ON THE WEIGHTS OF MOD p HILBERT MODULAR FORMS

TOBY GEE

Abstract. We prove many cases of a conjecture of Buzzard, Diamond and Jarvis on the possible weights of mod p Hilbert modular forms, by making use of modularity lifting theorems and computations in p-adic Hodge theory.

Contents

1. Introduction

If a representation

$$\overline{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is continuous, odd, and irreducible, then a conjecture of Serre (now a theorem of Khare-Wintenberger and Kisin) predicts that $\overline{\rho}$ is modular. More precisely, Serre predicted a minimal weight $k(\overline{\rho})$ and a minimal level $N(\overline{\rho})$ for a modular form giving rise to $\overline{\rho}$.

It is natural to try to extend these results to totally real fields F. The natural generalisation of Serre's conjecture is to conjecture that if

$$\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is continuous, irreducible and totally odd, then it is modular (in the sense that it arises from a Hilbert modular form). It is straightforward to generalise the definition of $N(\overline{\rho})$ to this setting, and there has been much progress on "level-lowering" for Hilbert modular forms. It is, however, much harder to generalise the definition of $k(\overline{\rho})$. For example, there is no longer a total ordering on the weights, and the p-adic Hodge theory is much more complicated than in the classical case.

Suppose that p is unramified in F. Recently (see [BDJ05]), Buzzard, Diamond and Jarvis have proposed a conjectural set $W(\overline{\rho})$ of weights attached to $\overline{\rho}$, from which in the classical case one can deduce the weight part of Serre's conjecture (see [BDJ05] for more details). In this paper we prove many cases of a closely related conjecture (we work with a definite, rather than indefinite quaternion algebra; as we discuss below, it should be straightforward to prove the corresponding results in

2000 *Mathematics Subject Classification.* 11F33.

the setting of [BDJ05]). To be precise, a weight is an irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_2(\mathcal{O}_F/p)$, and such a representation factors as a tensor product

$$\otimes_{v|p}\sigma_{\vec{a},\vec{b}}$$

where $\vec{a}$, $\vec{b}$ are $[k_v:\mathbb{F}_p]$-tuples indexed by embeddings $\tau : k_v \hookrightarrow \overline{\mathbb{F}}_p$, and $0 \le a_\tau \le p-1$, $1 \le b_\tau \le p$. Then we say that a weight is *regular* if in fact $2 \le b_\tau \le p-2$ for all τ. Our main theorem requires a technical condition which we prefer to state later, that of a weight being partially ordinary of type I for $\overline{\rho}$, I a set of places of F dividing p; see section 2. Note that generically a weight is non-ordinary (that is to say, it is partially ordinary of type $\emptyset$).

Theorem. *Suppose that $\overline{\rho}$ is modular, that $p>2$, and that $\overline{\rho}(G_{F(\zeta_p)})$ is irreducible. Then if σ is a regular weight and $\overline{\rho}$ is modular of weight σ then $\sigma \in W(\overline{\rho})$. Conversely, if $\sigma \in W(\overline{\rho})$ and σ is non-ordinary for $\overline{\rho}$, then $\overline{\rho}$ is modular of weight σ. If σ is partially ordinary of type I for $\overline{\rho}$ and $\overline{\rho}$ has a partially ordinary modular lift of type I then $\overline{\rho}$ is modular of weight σ.*

Before we discuss the proof, we make some remarks about the assumptions in the theorem. The assumption that $\overline{\rho}$ is modular is essential to our methods. The assumption that $p>2$ seems to be crucial, as at present 2-adic Hodge theory is not available in sufficient generality. The assumption that $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible, and the assumption on partial ordinarity, are needed in order to apply $R=T$ theorems.

The main idea of our proof is the same as that for our proof of a companion forms theorem for totally real fields (see [Gee07]), namely that we use a lifting theorem to construct lifts of $\overline{\rho}$ satisfying certain local properties at places $v|p$, and then use a modularity lifting theorem of Kisin to prove that these representations are modular. In fact, Kisin's theorem is not general enough for our applications, and we need to use the main theorem of [Gee06]. The arguments are much more complicated than those in [Gee07] because we need to construct liftings with more delicate local properties; rather than just considering ordinary lifts, we must consider potentially Barsotti-Tate lifts of specified type.

The other complication which intervenes is that the connection between being modular of a certain weight and having a lift of a certain type is rather subtle, and this is the reason for our hypothesis that the weight be regular. One needs to consider many liftings for each weight, and we have only obtained the necessary combinatorial results in the case where the weight is regular. However, while these results appear to hold for most non-regular weights, there are cases where they do not hold, so it seems that it is not possible to give a general proof that the list of weights is correct by simply considering the types of potentially Barsotti-Tate lifts. It is possible to give a complete proof in the case where p splits completely in F, and we do this in [Gee08].

We now outline the structure of the paper. Rather than working with the "geometric" conventions of [BDJ05], we prefer to work with more "arithmetic" ones. In particular, we normalise the isomorphism of local class field theory so that a uniformiser corresponds to an arithmetic Frobenius element, and we work with automorphic forms on definite quaternion algebras. We set out our conventions in section 2, and we state the appropriate reformulation of the conjectures of [BDJ05] here. In section 3 we carry out the required local analysis in the case where the local representation is reducible. Sections 3.1 and 3.2 use Breuil modules and strongly divisible modules to determine when reducible representations arise as the generic fibres of certain finite flat group schemes. In section 3.4 we relate these finite flat group schemes to certain crystalline representations considered in [BDJ05], and in section 3.5 we prove the necessary combinatorial results relating types and regular weights.

