Literal notes

0.1. **Where are we?** Recall that for each $I \in \mathbf{fSet}$ and $W^I \in \mathbf{Rep}(\check{G}^I)$, we have produced an object $\widetilde{\mathcal{F}}_{I,W^I} \in D^{\text{ind}}(X^I)$. It is the proper pushforward

$$\widetilde{\mathcal{F}}_{I,W^I} := (\mathfrak{p}_I)_! \operatorname{Sat}(W^I)$$

along paws of the moduli of Shtukas:

$$\begin{array}{ccccc} \operatorname{Sht}_I & \longrightarrow & \operatorname{Hecke}_I & \longrightarrow & X^I \\ \downarrow & & \downarrow & & \\ \operatorname{Bun}_G & \xrightarrow{1\times\operatorname{Frob}} & \operatorname{Bun}_G \times \operatorname{Bun}_G . & & \end{array} \qquad (\mathfrak{p}_I : \operatorname{Sht}_I \to X^I)$$

where $\operatorname{Sat}(W^I)$ is regarded as an object in the constructible derived category of Sht_I.

Remark 0.1. The geometry of Sht_I equips $\widetilde{\mathcal{F}}_{I,W^I}$ with equivariance structure with respect to the partial Frobenii:

$$F_{\{i\}} : \operatorname{Frob}^*_{\{i\}}(\widetilde{\mathcal{F}}_{I,W^I}) \to \widetilde{\mathcal{F}}_{I,W^I}.$$

0.2. **Cayley-Hamilton.** Suppose x is a closed point of X, and $V \in \mathbf{Rep}(\check{G})$. We would like to study

$$\widetilde{\mathcal{F}}_{K\sqcup\{0\},W^K\boxtimes V}\big|_{X^K\times\{x\}} \in D^{\text{ind}}(X^K \times \{x\}).$$

Hence, we consider $K \in \mathbf{fSet}$ and $W^K \in \mathbf{Rep}(\check{G})$ fixed as well from now on.

The equivariance data for the partial Frobenius $\operatorname{Frob}^*_{\{0\}}$ now give rise to an endomorphism:

$$F := F^{\deg(x)}_{\{0\}} : \widetilde{\mathcal{F}}_{K\sqcup\{0\},W^K\boxtimes V}\big|_{X^K\times\{x\}} \to \widetilde{\mathcal{F}}_{K\sqcup\{0\},W^K\boxtimes V}\big|_{X^K\times\{x\}}$$

since $\operatorname{Frob}^{\deg(x)}_{\{0\}}$ acts as identity on $\{x\}$.

The way we will interpret Cayley-Hamilton can be summarized by the slogan: <u>F satisfies its own characteristic equation.</u>

0.3. **The trace operator.** The usual statement of Cayley-Hamilton is the following equation for every endomorphism T acting on a vector space V of dimension n:

$$\sum_{i=0}^{n} (-1)^i \operatorname{Tr}(T\big|\Lambda^{n-i}V) \cdot T^i = 0. \tag{0.1}$$

In our setting, the role of the trace operators $\operatorname{Tr}(T\big|\Lambda^{n-i}V)$ will be played by the endomorphism $S_{\Lambda^{n-i}V,x}$.

Proposition 0.2 (V. Lafforgue's Proposition 7.1)**.** *Suppose $n = \dim(V)$. Then the following identity holds in the endomorphism ring of $\widetilde{\mathcal{F}}_{K\sqcup\{0\},W^K\boxtimes V}\big|_{X^K\times\{x\}}$:*

$$\sum_{i=0}^{n} (-1)^i S_{\Lambda^{n-i}V,x} \circ F^i = 0. \tag{0.2}$$

Remark 0.3. (a) The endomorphism $S_{\Lambda^{n-i}V,x}$ commutes with F (hence F^i).
(b) The identity (0.2) is useful because it imposes a finiteness condition on the operator F.

(c) You may wonder: the endomorphism F is only defined when we restrict to a closed point—how can it be useful in practice?

