonic forms implies the Hodge decomposition

$$\mathbb{H}^n(X^h,\Omega^\bullet_{X^h}) \cong H^n(X^h,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X^h) \tag{2.7}$$

where the first equality is nothing but the holomorphic de Rham theorem. Furthermore, since $X^h$ is a Kähler manifold, there is an isomorphism

$$H^{p,q}(X^h) \cong H^q_{\bar\partial}(X^h,\Omega^{p,\bullet}_{X^h}) \cong H^q(X^h,\Omega^p_{X^h}) \tag{2.8}$$

where $(\Omega^{p,\bullet}_{X^h},\bar\partial)$ is the complex of smooth $(p,\bullet)$-forms, and the second isomorphism follows from the fact that $(\Omega^{p,\bullet}_{X^h},\bar\partial)$ is an acyclic resolution of $\Omega^p_{X^h}$. On the other hand,

$$\dim \mathbb{H}^n(X^h,\Omega^\bullet_{X^h}) = \sum_{p+q=n} \dim {}^hE^{p,q}_\infty \le \sum_{p+q=n} \dim {}^hE^{p,q}_1 = \sum_{p+q=n} \dim H^q(X^h,\Omega^p_{X^h}) \tag{2.9}$$

where equality holds if and only if ${}^hE_r$ degenerates at $E_1$. Hence (2.7), (2.8), and (2.9) together imply that ${}^hE_r$ degenerates at $E_1$. It now follows from the Lemma 2.2 that the algebraic de Rham spectral sequence $E_r$ associated to $X$ also degenerates at $E_1$. Furthermore, we have the following version of diagram (2.6):

$$\begin{tikzcd} F^p\mathbb{H}^{p+q}(X,\Omega^\bullet_{X/\mathbb{C}}) \arrow[r] \arrow[d, equal] & E^{p,q}_\infty = H^q(X,\Omega^p_{X/\mathbb{C}}) \arrow[d, equal] \\ F^p\mathbb{H}^{p+q}(X^h,\Omega^\bullet_{X^h}) \arrow[r] & {}^hE^{p,q}_\infty = H^q(X^h,\Omega^p_{X^h}) \end{tikzcd} \tag{2.10}$$

In summary, we have proved

**Lemma 2.4.** *Let $X$ be a smooth, projective scheme over $\mathbb{C}$. Then the algebraic Hodge-de Rham spectral sequence* (2.3) *associated to $X \to \operatorname{Spec}(\mathbb{C})$ degenerates at $E_1$.*

The degeneration of (2.3) allows one to derive Hodge symmetry and Hodge filtration for the algebraic cohomology groups $H^q(X,\Omega^p_{X/\mathbb{C}})$, as worked out in the following remarks.

**Remark 2.5.** The complex conjugation map on $H^n(X^h,\mathbb{C})$ induces an isomorphism $H^{p,q}(X^h) = \overline{H^{p,q}(X^h)}$. By (2.8), we have an induced complex conjugation map

$$c : H^q(X^h,\Omega^p_{X^h}) \xrightarrow{\sim} H^p(X^h,\Omega^q_{X^h})$$

Via the compatibility condition (2.10), it gives rise to an isomorphism:

$$c : H^q(X,\Omega^p_X) \xrightarrow{\sim} H^p(X,\Omega^q_X) \tag{2.11}$$

This isomorphism is of transcendental nature, as complex conjugation depends on the underlying real structure of $H^n(X^h,\mathbb{C})$.

**Remark 2.6.** It is a fact in Kähler geometry that the induced filtration $F^\bullet$ on $\mathbb{H}^n(X^h,\Omega^\bullet_{X^h}) = H^n(X^h,\mathbb{C})$ satisfies

$$H^{p,q}(X^h) = F^pH^{p+q}(X^h,\mathbb{C}) \cap \overline{F^qH^{p+q}(X^h,\mathbb{C})} \tag{2.12}$$