We then repeat this analysis in the irreducible case in section 4, and finally in section 5 we combine these results with the lifting theorems mentioned above to deduce our main results. Firstly, we use our local results to show that if $\overline{\rho}$ is modular of weight σ with σ regular, then $\sigma \in W(\overline{\rho})$. For each regular weight $\sigma \in W(\overline{\rho})$ we then produce a modular lift of $\overline{\rho}$ which is potentially Barsotti-Tate of a specific type, so that $\overline{\rho}$ must be modular of some weight occurring in the mod p reduction of this type. We then check that σ is the only element of $W(\overline{\rho})$ occurring in this reduction, so that $\overline{\rho}$ is modular of weight σ, as required. In fact, we do not quite do this; the combinatorics is slightly more involved, and we are forced to make use of a notion of a "weakly regular" weight. See section 5 for the details.

It is a pleasure to thank Fred Diamond for numerous helpful discussions regarding this work; without his patient advice this paper could never have been written. We would like to thank David Savitt for pointing out several errors and omissions in an earlier version of this paper, and for writing [Sav08]. We would also like to thank Florian Herzig for pointing out an inconsistency between our conventions and those of [BDJ05], which led to the writing of section 2. We are extremely grateful to Xavier Caruso and Christophe Breuil for their many helpful comments and corrections; in particular, the material in section 3.4 owes a considerable debt to Caruso's efforts to correct a number of misunderstandings and inaccuracies, and the proof of Lemma 3.3.2 is based on an argument of his. We would also like to that the anonymous referee for a careful reading, and for pointing out a number of serious errors in an earlier version of the paper.

2. Definitions

2.1. Rather than use the conventions of [BDJ05], we choose to state a closely related variant of their conjectures by working on totally definite quaternion algebras. This formulation is more suited to applications to modularity lifting theorems, and indeed to the application of modularity lifting theorems to proving cases of the conjecture.

We begin by recalling some standard facts from the theory of quaternionic modular forms; see either [Tay06], section 3 of [Kis08a] or section 2 of [Kis08b] for more details, and in particular the proofs of the results claimed below. We will follow Kisin's approach closely. Let F be a totally real field in which p is unramified, and let D be a quaternion algebra with center F which is ramified at all infinite places of F and at a set Σ of finite places, which contains no places above p. Fix a maximal order $\mathcal{O}_D$ of D and for each finite place $v \notin \Sigma$ fix an isomorphism $(\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$. For any finite place v let π_v denote a uniformiser of F_v.

Let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact subgroup, with each $U_v \subset (\mathcal{O}_D)_v^\times$. Furthermore, assume that $U_v = (\mathcal{O}_D)_v^\times$ for all $v \in \Sigma$, and that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ if $v|p$.

Take A a topological $\mathbb{Z}_p$-algebra. For each $v|p$, fix a continuous representation $\sigma_v : U_v \to \mathrm{Aut}(W_{\sigma_v})$ with W_{σ_v} a finite free A-module. Write $W_\sigma = \otimes_{v|p,A} W_{\sigma_v}$ and let $\sigma = \prod_{v|p} \sigma_v$. We regard σ as a representation of U in the obvious way (that is, we let U_v act trivially if $v \nmid p$). Fix also a character $\psi : (\mathbb{A}_F^f)^\times / F^\times \to A^\times$ such that for any place v of F, $\sigma|_{U_v\cap\mathcal{O}_{F_v}^\times}$ is multiplication by ψ^{-1}. Then we can think of W_σ as a $U(\mathbb{A}_F^f)^\times$-module by letting $(\mathbb{A}_F^f)^\times$ act via ψ^{-1}.

Let $S_{\sigma,\psi}(U,A)$ denote the set of continuous functions

$$f : D^\times\backslash(D\otimes_F\mathbb{A}_F^f)^\times \to W_\sigma$$

such that for all $g\in(D\otimes_F\mathbb{A}_F^f)^\times$ we have

$$f(gu)=\sigma(u)^{-1}f(g) \text{ for all } u\in U,$$

$$f(gz)=\psi(z)f(g) \text{ for all } z\in(\mathbb{A}_F^f)^\times.$$

We can write $(D\otimes_F\mathbb{A}_F^f)^\times=\coprod_{i\in I}D^\times t_iU(\mathbb{A}_F^f)^\times$ for some finite index set I and some $t_i\in(D\otimes_F\mathbb{A}_F^f)^\times$. Then we have

$$S_{\sigma,\psi}(U,A)\xrightarrow{\sim}\oplus_{i\in I}W_\sigma^{(U(\mathbb{A}_F^f)^\times\cap t_i^{-1}D^\times t_i)/F^\times},$$

the isomorphism being given by the direct sum of the maps $f\mapsto\{f(t_i)\}$. From now on we make the following assumption:

$$\text{For all } t\in(D\otimes_F\mathbb{A}_F^f)^\times \text{ the group } (U(\mathbb{A}_F^f)^\times\cap t^{-1}D^\times t)/F^\times=1.$$

One can always replace U by a subgroup (obeying the assumptions above) for which this holds (c.f. section 3.1.1 of [Kis07]). Under this assumption, which we make from now on, $S_{\sigma,\psi}(U,A)$ is a finite projective A-module, and the functor $W_\sigma\mapsto S_{\sigma,\psi}(U,A)$ is exact in W_σ.