The answer is that ultimately we are interested in the (Hecke-finite part of the) restriction of $\widetilde{\mathcal{F}}_{I,V^I}$ to the generic point η^I of X^I:

$$\mathcal{F}_{I,V^I} := (\widetilde{\mathcal{F}}_{I,V^I})^{\mathrm{Hf}}_{\eta^I}.$$

One may then use generality of lisse sheaves to pass from properties at a closed point to those at η^I.

0.4. **Proof of the usual Cayley-Hamilton.** Assume that T is an endomorphism on the vector space V of dimension n. We want to prove (0.1).

The proof will explain exactly what the endomorphism appearing on the left-hand-side *is*. In fact, we claim:

$$\frac{1}{n+1}\cdot\sum_{i=0}^{n}(-1)^i\,\mathrm{Tr}(T\big|\Lambda^{n-i}V)\cdot T^i = \mathfrak{C}_n(A_{n+1}).$$

Now, what is the right-hand-side? Fix an integer $m \geq 0$, and let L be an endomorphism of $V^{\otimes(1+m)}$. The notation $\mathfrak{C}_m(L)$ denotes the following endomorphism of V:

[a]

$$\mathfrak{C}_m(L) : V \xrightarrow{1\otimes\mathrm{unit}} V\otimes V^{\otimes m}\otimes(V^{\otimes m})^* \xrightarrow{L\otimes 1} V\otimes V^{\otimes m}\otimes(V^{\otimes m})^* \to \xrightarrow{1\otimes T^{\otimes m}\otimes 1} V\otimes V^{\otimes m}\otimes(V^{\otimes m})^* \xrightarrow{1\otimes\text{co-unit}} V$$

[a]Do not erase. Later we will modify this diagram.

and the endomorphism A_m denotes the *anti-symmetrizer* on $V^{\otimes m}$:

$$A_m := \frac{1}{m!}\sum_{\sigma\in S_m}\mathrm{sgn}(\sigma)\cdot\sigma.$$

Then, it is clear that the usual Cayley-Hamilton follows from the above identity, as $A_{n+1} = 0$.

In fact, we claim something even stronger. Let $S^{(i+1)}_{n+1}$ denote the subset of S_{n+1} consisting of permutations for which the cycle containing $\{0\}$ has length $i+1$. Then we may decompose A_{n+1} into:

$$A = \sum_{i=0}^{n} A^{(i+1)}_{n+1},\ \text{ where } A^{(i+1)}_{n+1} := \frac{1}{(n+1)!}\sum_{\sigma\in S^{(i+1)}_{n+1}}\mathrm{sgn}(\sigma)\cdot\sigma,$$

and the claim is that

$$\frac{(-1)^i}{n+1}\cdot\mathrm{Tr}(T\big|\Lambda^{n-i}V)\cdot T^i = \mathfrak{C}_n(A^{(i+1)}_{n+1}). \tag{0.3}$$

The proof is a straightforward computation based on the following:

Observation 0.4. [a]

(a) Suppose $\rho\in S_m$ is a permutation fixing $\{0\}$. Then

$$\mathfrak{C}_m(\rho L\rho^{-1}) = \mathfrak{C}_m(L).$$

(b) Suppose $L = L_0\otimes L_1$, where L_0 applies to the first $(i+1)$-factors and L_1 to the last $(m-i)$-factors. Then

$$\mathfrak{C}_m(L) = \mathfrak{C}_{m-i}(1\otimes L_1)\circ\mathfrak{C}_i(L_0).$$

(c) Suppose $\sigma_0 \in S_{i+1}$ is the permutation $(0 \to 1 \to \cdots \to i)$. Then

$$\mathfrak{C}_i(\sigma_0) = T^i.$$

(d) Suppose A_m is the anti-symmetrizer on $V^{\otimes m}$. Then

$$\mathfrak{C}_m(1 \otimes A_m) = \mathrm{Tr}(T\big|\Lambda^m V).$$

[a] Do not erase. Later we will modify these statements.