Indeed, one may compute the spectral sequence ${}^hE_r$ using the double complex $(\Omega^{\bullet,\bullet}_{X^h},\partial,\bar\partial)$, filtered by the first degree (cf. [8]). Thus $F^pH^{p+q}(X^h,\mathbb{C})$ consists of cohomology classes admitting a representative $\alpha$ which only has nonzero $(r,s)$-components for $r \ge p$. So the right-hand-side consists of cohomology classes admitting representatives $\alpha_1$ and $\alpha_2$, having nonzero $(r,s)$-components only for $r \ge p$, respectively $s \ge q$. Write $\alpha_1 - \alpha_2 = d\beta$, for some form $\beta$. Suppose $\beta = \beta_1 + \beta_2$, where $\beta_1$ (resp. $\beta_2$) only has nonzero $(r,s)$-components for $r \ge p$ (resp. $s \ge q$). Then let

$$\alpha' = \alpha_1 - d\beta_1 = \alpha_2 + d\beta_2$$

This is a $(p,q)$-form cohomologous to $\alpha_1$, $\alpha_2$, as $\alpha_1 - d\beta_1$ (resp. $\alpha_2 + d\beta_2$) only has nonzero $(r,s)$-components for $r \ge p$ (resp. $s \ge q$). The converse is clear.

Furthermore, $F^\bullet$ satisfies an orthogonality relation:

$$F^pH^{p+q}(X^h,\mathbb{C})\cap\overline{F^{q+1}H^{p+q}(X^h,\mathbb{C})}=\{0\} \tag{2.13}$$

It is clear from (2.6), together with the holomorphic de Rham theorem, that the same relation holds for the algebraic de Rham cohomology groups:

$$F^p\mathbb{H}^{p+q}(X,\Omega^\bullet_{X/\mathbb{C}})\cap\overline{F^{q+1}\mathbb{H}^{p+q}(X,\Omega^\bullet_{X/\mathbb{C}})}=\{0\} \tag{2.14}$$

and the natural way to define the algebraic analogues for $H^{p,q}(X^h)$ will be to use (2.12). We will discuss this in greater details in the next section. One can also deduce that

$$F^pH^n(X^h,\mathbb{C})=\bigoplus_{\substack{r+q=n\\ r\geq p}}H^{r,q}(X^h)$$

but the argument is essentially contained in the algebraic case (Corollary 2.9), so we will not repeat it here.

2.3. **General absolute case.** The case for a smooth, proper scheme over $\mathbb{C}$ is essentially due to the following fact, together with resolution of singularities.

**Lemma 2.7.** *Let $X,Y$ be smooth schemes over of finite type over $\mathbb{C}$. Suppose $g:X\to Y$ is a proper morphism, inducing a birational equivalence. Then the pullback on cohomology:*

$$g^*:H^q(Y,\Omega^p_{Y/\mathbb{C}})\to H^q(X,\Omega^p_{X/\mathbb{C}})$$

*is injective.*

*Proof.* Let $\dim X=\dim Y=n$. Consider first the top cohomology group of the canonical sheaf. Serre duality shows that

$$\dim H^n(X,\Omega^n_{X/\mathbb{C}})=\dim H^0(X,\mathcal{O}_X)=1$$

since $X$ is proper. Similarly, $\dim H^n(Y,\Omega^n_{Y/\mathbb{C}})=1$. Since $g$ induces a birational equivalence, the pullback $g^*:H^n(Y,\Omega^n_{Y/\mathbb{C}})\to H^n(X,\Omega^n_{X/\mathbb{C}})$ is nonzero. Hence it is injective. Now given any $\alpha\in H^q(Y,\Omega^p_{Y/\mathbb{C}})$, by Serre duality, there exists some $\beta\in H^{n-q}(Y,\Omega^{n-p}_{Y/\mathbb{C}})$ such that $\alpha\cup\beta\neq 0\in H^n(Y,\Omega^n_{Y/\mathbb{C}})$. Since pullback commutes with cup product,

$$g^*\alpha\cup g^*\beta=g^*(\alpha\cup\beta)\neq 0\in H^n(X,\Omega^n_{X/\mathbb{C}})$$

and it follows that $g^*\alpha\neq 0\in H^q(X,\Omega^p_{X/\mathbb{C}})$. □

The following results are contained in [1], Prop. 5.3.