We now define some Hecke algebras. Let S be a set of finite places containing Σ, the places dividing p, and the primes of F such that U_v is not a maximal compact subgroup of $D_v^\times$. Let $\mathbb{T}^{\mathrm{univ}}_{S,A}=A[T_v]_{v\notin S}$ be the commutative polynomial ring in the formal variables T_v. Consider the left action of $(D\otimes_F\mathbb{A}_F^f)^\times$ on W_σ-valued functions on $(D\otimes_F\mathbb{A}_F^f)^\times$ given by $(gf)(z)=f(zg)$. For each finite place v of F we fix a uniformiser π_v of F_v. Then we make $S_{\sigma,\psi}(U,A)$ a $\mathbb{T}^{\mathrm{univ}}_{S,A}$-module by letting T_v act via the double coset $U\left(\begin{smallmatrix}\pi_v&0\\0&1\end{smallmatrix}\right)U$. These are independent of the choices of π_v. We will write $\mathbb{T}_{\sigma,\psi}(U,A)$ or $\mathbb{T}_{\sigma,\psi}(U)$ for the image of $\mathbb{T}^{\mathrm{univ}}_{S,A}$ in $\operatorname{End}S_{\sigma,\psi}(U,A)$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S,A}$. We say that $\mathfrak{m}$ is in the support of (σ,ψ) if $S_{\sigma,\psi}(U,A)_\mathfrak{m}\neq0$. Now let $\mathcal{O}$ be the ring of integers in $\overline{\mathbb{Q}}_p$, with residue field $\mathbb{F}=\overline{\mathbb{F}}_p$, and suppose that $A=\mathcal{O}$ in the above discussion, and that σ has open kernel. Consider a maximal ideal $\mathfrak{m}\subset\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$ which is induced by a maximal ideal of $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$. Then there is a semisimple Galois representation $\overline{\rho}_\mathfrak{m}:G_F\to\mathrm{GL}_2(\mathbb{F})$ associated to $\mathfrak{m}$ which is characterised up to equivalence by the property that if $v\notin S$ and Frob_v is an arithmetic Frobenius at v, then the trace of $\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ is the image of T_v in $\mathbb{F}$.

We are now in a position to define what it means for a Galois representation to be modular of some weight. Let F_v have ring of integers $\mathcal{O}_v$ and residue field k_v, and let σ be an irreducible $\overline{\mathbb{F}}_p$-representation of $G:=\prod_{v|p}\mathrm{GL}_2(k_v)$. We also denote by σ the representation of $\prod_{v|p}\mathrm{GL}_2(\mathcal{O}_v)$ induced by the surjections $\mathcal{O}_v\twoheadrightarrow k_v$.

Definition 2.1.1. We say that an irreducible representation $\overline{\rho}:G_F\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is modular of weight σ if for some D, S, U, ψ, and $\mathfrak{m}$ as above we have $S_{\sigma,\psi}(U,\mathbb{F})_\mathfrak{m}\neq0$ and $\overline{\rho}_\mathfrak{m}\otimes\overline{\mathbb{F}}_p\cong\overline{\rho}$.

We now show how one can gain information about the weights associated to a particular Galois representation by considering lifts to characteristic zero.

Lemma 2.1.2. *Let $\psi : F^\times \backslash (\mathbb{A}_F)^\times \to \mathcal{O}^\times$ be a continuous character, and write $\overline{\psi}$ for the composite of ψ with the projection $\mathcal{O}^\times \to \mathbb{F}^\times$. Fix a representation σ on a finite free $\mathcal{O}$-module W_σ which corresponds as above to a choice of Galois type for each $v|p$, and an irreducible representation σ' on a finite free $\mathbb{F}$-module $W_{\sigma'}$. Suppose that for each $v|p$ we have $\sigma|_{U_v \cap \mathcal{O}_{F_v}^\times} = \psi^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$ and $\sigma'|_{U_v \cap \mathcal{O}_{F_v}^\times} = \overline{\psi}^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times}$*

Let $\mathfrak{m}$ be a maximal ideal of either $\mathbb{T}_{S,\mathcal{O}}^{univ}$ or $\mathbb{T}_{S^p,\mathcal{O}}^{univ}$.

Suppose that $W_{\sigma'}$ occurs as a $\prod_{v|p} U_v$-module subquotient of $W_{\overline{\sigma}} := W_\sigma \otimes \mathbb{F}$. If $\mathfrak{m}$ is in the support of $(\sigma', \overline{\psi})$, then $\mathfrak{m}$ is in the support of (σ, ψ).

Conversely, if $\mathfrak{m}$ is in the support of (σ, ψ), then $\mathfrak{m}$ is in the support of $(\sigma', \overline{\psi})$ for some irreducible $\prod_{v|p} U_v$-module subquotient $W_{\sigma'}$ of $W_{\overline{\sigma}}$.

Proof. The first part is proved just as in Lemma 3.1.4 of [Kis08a], and the second part follows from Proposition 1.2.3 of [AS86]. □

We note a special case of this result, relating the existence of potentially Barsotti-Tate lifts of a particular tame type to information about Serre weights. Firstly, we recall some particular representations of $\mathrm{GL}_2(k_v)$. For any pair of distinct characters $\chi_1, \chi_2 : k_v^\times \to \mathcal{O}^\times$ we let $I(\chi_1, \chi_2)$ denote the irreducible $(q+1)$-dimensional $\overline{\mathbb{Q}}_p$-representation of $\mathrm{GL}_2(k_v)$ induced from the character of B (the upper triangular matrices in $\mathrm{GL}_2(k_v)$) given by

$$\begin{pmatrix} x & w \\ 0 & y \end{pmatrix} \mapsto \chi_1(x)\chi_2(y).$$

We let σ_{χ_1,χ_2} denote the representation of $\mathrm{GL}_2(k_v)$ on an $\mathcal{O}$-lattice in $I(\chi_1, \chi_2)$; we also regard this as a representation of $\mathrm{GL}_2(\mathcal{O}_v)$ via the natural projection. Let $\tau(\sigma_{\chi_1,\chi_2})$ be the inertial type $\chi_1 \oplus \chi_2$ (regarded as a representation of I_{F_v} via class field theory, normalised so that a uniformiser corresponds to a geometric Frobenius element).

Let k_v' be the quadratic extension of k_v. For any character $\theta : k_v'^\times \to \mathcal{O}^\times$ which does not factor through the norm $k_v'^\times \to k^\times$, there is an irreducible $(q-1)$-dimensional cuspidal representation $\Theta(\theta)$ of $\mathrm{GL}_2(k)$ (see Section 1 of [Dia05] for the definition of $\Theta(\theta)$). Let $\sigma_{\Theta(\theta)}$ denote the representation of $\mathrm{GL}_2(k_v)$ on an $\mathcal{O}$-lattice in $\Theta(\theta)$; we also regard this as a representation of $\mathrm{GL}_2(\mathcal{O}_v)$ via the natural projection. Let q_v be the cardinality of K_v, and let $\tau(\sigma_{\Theta(\theta)})$ be the inertial type $\theta \oplus \theta^{q_v}$ (again regarded as a representation of I_{F_v} via class field theory).