We will now perform the calculation that proves (0.3).

$$\mathfrak{C}_n(A_{n+1}^{(i+1)}) = \frac{1}{(n+1)!} \sum_{\sigma \in S_{n+1}^{(i+1)}} \mathrm{sgn}(\sigma) \cdot \mathfrak{C}_n(\sigma)$$

$$\text{(a)} \qquad = \frac{1}{(n+1)!} \cdot \binom{n}{i} \cdot i! \sum_{\tau \in S_{n-i}} (-1)^i \, \mathrm{sgn}(\tau) \cdot \mathfrak{C}_n(\sigma_0 \otimes \tau)$$

$$\text{(b)} \qquad = \frac{1}{n+1} \cdot \frac{1}{(n-i)!} \sum_{\tau \in S_{n-i}} (-1)^i \, \mathrm{sgn}(\tau) \cdot \mathfrak{C}_i(\sigma_0) \circ \mathfrak{C}_{n-i}(1 \otimes \tau)$$

where σ_0 is the permutation $(0 \to 1 \to \cdots \to i)$. It follows that

$$\text{(c)} \qquad \mathfrak{C}_n(A_{n+1}^{(i+1)}) = \frac{1}{n+1} \cdot \frac{1}{(n-i)!} \sum_{\tau \in S_{n-i}} (-1)^i \, \mathrm{sgn}(\tau) \cdot F^i \circ \mathfrak{C}_{n-i}(1 \otimes \tau)$$

$$= \frac{(-1)^i}{n+1} \cdot F^i \circ \mathfrak{C}_{n-i}(1 \otimes A_{n-i})$$

$$\text{(d)} \qquad = \frac{(-1)^i}{n+1} \cdot F^i \cdot \mathrm{Tr}(T\big|\Lambda^{n-i} V).$$

0.5. **One word about (c).** This is a cool trick. For $i = 1$, the diagram corresponds to:

$$\mathfrak{C}_1(\sigma_0) : V \xrightarrow{1 \otimes \mathrm{unit}} V \otimes V \otimes V^* \xrightarrow{\sigma_0 \otimes 1} V \otimes V \otimes V^* \xrightarrow{1 \otimes T \otimes 1} V \otimes V \otimes V^* \xrightarrow{1 \otimes \text{co-unit}} V$$

which is equal to:

$$V \xrightarrow{1 \otimes \mathrm{unit}} V \otimes V^* \otimes V \xrightarrow{T \otimes 1 \otimes 1} V \otimes V^* \otimes V \xrightarrow{\text{co-unit} \otimes 1} V.$$

This latter composition is obviously equal to T.

The general case follow from induction—you will see it clearly from the following way of writing the composition:

$$\mathfrak{C}_i(\sigma_0) : V \xrightarrow{1 \otimes \mathrm{unit}^{\otimes i}} V \otimes (V^* \otimes V)^{\otimes i} \xrightarrow{T \otimes (1 \otimes T)^{\otimes (i-1)} \otimes 1} V \otimes (V^* \otimes V)^{\otimes i} \xrightarrow{\text{co-unit}^{\otimes i} \otimes 1} V.$$

0.6. **Generalization.** One would like to generalize the above "tensorial" proof of Cayley-Hamilton to the endomorphism F on $\widetilde{\mathcal{F}}_{K \sqcup \{0\}, W^K \boxtimes V}\big|_{X^K \times \{x\}}$.

(a) As a first observation, the endomorphism F takes place in the category $D^{\mathrm{ind}}(X^K \times \{x\})$, and yet we want to utilize constructions such as the unit map $V \to V \otimes V^{\otimes m} \otimes (V^{\otimes m})^*$ of $\check{G}$-representations.

Hence we should be studying functors:

$$\Phi : \mathbf{Rep}(\check{G}) \to D^{\mathrm{ind}}(X^K \times \{x\})_{\mathrm{aut}}, \quad V \rightsquigarrow \widetilde{\mathcal{F}}_{K \sqcup \{0\}, W^K \boxtimes V}\big|_{X^K \times \{x\}} \text{ with } F\text{-action}.$$

Here, $\mathcal{C}_{\mathrm{aut}}$ denotes the category of pairs (c, F) with $c \in \mathcal{C}$ and $F \in \mathrm{Aut}(c)$.