**Theorem 2.8.** *Let $X$ be a smooth, proper scheme over $\mathbb{C}$. Then the algebraic Hodge-de Rham spectral sequence (2.3) associated to $X\to\operatorname{Spec}(\mathbb{C})$ degenerates at $E_1$.*

*Proof.* It follows from Chow's lemma, and Hironaka's resolution of singularities that there exists a proper morphism $g:X'\to X$, inducing a birational equivalence, where $X'$ is a smooth projective scheme over $\mathbb{C}$ of finite type. Since $g^*$ commutes with the exterior differential, the following diagram commutes

$$\begin{tikzcd}
H^q(X,\Omega^p_{X/\mathbb{C}}) \arrow[r, "d"] \arrow[d, hook, "g^*"] & H^q(X,\Omega^{p+1}_{X/\mathbb{C}}) \arrow[d, hook, "g^*"] \\
H^q(X',\Omega^p_{X'/\mathbb{C}}) \arrow[r, "d"] & H^q(X',\Omega^{p+1}_{X'/\mathbb{C}})
\end{tikzcd}$$

where the vertical arrows are injective by Lemma 2.7. The lower horizontal arrow vanishes, because this is $d_1$ of the spectral sequence $E'_r$ associated to $X'$, which degenerates at $E'_1$ by Lemma 2.4. Hence $d=0$ on the upper row as well.

We now prove the degeneration of $E_r^{p,q}$, associated to $X$, by induction. The base case $d_1=0$ is precisely given above. Assume now that $d_1=\cdots=d_{r-1}=0$. Then $E_r^{p,q}=H^q(X,\Omega^p_{X/\mathbb{C}})$, and we again have a commutative diagram:

$$\begin{tikzcd}
E_r^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \arrow[r, "d_r"] \arrow[d, hook, "g^*"] & H^{q-r+1}(X, \Omega^{p+r}_{X/\mathbb{C}}) \arrow[d, hook, "g^*"] \\
(E'_r)^{p,q} = H^q(X', \Omega^p_{X'/\mathbb{C}}) \arrow[r, "d_r"] & H^{q-r+1}(X', \Omega^{p+r}_{X'/\mathbb{C}})
\end{tikzcd}$$

Thus injectivity of $g^*$ implies that $d_r = 0$ in the upper arrow. □

As a consequence, we derive the Hodge decomposition for $X$. The result is interesting because $X^h$ is not necessarily a Kähler manifold anymore.

**Corollary 2.9.** *Let $F^\bullet$ be the induced filtration on $\mathbb{H}^n(X, \Omega^\bullet_{X/\mathbb{C}})$, and define*

$$H^{p,q}(X) = F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \cap \overline{F^q\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}})} \tag{2.15}$$

*Then for each $n$ and $0 \leq p \leq n$,*

$$F^p\mathbb{H}^n(X, \Omega^\bullet_{X/\mathbb{C}}) = \bigoplus_{\substack{r+q=n \\ r\geq p}} H^{r,q}(X) \tag{2.16}$$

The definition (2.15) is clearly an imitation of the analytic case (2.12). Indeed, $H^{p,q}(X)$ corresponds, in $\mathbb{H}^{p+q}(X^h, \Omega^\bullet_{X^h}) = H^{p+q}(X^h, \mathbb{C})$, to $H^{p,q}(X^h)$.

*Proof of Corollary 2.9.* The proof is essentially a dimension counting. Let $g : X' \to X$ be as in the proof of Theorem 2.8. By degeneration of the spectral sequence, we have an injective homomorphism:

$$g^* : E_\infty^{p,q} = H^q(X, \Omega^p_{X/\mathbb{C}}) \to H^q(X', \Omega^p_{X'/\mathbb{C}}) = (E'_\infty)^{p,q}$$

Hence $g^*$ is compatible with the induced filtrations $F^\bullet$ on $\mathbb{H}(X, \Omega^\bullet_{X/\mathbb{C}})$ and $\mathbb{H}(X', \Omega^\bullet_{X'/\mathbb{C}})$. The filtration on $\mathbb{H}(X', \Omega^\bullet_{X'/\mathbb{C}})$ satisfies the orthogonality relation (2.14):

$$F^p\mathbb{H}^{p+q}(X', \Omega^\bullet_{X'/\mathbb{C}}) \cap \overline{F^{q+1}\mathbb{H}^{p+q}(X', \Omega^\bullet_{X'/\mathbb{C}})} = \{0\} \tag{2.17}$$