Lemma 2.1.3. *For each $v|p$, fix a representation σ_v of the type just considered (that is, isomorphic to σ_{χ_1,χ_2} or to $\sigma_{\Theta(\theta)}$). Let $\tau_v = \tau(\sigma_v)$ be the corresponding inertial type. Suppose that $\overline{\rho}$ is modular of weight σ, and that σ is a $\prod_{v|p} \mathrm{GL}_2(k_v)$-subquotient of $\otimes_{v|p} \sigma_v \otimes_{\mathcal{O}} \mathbb{F}$. Then $\overline{\rho}$ lifts to a modular Galois representation which is potentially Barsotti-Tate of type τ_v for each $v|p$.*

Conversely, suppose that $\overline{\rho}$ lifts to a modular Galois representation which is potentially Barsotti-Tate of type τ_v for each $v|p$. Then $\overline{\rho}$ is modular of weight σ for some $\prod_{v|p} \mathrm{GL}_2(k_v)$-subquotient σ of $\otimes_{v|p} \sigma_v \otimes_{\mathcal{O}} \mathbb{F}$.

Proof. This follows from Lemma 2.1.2, the Jacquet-Langlands correspondence, and the compatibility of the local and global Langlands correspondences at places dividing p (see [Kis08b]).

□

We now state a conjecture on Serre weights, following [BDJ05]. Note that our conjecture is only valid for regular weights; there are some additional complications when dealing with non-regular weights. Let $\overline{\rho}: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be modular. We propose a conjectural set of regular weights $W(\overline{\rho})$ for $\overline{\rho}$. Our formulation follows that of [BDJ05].

In fact, for each place $v|p$ we propose a set of weights $W(\overline{\rho}|_{G_{F_v}})$, and we define

$$W(\overline{\rho}) := \left\{\otimes_{v|p}\sigma_v | \sigma_v \in W(\overline{\rho}|_{G_{F_v}})\right\}.$$

Let S_v be the set of embeddings $k_v \hookrightarrow \overline{\mathbb{F}}_p$. A weight for $\mathrm{GL}_2(k_v)$ is an isomorphism class of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(k_v)$, which automatically contains one of the form

$$\sigma_{\vec{a},\vec{b}} = \otimes_{\tau\in S_v} \det{}^{a_\tau} \mathrm{Sym}^{b_\tau - 1} k_v^2 \otimes_\tau \overline{\mathbb{F}}_p,$$

with $0 \le a_\tau \le p-1$ and $1 \le b_\tau \le p$ for each $\tau \in S_v$. We demand further that some $a_\tau < p-1$, in which case the representations $\sigma_{\vec{a},\vec{b}}$ are pairwise non-isomorphic.

Definition 2.1.4. We say that a weight $\sigma_{\vec{a},\vec{b}}$ is *regular* if $2 \le b_\tau \le p-2$ for all τ. We say that it is *weakly regular* if $1 \le b_\tau \le p-1$ for all τ.

For each $\tau \in S_v$ we have the fundamental character ω_τ of I_{F_v} given by composing τ with the homomorphism $I_{F_v} \to k_v^\times$ given by local class field theory, normalised so that uniformisers correspond to geometric Frobenius elements. Let k_v' denote the quadratic extension of k_v. Let S_v' denote the set of embeddings $\sigma : k_v' \hookrightarrow \overline{\mathbb{F}}_p$, and let ω_σ denote the fundamental character corresponding to σ.

Suppose firstly that $\overline{\rho}|_{G_{F_v}}$ is irreducible. There is a natural $2-1$ map $\pi : S_v' \to S_v$ given by restriction to k_v, and we say that a subset $J \subset S_v'$ is a *full subset* if $|J| = |\pi(J)| = |S_v|$. Then we have

Definition 2.1.5. Let $\sigma_{\vec{a},\vec{b}}$ be a regular weight for $\mathrm{GL}_2(k_v)$. Then $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})$ if and only if there exists a full subset $J \subset S_v'$ such that

$$\overline{\rho}|_{I_{F_v}} \sim \prod_{\tau\in S_v} \omega_\tau^{a_\tau} \begin{pmatrix} \prod_{\sigma\in J} \omega_\sigma^{b_{\sigma|_{k_v}}} & 0 \\ 0 & \prod_{\sigma\notin J} \omega_\sigma^{b_{\sigma|_{k_v}}} \end{pmatrix}.$$

Suppose now that $\overline{\rho}|_{G_{F_v}}$ is reducible, say $\overline{\rho}|_{G_{F_v}} \sim \left(\begin{smallmatrix} \psi_1 & * \\ 0 & \psi_2 \end{smallmatrix}\right)$. We define the set $W(\overline{\rho}|_{G_{F_v}})$ in two stages. Firstly, define a set $W(\overline{\rho}|_{G_{F_v}})'$ of regular weights as follows.

Definition 2.1.6. A regular weight $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})'$ if and only if there exists $J \subset S_v$ such that $\psi_1|_{I_{F_v}} = \prod_{\tau\in S_v} \omega_\tau^{a_\tau} \prod_{\tau\in J} \omega_\tau^{b_\tau}$ and $\psi_2|_{I_{F_v}} = \prod_{\tau\in S_v} \omega_\tau^{a_\tau} \prod_{\tau\notin J} \omega_\tau^{b_\tau}$. We say that $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})'$ is *ordinary for* $\overline{\rho}$ if furthermore $J = S_v$ or $J = \emptyset$.