(b) The usual Cayley-Hamilton would correspond to the functor

$$\Phi : \mathbf{Rep}(\mathrm{GL}_n) \to \mathbf{Vect}_{\mathrm{aut}}, \quad V \rightsquigarrow \mathrm{oblv}(V) \text{ with } T\text{-action}.$$

(c) In order to formulate "trace" (at least in a way applicable for us), we will need additional data that I will presently explain.

Suppose $\mathcal{C}_I$ is a family of categories parametrized by $I \in \mathbf{fSet}$. For us,

$$\mathcal{C}_I := D^{\mathrm{ind}}(X^K \times \{x\}^I).$$

For each $I \in \mathbf{fSet}$, let $\mathrm{Rep}(\mathbb{Z}^I, \mathcal{C}_I)$ denote the category of pairs $(c, \{F_i\}_{i \in I})$ where $c \in \mathcal{C}_I$ and $\{F_i\}_{i\in I}$ is an I-family of automorphisms of c.

Suppose we have a system of functors (where G is any linear algebraic group):

$$\Phi_I : \mathbf{Rep}(G^I) \to \mathbf{Rep}(\mathbb{Z}^I, \mathcal{C}_I)$$

with functoriality along $I \to J$, i.e., a commutativity constraint:

$$\begin{array}{ccc} \mathbf{Rep}(G^I) & \longrightarrow & \mathbf{Rep}(\mathbb{Z}^I, \mathcal{C}_I) \\ \big\downarrow & & \big\downarrow R \\ \mathbf{Rep}(G^J) & \longrightarrow & \mathbf{Rep}(\mathbb{Z}^J, \mathcal{C}_J) \end{array}$$

satisfying a compatibility condition for compositions: $I \to J \to K$. Fixing a G-representation V of dimension n, there is a well-defined notion of "trace of F on $U \in \mathbf{Rep}(G)$" as an operator:

$$\begin{aligned} S_U : \Phi_{\{1\}}(V) &\xrightarrow{\Phi_{\{1\}}(1\otimes \mathrm{unit})} \Phi_{\{1\}}(V \otimes U \otimes U^*) \cong R \circ \Phi_{\{1,2,3\}}(V \boxtimes U \boxtimes U^*) \\ &\xrightarrow{R(F_{(0,1,0)})} R \circ \Phi_{\{1,2,3\}}(V \boxtimes U \boxtimes U^*) \cong \Phi_{\{1\}}(V \otimes U \otimes U^*) \\ &\xrightarrow{\Phi_{\{1\}}(1\otimes \text{co-unit})} \Phi_{\{1\}}(V). \end{aligned}$$

We can state Cayley-Hamilton in this context:

Theorem 0.5. *The following equality holds in* $\mathrm{End}(\Phi_{\{1\}}(V))$*:*

$$\sum_{i=0}^{n} (-1)^i S_{\Lambda^{n-i}V} \circ F^i = 0.$$

The application we have in mind is for $\Phi_I : \mathbf{Rep}(\check{G}^I) \to \mathbf{Rep}(\mathbb{Z}^I, D^{\mathrm{ind}}(X^K \times \{x\}^I))$ given by the functor:

$$V^I \rightsquigarrow \widetilde{\mathcal{F}}_{K \sqcup \{0\}, W^K \boxtimes V^I}\big|_{X^K \times \{x\}} \text{ with } \{F_i\}_{i\in I}\text{-action}.$$

0.7. **Proof of the general theorem.** The proof of the general theorem amounts to establishing the analogues of Observations 0.4, with the following definition of $\mathfrak{C}_m(L)$ (now for L an endomorphism of the G-representation $V^{\otimes(1+m)}$):

$$\begin{aligned} \mathfrak{C}_m(L) : \Phi_{\{1\}}(V) &\xrightarrow{\Phi_{\{1\}}(1\otimes \mathrm{unit})} \Phi_{\{1\}}(V \otimes V^{\otimes m} \otimes (V^{\otimes m})^*) \xrightarrow{\Phi_{\{1\}}(L\otimes 1)} \Phi_{\{1\}}(V \otimes V^{\otimes m} \otimes (V^{\otimes m})^*) \\ &\cong R \circ \Phi_{\{1,2,3\}}(V \boxtimes V^{\otimes m} \boxtimes (V^{\otimes m})^*) \xrightarrow{R(F_{(0,1,0)})} R \circ \Phi_{\{1,2,3\}}(V \boxtimes V^{\otimes m} \boxtimes (V^{\otimes m})^*) \\ &\cong \Phi_{\{1\}}(V \otimes V^{\otimes m} \otimes (V^{\otimes m})^*) \xrightarrow{\Phi_{\{1\}}(1\otimes \text{co-unit})} \Phi_{\{1\}}(V). \end{aligned}$$