Since $g^*$ is also compatible with complex conjugation, it follows from its injectivity that

$$F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \cap \overline{F^{q+1}\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}})} = \{0\} \tag{2.18}$$

hence an inequality

$$\dim F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) + \dim F^{q+1}\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \leq \dim \mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \tag{2.19}$$

By letting $h^{p,q} = \dim H^q(X, \Omega^p_{X/\mathbb{C}}) = \dim \mathrm{Gr}^p_{F^\bullet}\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}})$, we have

$$F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) = \sum_{i\geq p} h^{i,p+q-i}$$

Hence (2.19) can be rewritten as

$$\sum_{i\geq p} h^{i,p+q-i} + \sum_{i\geq q+1} h^{i,p+q-i} \leq \sum_{i\geq 0} h^{i,p+q-i}$$

which simplifies to

$$\sum_{i\geq p} h^{i,p+q-i} \leq \sum_{0\leq i\leq q} h^{i,p+q-i} \tag{2.20}$$

By Serre duality $h^{p,q} = h^{\dim X-p,\dim X-q}$, the inequality (2.20) implies its own opposite. We thus obtain equality in (2.19), and together with (2.18), there holds

$$\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) = F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \oplus \overline{F^{q+1}\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}})} \tag{2.21}$$

Intersecting both sides with $F^{p-1}\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}})$, and using the definition of $H^{p,q}$, we obtain

$$F^{p-1}\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) = F^p\mathbb{H}^{p+q}(X, \Omega^\bullet_{X/\mathbb{C}}) \oplus H^{p-1,q+1} \tag{2.22}$$

A repeated application of (2.22) for $p+q=n$ yields

$$F^p\mathbb{H}^n(X,\Omega^\bullet_{X/\mathbb{C}}) = F^{p+1}\mathbb{H}^n(X,\Omega^\bullet_{X/\mathbb{C}}) \oplus H^{p,n-p} = \cdots = \bigoplus_{\substack{r+q=n \\ r\geq p}} H^{r,q}(X)$$

as desired. □

**Remark 2.10.** Note that the Hodge decomposition of $\mathbb{H}^n(X,\Omega^\bullet_{X/\mathbb{C}})$ follows from (2.16) for $p=0$:

$$\mathbb{H}^n(X,\Omega^\bullet_{X/\mathbb{C}}) = \bigoplus_{p+q=n} H^{p,q}(X) \tag{2.23}$$

Furthermore, (2.16) applied to $p$ and $p+1$ gives

$$\dim H^{p,q}(X) = \dim F^p\mathbb{H}^n(X,\Omega^\bullet_{X/\mathbb{C}}) - \dim F^{p+1}\mathbb{H}^n(X,\Omega^\bullet_{X/\mathbb{C}}) = \dim H^q(X,\Omega^p_{X/\mathbb{C}})$$

Since $H^{p,q}(X) = \overline{H^{q,p}(X)}$, we obtain the Hodge symmetry

$$\dim H^q(X,\Omega^p_{X/\mathbb{C}}) = \dim H^p(X,\Omega^q_{X/\mathbb{C}}) \tag{2.24}$$

### 2.4. Genearl relative case.

It is worth mentioning the following vastly more general result, although we will not prove it in this paper.

**Theorem 2.11** ([1], Thm. 5.5)**.** *Let $f : X \to S$ be a proper, smooth morphism of schemes of finite type over $\mathbb{C}$. Then*

- *(i) The sheaves $R^qf_*\Omega^p_{X/S}$ is locally free, of finite type, and is compatible with base change.*
- *(ii) The spectral sequence (2.2) degenerates at $E_1$.*
- *(iii) At each point of $S$, the sheaves $R^qf_*\Omega^p_{X/S}$ and $R^pf_*\Omega^q_{X/S}$ are of the same rank.*

Lemma 2.4, and Theorem 2.8 are special cases for the above theorem, with $S = \mathrm{Spec}(\mathbb{C})$, and $f$ is projective, respectively proper.

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