Suppose that we have a regular weight $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})'$ and a corresponding subset $J \subset S_v$. We now define crystalline lifts $\widetilde{\psi}_1$, $\widetilde{\psi}_2$ of ψ_1, ψ_2. If ψ is a crystalline character of G_{F_v} and $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p$ we say that the Hodge-Tate weight of ψ with respect to τ is the i for which $gr^{-i}((\psi \otimes_{\mathbb{Q}_p} B_{dR})^{G_{F_v}} \otimes_{\overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} F_v, 1\otimes\tau} \overline{\mathbb{Q}}_p) \neq 0$. Then we demand that for some Frobenius element Frob_p of G_{F_v}, $\widetilde{\psi}_i(\mathrm{Frob}_p)$ is the Teichmüller lift of $\psi_i(\mathrm{Frob}_p)$, and that:

- $\widetilde{\psi}_1$ is crystalline, and the Hodge-Tate weight of $\widetilde{\psi}_1$ with respect to τ is $a_\tau + b_\tau$ if $\tau \in J$, and a_τ if $\tau \notin J$.

- $\widetilde{\psi}_2$ is crystalline, and the Hodge-Tate weight of $\widetilde{\psi}_2$ with respect to τ is $a_\tau + b_\tau$ if $\tau \notin J$, and a_τ if $\tau \in J$.

The existence and uniqueness of $\widetilde{\psi}_1$, $\widetilde{\psi}_2$ is straightforward (see [BDJ05]). Then we have

Definition 2.1.7. $\sigma_{\vec{a},\vec{b}} \in W(\overline{\rho}|_{G_{F_v}})$ if and only if $\overline{\rho}|_{G_{F_v}}$ has a lift to a crystalline representation $\left(\begin{smallmatrix} \widetilde{\psi}_1 & * \\ 0 & \widetilde{\psi}_2 \end{smallmatrix}\right)$.

For future reference, we say that a weight σ is I-ordinary for $\overline{\rho}$ if I is the set of places $v|p$ for which σ_v is ordinary for $\overline{\rho}$. We say that $\overline{\rho}$ has an I-ordinary modular lift if it has a potentially Barsotti-Tate modular lift which is potentially ordinary at precisely the places in I.

2.2. Relation to the Buzzard-Diamond-Jarvis conjecture. Our conjectured sets of regular weights are exactly the same as the regular weights predicted in [BDJ05]. However, they work with indefinite quaternion algebras rather than the definite ones of this paper, and in the absence of a mod p Jacquet-Langlands correspondence our results do not automatically prove cases of their conjectures. That said, our arguments are for the most part purely local, with the only global input being in characteristic zero, where one does have a Jacquet-Langlands correspo-nence. In particular, given the analogue of Lemma 2.1.2 in the setting of [BDJ05], which we anticipate will be present in the final version of that paper, our arguments will go over unchanged to their setting.

3. Local analysis - the reducible case

3.1. Breuil Modules. Let $p > 2$ be prime, let k be a finite extension of $\mathbb{F}_p$, let $K_0 = W(k)[1/p]$, and let K be a finite Galois totally tamely ramified extension of K_0, of degree e. Assume that there is a uniformiser π of $\mathcal{O}_K$ such that $\pi^e \in M$, where M is a subfield of K_0, and fix such a π. Since K/M is tamely ramified, the category of Breuil modules with coefficients and descent data is easy to describe (see [Sav08]). Let $k \in [2, p-1]$ be an integer. Let E be a finite extension of $\mathbb{F}_p$. The category $\mathrm{BrMod}_{dd,M}^{k-1}$ consists of quintuples $(\mathcal{M}, \mathcal{M}_{k-1}, \phi_{k-1}, \hat{g}, N)$ where:

- $\mathcal{M}$ is a finitely generated $(k \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module, free over $k[u]/u^{ep}$.
- $\mathcal{M}_{k-1}$ is a $(k \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-submodule of $\mathcal{M}$ containing $u^{e(k-1)}\mathcal{M}$.
- $\phi_{k-1} : \mathcal{M}_{k-1} \to \mathcal{M}$ is E-linear and ϕ-semilinear (where $\phi : k[u]/u^{ep} \to k[u]/u^{ep}$ is the p-th power map) with image generating $\mathcal{M}$ as a $(k \otimes_{\mathbb{F}_p} E)[u]/u^{ep}$-module.
- $N : \mathcal{M} \to \mathcal{M}$ is $(k \otimes_{\mathbb{F}_p} E)$-linear and satisfies $N(ux) = uN(x) - ux$ for all $x \in \mathcal{M}$, $u^e N(\mathcal{M}_{k-1}) \subset \mathcal{M}_{k-1}$, and $\phi_{k-1}(u^e N(x)) = N(\phi_{k-1}(x))$ for all $x \in \mathcal{M}_{k-1}$.
- $\hat{g} : \mathcal{M} \to \mathcal{M}$ are additive bijections for each $g \in \mathrm{Gal}(K/M)$, preserving $\mathcal{M}_{k-1}$, commuting with the ϕ-, E-, and N-actions, and satisfying $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/M)$, and $\hat{1}$ is the identity. Furthermore, if $a \in k \otimes_{\mathbb{F}_p} E$, $m \in \mathcal{M}$ then $g(au^i m) = g(a)((g(\pi)/\pi)^i \otimes 1)u^i g(m)$.

We will omit the M from the notation in the case $M = K_0$. We write $\mathrm{BrMod}_{dd,M} = \mathrm{BrMod}_{dd,M}^1$. The category $\mathrm{BrMod}_{dd,M}$ is equivalent to the category of finite flat group schemes over $\mathcal{O}_K$ together with an E-action and descent data on the generic fibre from K to M (this equivalence depends on π). In this case it follows from the other axioms that there is always a unique N with the required properties, and we will frequently omit the details of this operator when we are working in the case $k=2$. In section 3.4 we will also use the case $k=p-2$, and here we will make the operators N explicit.