Among the four statements, (a) and (b) remain the same. The analogues of (c) and (d) state:

(c') Suppose $\sigma_0 \in S_{i+1}$ is the permutation $(0 \to 1 \to \cdots \to i)$. Then

$$\mathfrak{C}_i(\sigma_0) = F^i.$$

(d') Suppose A_m is the anti-symmetrizer on $V^{\otimes m}$. Then

$$\mathfrak{C}_m(1 \otimes A_m) = S_{\Lambda^m V}.$$

I will present the proofs of (d') and (c'). The proof of (d') is straightforward, and after seeing it, you will be convinced that the analogous of (a) and (b) can be carried out in a similar manner. The proof of (c') is more substantive.

0.8. **Proof of (d').** I first claim that there exists an embedding:

$$\iota : \Lambda^m V \boxtimes (\Lambda^m V)^* \to V^{\otimes m} \boxtimes (V^{\otimes m})^*$$

of $G^{\{1,2\}}$-representations, such that the following diagram *of $G^{\{1\}}$-representations* commute[1]:

$$\begin{array}{ccc} \text{triv} & \xrightarrow{\text{unit}} & V^{\otimes m} \otimes (V^{\otimes m})^* \\ \Big\downarrow{\scriptstyle \text{unit}} & & \Big\downarrow{\scriptstyle A_m \otimes 1} \\ \Lambda^m V \otimes (\Lambda^m V)^* & \xrightarrow{\iota} & V^{\otimes m} \otimes (V^{\otimes m})^* \end{array} \qquad \begin{array}{ccc} \Lambda^m V \otimes (\Lambda^m V)^* & \xrightarrow{\iota} & V^{\otimes m} \otimes (V^{\otimes m})^* \\ \Big\downarrow{\scriptstyle \text{co-unit}} & & \Big\downarrow{\scriptstyle \text{co-unit}} \\ \text{triv} & \xrightarrow{1} & \text{triv}\,. \end{array}$$

(Aside: how do we write such an ι? In general, given maps between G-representations $V \xrightarrow{T} W \xrightarrow{S} V$, such that $T \circ S = 1$, the map $S \boxtimes T^* : W \boxtimes W^* \to V \boxtimes V^*$ does the job.)

Now, the statement (d') follows from the commutative diagram (left-hand-side = red, right-hand-side = blue):

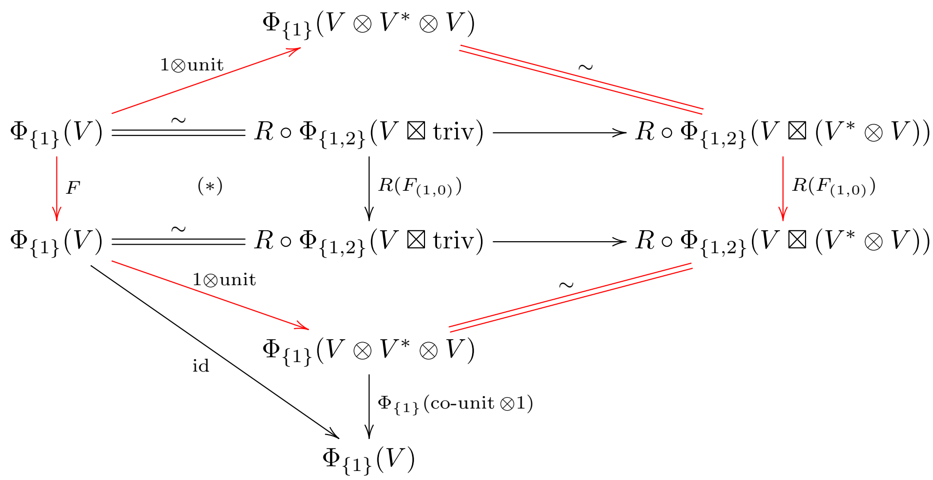
$$\begin{array}{ccc} \Phi_{\{1\}}(V) & \xrightarrow{1 \otimes \text{unit}} & \Phi_{\{1\}}(V \otimes V^{\otimes m} \otimes (V^{\otimes m})^*) \\ \Big\downarrow{\scriptstyle 1 \otimes \text{unit}} & & \Big\downarrow{\scriptstyle 1 \otimes A_m \otimes 1} \\ \Phi_{\{1\}}(V \otimes \Lambda^m V \otimes (\Lambda^m V)^*) & \xrightarrow{1 \otimes \iota} & \Phi_{\{1\}}(V \otimes V^{\otimes m} \otimes (V^{\otimes m})^*) \\ \vdots & & \vdots \\ \Phi_{\{1\}}(V \otimes \Lambda^m V \otimes (\Lambda^m V)^* & \xrightarrow{1 \otimes \iota} & \Phi_{\{1\}}(V \otimes V^{\otimes m} \otimes (V^{\otimes m})^*) \\ \Big\downarrow{\scriptstyle \text{co-unit}} & & \Big\downarrow{\scriptstyle \text{co-unit}} \\ \Phi_{\{1\}}(V) & \xrightarrow{1} & \Phi_{\{1\}}(V). \end{array}$$

The middle square commutes by functoriality of $\Phi_{\{1,2,3\}}$ because ι is induced from a morphism of $G^{\{1,2\}}$-representations.

0.9. **Proof of (c').** We proceed by three steps:

[1] Draw the second square under the first, then modify into the following diagram.

0.9.1. *The case $i = 1$.* We apply the trick from before in re-organizing the composition to get rid of σ_0. The objective is to show that the red hexagon commutes.



Now, all squares obviously commute *except* the one with $(*)$. To get that one to work out, we need compatibility of the commutativity constraint along $\{1\} \to \{1,2\} \to \{1\}$:

$$\begin{array}{ccccc}
\Phi_{\{1\}}(V) & \overset{\sim}{=\!=} & R_{\{1,2\}\to\{1\}} \circ R_{\{1\}\to\{1,2\}}(V) & \overset{\sim}{=\!=} & R_{\{1,2\}\to\{1\}} \circ \Phi_{\{1,2\}}(V \boxtimes \mathrm{triv}) \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle R_{\{1,2\}\to\{1\}}\circ R_{\{1\}\to\{1,2\}}(F)} & & \downarrow{\scriptstyle R_{\{1,2\}\to\{1\}}(F_{(1,0)})} \\
\Phi_{\{1\}}(V) & \overset{\sim}{=\!=} & R_{\{1,2\}\to\{1\}} \circ R_{\{1\}\to\{1,2\}}(V) & \overset{\sim}{=\!=} & R_{\{1,2\}\to\{1\}} \circ \Phi_{\{1,2\}}(V \boxtimes \mathrm{triv}).
\end{array}$$

0.9.2. *The induction step (modulo something).* We would now like to perform the induction step, assuming the result for $i-1$, i.e., commutativity of:

$$\begin{array}{ccccc}
\Phi_{\{1\}}(V) & \xrightarrow{1\otimes \mathrm{unit}^{\otimes(i-1)}} & \Phi_{\{1\}}(V \otimes (V^* \otimes V)^{\otimes(i-1)}) & \overset{\sim}{=\!=} & R \circ \Phi_{\{1,\cdots,2i-1\}}(V \boxtimes (V^* \boxtimes V)^{\boxtimes(i-1)}) \\
\downarrow{\scriptstyle F^{i-1}} & & & & \downarrow{\scriptstyle R(F_{(1,0,1,\cdots,1,0,0)})} \\
\Phi_{\{1\}}(V) & \xrightarrow{1\otimes \mathrm{unit}^{\otimes(i-1)}} & \Phi_{\{1\}}(V \otimes (V^* \otimes V)^{\otimes(i-1)}) & \overset{\sim}{=\!=} & R \circ \Phi_{\{1,\cdots,2i-1\}}(V \boxtimes (V^* \boxtimes V)^{\boxtimes(i-1)}).
\end{array}$$

In fact, we will need the commutativity of a similar diagram for

$$\Phi_{\{1,2\}}(U \boxtimes V) \xrightarrow{F_{(0,i-1)}} \Phi_{\{1,2\}}(U \boxtimes V) \tag{0.4}$$

which will be established in the next step. We assume this for now.