We choose in this paper (except in section 3.4) to adopt the conventions of [BM02] and [Sav05], rather than those of [BCDT01]; thus rather than working with the usual contravariant equivalence of categories, we work with a covariant version of it, so that our formulae for generic fibres will differ by duality and a twist from those following the conventions of [BCDT01]. To be precise, we obtain the associated G_L-representation (which we will refer to as the generic fibre) of an object of BrMod_{dd} via the functor $T_{st,2}^M$.

Let $\rho : G_{K_0} \to \mathrm{GL}_2(E)$ be a continuous representation. We assume from now on that E contains k. Suppose for the rest of this section that ρ is reducible but not scalar, say $\rho \sim \left(\begin{smallmatrix}\psi_1 & * \\ 0 & \psi_2\end{smallmatrix}\right)$. Fix $\pi = (-p)^{1/(p^r-1)}$, where $r=[k:\mathbb{F}_p]$, and fix $K = K_0(\pi)$, so that π is a uniformiser of $\mathcal{O}_K$, the ring of integers of K. By class field theory $\psi_1|_{I_K}, \psi_{2|_{I_K}}$ are trivial.

We fix some general notation for elements of BrMod_{dd}. Let S denote the set of embeddings $\tau : k \hookrightarrow E$. We have an isomorphism $k \otimes_{\mathbb{F}_p} E \xrightarrow{\sim} \oplus_S E_\tau$, where $E_\tau := k \otimes_{k,\tau} E$, and we let ϵ_τ denote the idempotent corresponding to the embedding τ. Then any element $\mathcal{M}$ of BrMod_{dd} can be decomposed into $E[u]/u^{ep}$-modules $\mathcal{M}^\tau := \epsilon_\tau \mathcal{M}$, $\tau \in S$, so that $\hat{g} : \mathcal{M}^\tau \to \mathcal{M}^\tau$, and $\phi_1 : \mathcal{M}_1^\tau \to \mathcal{M}^{\tau\circ\phi^{-1}}$. We now write $S = \{\tau_1, \ldots, \tau_r\}$, numbered so that $\tau_{i+1} = \tau_i \circ \phi^{-1}$, where we identify τ_{r+1} with τ_1. In fact, it will often be useful to consider the indexing set of S to be $\mathbb{Z}/r\mathbb{Z}$, and we will do so without further comment.

Fix $J \subset S$. We wish to single out particular representations ρ depending on J. Firstly, we need some notation. Recall that (as in appendix B of [CDT99]) if $\rho' : G_{K_0} \to \mathrm{GL}_2(\mathcal{O}_L)$ is potentially Barsotti-Tate, where L is a finite extension of $W(E)[1/p]$, then there is a Weil-Deligne representation $WD(\rho') : W_{K_0} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, and we say that ρ' has type $WD(\rho')|_{I_{K_0}}$.

Definition 3.1.1. We say that ρ has a *lift of type* J if there is a representation $\rho' : G_{K_0} \to \mathrm{GL}_2(\mathcal{O}_L)$ lifting ρ, where L is a finite extension of $W(E)[1/p]$, such that ρ' becomes Barsotti-Tate over K, with $\varepsilon^{-1}\det\rho'$ equal to the Teichmüller lift of $\varepsilon^{-1}\det\rho$ (with ε denoting the cyclotomic character) and $\tau(\rho') \sim \widetilde{\psi}_1|_{I_{K_0}} \prod_{\tau\in J} \widetilde{\omega}_\tau^{-p} \oplus \widetilde{\psi}_2|_{I_{K_0}} \prod_{\tau\notin J} \widetilde{\omega}_\tau^{-p}$. Here a tilde denotes the Teichmüller lift.

Definition 3.1.2. For any subset $H \subset S$, we say that an element $\mathcal{M}$ of BrMod_{dd} is of class H if for all $\tau \in S$ we can choose a basis e_τ of $\mathcal{M}^\tau$ such that $\mathcal{M}_1^\tau$ is generated by $u^{j_\tau} e_\tau$, where

$$j_\tau = \begin{cases} 0 \text{ if } \tau\circ\phi^{-1} \notin H \\ e \text{ if } \tau\circ\phi^{-1} \in H \end{cases}$$

Definition 3.1.3. We say that an element $\mathcal{M}$ of BrMod_{dd} is of type J if $\mathcal{M}$ is an extension of an element of class J^c by an element of class J, and we say that ρ has a model of type J if there is an element of BrMod_{dd} of type J with generic fibre ρ.

We will also refer to finite flat group schemes with descent data as being of class J or of type J if they correspond to Breuil modules with descent data of this kind. The notions of having a model of type J, and having a lift of type J are closely related, although not in general equivalent. We will see in section 3.2 that in sufficiently generic cases, if ρ has a model of type J then it has a lift of type J, and in section 3.5 we prove a partial converse (see Proposition 3.5.5).

3.2. Strongly divisible modules. In this section we prove that if ρ has a model of type J then it has a lift of type J. We begin by recalling the definition and basic properties of strongly divisible modules from [Sav05]. For the purpose of giving these definitions we return briefly to the general setting of K_0 an unramified finite extension of $\mathbb{Q}_p$ and K a totally tamely ramified Galois extension of K_0 of degree e, with uniformiser π, satisfying $\pi^e \in M$ for some subfield M of K_0.

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and residue field E. Let S_K be the ring

$$\left\{ \sum_{j=0}^{\infty} r_j \frac{u^j}{\lfloor j/e \rfloor !},\ r_j \in W(k),\ r_j \to 0 \ \mathfrak{m}_R\text{-adically as } j \to \infty \right\},$$

and let $S_{K,\mathcal{O}_L} = S_K \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. Let $\mathrm{Fil}^1 S_{K,\mathcal{O}_L}$ be the $\mathfrak{m}_R$-adic completion of the ideal generated by $E(u)^j/j!$, $j \geq 1$, where $E(u)$ is the minimal polynomial of π over K_0. Let $\phi : S_{K,\mathcal{O}_L} \to S_{K,\mathcal{O}_L}$ be the unique R-linear, $W(k)$-semilinear ring homomorphism with $\phi(u) = u^p$, and let N be the unique $W(k)\otimes R$-linear derivation such that $N(u) = -u$ (so that $N\phi = p\phi N$). One can check that $\phi(\mathrm{Fil}^1 S_{K,\mathcal{O}_L}) \subset pS_{K,\mathcal{O}_L}$, and we define $\phi_1 : \mathrm{Fil}^1 S_{K,\mathcal{O}_L} \to S_{K,\mathcal{O}_L}$ by $\phi_1 = (\phi|_{\mathrm{Fil}^1 S_{K,\mathcal{O}_L}})/p$. One can check (see section 4 of [Sav05]) that if I is an ideal of $\mathcal{O}_L$, then $IS_{K,\mathcal{O}_L} \cap \mathrm{Fil}^1 S_{K,\mathcal{O}_L} = I \,\mathrm{Fil}^1 S_{K,\mathcal{O}_L}$. We give S_K an action of $\mathrm{Gal}(K/M)$ via ring isomorphisms via the usual action on $W(k)$, and by letting $\hat{g}(u) = (g(\pi)/\pi)u$.

We now define the category $\mathcal{O}_L - \mathrm{Mod}_{cris,dd,M}$, the category of strongly divisible $\mathcal{O}_L$-modules with descent data from K to M.

Definition 3.2.1. A strongly divisible $\mathcal{O}_L$-module with descent data from K to M is a finitely generated free $S_{K,\mathcal{O}_L}$-module $\mathcal{M}$, together with a sub-$S_{K,\mathcal{O}_L}$-module $\mathrm{Fil}^1 \mathcal{M}$ and a map $\phi : \mathcal{M} \to \mathcal{M}$, and additive bijections $\hat{g} : \mathcal{M} \to \mathcal{M}$ for each $g \in \mathrm{Gal}(K/M)$, satisfying the following conditions:

1. $\mathrm{Fil}^1 \mathcal{M}$ contains $(\mathrm{Fil}^1 S_{K,\mathcal{O}_L})\mathcal{M}$,
2. $\mathrm{Fil}^1 \mathcal{M} \cap I\mathcal{M} = I \,\mathrm{Fil}^1 \mathcal{M}$ for all ideals I in $\mathcal{O}_L$,
3. $\phi(sx) = \phi(s)\phi(x)$ for $s \in S_{K,\mathcal{O}_L}$ and $x \in \mathcal{M}$,
4. $\phi(\mathrm{Fil}^1 \mathcal{M})$ is contained in $p\mathcal{M}$ and generates it over $S_{K,\mathcal{O}_L}$,
5. $\hat{g}(sx) = \hat{g}(s)\hat{g}(x)$ for all $s \in S_{K,\mathcal{O}_L}$, $x \in \mathcal{M}$, $g \in \mathrm{Gal}(K/M)$,
6. $\hat{g}_1 \circ \hat{g}_2 = \widehat{g_1 \circ g_2}$ for all $g_1, g_2 \in \mathrm{Gal}(K/M)$,
7. $\hat{g}(\mathrm{Fil}^1 \mathcal{M}) \subset \mathrm{Fil}^1 \mathcal{M}$ for all $g \in \mathrm{Gal}(K/M)$, and
8. ϕ commutes with $\hat{g}$ for all $g \in \mathrm{Gal}(K/M)$.

Note that it is not immediately obvious that this definition is equivalent to Definition 4.1 of [Sav05], as we have made no mention of the operator N of *loc. cit.* However, since $\mathcal{O}_L$ is finite over $\mathbb{Z}_p$, it follows from part (1) of Proposition 5.1.3 of [Bre00] that any such operator N is unique. The existence of an operator N satisfying all of the conditions of Definition 4.1 of [Sav05] except possibly for $\mathcal{O}_L$-linearity follows from the argument at the beginning of section 3.5 of [Sav05]. To check $\mathcal{O}_L$-linearity it is enough (by $\mathbb{Z}_p$-linearity) to check that N is compatible with the action of the units in $\mathcal{O}_L$, but this is clear from the uniqueness of N.

By Proposition 4.13 of [Sav05] (and the remarks immediately preceding it), there is a functor $T^M_{st,2}$ from the category $\mathcal{O}_L-\mathrm{Mod}^1_{cris,dd,M}$ to the category of G_M-stable $\mathcal{O}_L$-lattices in representations of G_M which become Barsotti-Tate on restriction to G_K. This functor preserves dimensions in the obvious sense.

Recall also from section 4.1 of [Sav05]) that there is a functor T_0, compatible with $T^M_{st,2}$, from $\mathcal{O}_L-\mathrm{Mod}^1_{cris,dd,M}$ to $\mathrm{BrMod}_{dd,M}$. The functor T_0 is given by $\mathcal{M}\mapsto(\mathcal{M}/\mathfrak{m}_L\mathcal{M})\otimes k[u]/u^{ep}$.

3.3. Models of type J. We now wish to discuss the relationships between models of type J and lifts of type J. With an eye to our future applications, we will often make a simplifying assumption.

Definition 3.3.1. Say that ρ is *J-regular* if $\psi_1\psi_2^{-1}|_{I_{K_0}}=\prod_{\tau\in J}\omega_\tau^{b_\tau}\prod_{\tau\in J^c}\omega_\tau^{-b_\tau}$ for some $2\le b_\tau\le p-2$.