It remains to show that the following (red) diagram commutes:

$$\begin{array}{ccccccccc}
\Phi_{\{1\}}(V) & \xrightarrow{1\otimes\mathrm{unit}} & \Phi_{\{1\}}(V\otimes V^*\otimes V) & \overset{\sim}{=\!=} & R\circ\Phi_{\{1,2,3\}}(V\boxtimes V^*\boxtimes V) & \longrightarrow & R\circ\Phi_{\{1,\cdots,2i+1\}}(V\boxtimes(V^*\boxtimes V)^{\boxtimes i}) \\
\downarrow{\scriptstyle F^{i-1}} & & & & \downarrow{\scriptstyle R(F_{(0,0,i-1)})} & (*) & \downarrow{\scriptstyle R(F_{(0,0,1,0,\cdots,1,0,0)})} \\
\Phi_{\{1\}}(V) & \xrightarrow{1\otimes\mathrm{unit}} & \Phi_{\{1\}}(V\otimes V^*\otimes V) & \overset{\sim}{=\!=} & R\circ\Phi_{\{1,2,3\}}(V\boxtimes V^*\boxtimes V) & \longrightarrow & R\circ\Phi_{\{1,\cdots,2i+1\}}(V\boxtimes(V^*\boxtimes V)^{\boxtimes i}) \\
\downarrow{\scriptstyle F} & & (**) & & \downarrow{\scriptstyle R(F_{(1,0,0)})} & & \downarrow{\scriptstyle R(F_{(1,0,\cdots,1,0,0)})} \\
\Phi_{\{1\}}(V) & \xrightarrow{1\otimes\mathrm{unit}} & \Phi_{\{1\}}(V\otimes V^*\otimes V) & \overset{\sim}{=\!=} & R\circ\Phi_{\{1,2,3\}}(V\boxtimes V^*\boxtimes V) & \longrightarrow & R\circ\Phi_{\{1,\cdots,2i+1\}}(V\boxtimes(V^*\boxtimes V)^{\boxtimes i})
\end{array}$$

where

(a) the $(*)$ square commutes by induction hypothesis, applied to $U = V \otimes V^*$. (Along horizontal arrows, we are growing $(i+1)$-copies of $V^* \boxtimes V$ on the last copy of V.)

(b) the $(**)$ square commutes by an argument as in step 1 (applied to $\{2\} \to \{1,2\} \to \{2\}$) shows that $R(F_{(0,0,1)})$ acts as F. Hence $R(F_{(1,0,i-1)})$ acts as F^i.

0.9.3. *The missing link.* We now justify the missing link, i.e., the commutativity with excursion for (0.4).

Indeed, for each i, the statement for

$$\Phi_{\{1,2\}}(U \boxtimes V) \xrightarrow{F_{(0,i)}} \Phi_{\{1,2\}}(U \boxtimes V)$$

follows from that for

$$\Phi_{\{1\}}(V) \xrightarrow{F^i} \Phi_{\{1\}}(V)$$

by defining $\widetilde{\mathcal{C}}_I := \mathcal{C}_{\{1\} \sqcup I}$ and $\widetilde{\Phi}_I : \mathbf{Rep}(G^I) \to \mathbf{Rep}(\mathbb{Z}^I, \widetilde{\mathcal{C}}_I)$ as

$$\widetilde{\Phi}_I(V^I) := \Phi_{\{1\} \sqcup I}(U \boxtimes V^I)$$

which again gives a compatible system of functors.

Remark 0.6. Note that even if $\{\Phi_I\}_{I \in \mathbf{fSet}}$ arose from a tensor functor between symmetric monoidal categories, $\{\widetilde{\Phi}_I\}_{I \in \mathbf{fSet}}$ will no longer form a tensor functor.