Suppose now that ρ has a model of type J. Recall that this means that, with the notation of Section 3.1, we can write down a Breuil module $\mathcal{M}$ with descent data whose generic fibre is ρ, which is an extension of a Breuil module with descent data $\mathcal{B}$ by a Breuil module with descent data $\mathcal{A}$, where $\mathcal{A}$ is of class J and $\mathcal{B}$ is of class J^c. Let ψ_i' denote the unique extension of $\psi_i|_{I_{K_0}}$ to a character of $\mathrm{Gal}(K/K_0)$. By Theorem 3.5 and Example 3.7 of [Sav08] we see that we can choose bases for $\mathcal{A}$ and $\mathcal{B}$ so that they take the following form:

$$\mathcal{A}^{\tau_i}=E[u]/u^{ep}\cdot e_{\tau_i}$$
$$\mathcal{A}_1^{\tau_i}=E[u]/u^{ep}\cdot u^{j_{\tau_i}}e_{\tau_i}$$
$$\phi_1(u^{j_{\tau_i}}e_{\tau_i})=(a^{-1})_i e_{\tau_{i+1}}$$
$$\hat{g}(e_{\tau_i})=\left(\left(\psi_1'\prod_{\sigma\in J}\omega_\sigma^{-p}\right)(g)\right)e_{\tau_i}$$
$$\mathcal{B}^{\tau_i}=E[u]/u^{ep}\cdot\overline{f}_{\tau_i}$$
$$\mathcal{B}_1^{\tau_i}=E[u]/u^{ep}\cdot u^{e-j_{\tau_i}}\overline{f}_{\tau_i}$$
$$\phi_1(u^{e-j_{\tau_i}}\overline{f}_{\tau_i})=(b^{-1})_i\overline{f}_{\tau_{i+1}}$$
$$\hat{g}(\overline{f}_{\tau_i})=\left(\left(\psi_2'\prod_{\sigma\notin J}\omega_\sigma^{-p}\right)(g)\right)\overline{f}_{\tau_i}$$

where a, $b\in E^\times$, the notation $(x)_i$ means x if $i=1$ and 1 otherwise, and

$$j_{\tau_i}=\begin{cases} e \text{ if } \tau_{i+1}\in J\\ 0 \text{ if } \tau_{i+1}\notin J.\end{cases}$$

We now seek to choose a basis for $\mathcal{M}$ extending the basis $\{e_\tau\}$ for $\mathcal{A}$. Such a basis will be given by lifting the $\overline{f}_\tau$ to elements f_τ (where we mean lifting under the map $e_\tau\mapsto 0$).

Lemma 3.3.2. *Assume that ρ is J-regular and has a model of type J. Then for some choice of basis, we can write*

$$\mathcal{M}^{\tau_i}=E[u]/u^{ep}\cdot e_{\tau_i}+E[u]/u^{ep}\cdot f_{\tau_i}$$
$$\mathcal{M}_1^{\tau_i}=E[u]/u^{ep}\cdot u^{j_{\tau_i}}e_{\tau_i}+E[u]/u^{ep}\cdot(u^{e-j_{\tau_i}}f_{\tau_i}+\lambda_{\tau_i}u^{i_{\tau_i}}e_{\tau_i})$$
$$\phi_1(u^{j_{\tau_i}}e_{\tau_i})=(a^{-1})_i e_{\tau_{i+1}}$$
$$\phi_1(u^{e-j_{\tau_i}}f_{\tau_i}+\lambda_{\tau_i}u^{i_{\tau_i}}e_{\tau_i})=(b^{-1})_i f_{\tau_{i+1}}$$

$$\hat{g}(e_{\tau_i}) = \left(\left(\psi_1' \prod_{\sigma\in J} \omega_\sigma^{-p}\right)(g)\right) e_{\tau_i}$$

$$\hat{g}(f_{\tau_i}) = \left(\left(\psi_2' \prod_{\sigma\notin J} \omega_\sigma^{-p}\right)(g)\right) f_{\tau_i}$$

where $\lambda_{\tau_i} \in E$, with $\lambda_{\tau_i} = 0$ if $\tau_{i+1} \notin J$, the i_{τ_i} are such that $\mathcal{M}_1$ is Galois-stable and $0 \le i_{\tau_i} \le e-1$, and

$$j_{\tau_i} = \begin{cases} e \text{ if } \tau_{i+1} \in J \\ 0 \text{ if } \tau_{i+1} \notin J. \end{cases}$$

Proof. Assume firstly that $J \neq S$, and choose k so that $\tau_{k+1} \notin J$. One can lift $\overline{f}_{\tau_k}$ to an element f_{τ_k} of $\phi_1(\mathcal{M}^{\tau_{k-1}})$, and in fact one can choose f_{τ_k} so that for all $g \in \mathrm{Gal}(K/K_0)$ we have

$$\hat{g}(f_{\tau_k}) = \left(\left(\psi_2' \prod_{\sigma\notin J} \omega_\sigma^{-p}\right)(g)\right) f_{\tau_k}$$

(the obstruction to doing this is easily checked to vanish, as the degree of K/K_0 is prime to p). As $k+1 \notin J$, we have $j_{\tau_k} = 0$, so that e_{τ_k} and $u^e f_{\tau_k}$ must generate $\mathcal{M}_1^{\tau_k}$.

Now, suppose inductively that for some i we have chosen f_{τ_i} and λ_{τ_i} so that $\mathcal{M}_1^{\tau_i}$ is generated by $u^{j_{\tau_i}} e_{\tau_i}$ and $(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i})$. Then we put $f_{\tau_{i+1}} = \phi_1(u^{e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{i_{\tau_i}} e_{\tau_i})/(b^{-1})_i$. Then $f_{\tau_{i+1}}$ is a lift of $\overline{f}_{\tau_{i+1}}$, and the commutativity of ϕ_1 and the action of $\mathrm{Gal}(K/K_0)$ ensures that

$$\hat{g}(f_{\tau_{i+1}}) = \left(\left(\psi_2' \prod_{\sigma\notin J} \omega_\sigma^{-p}\right)(g)\right) f_{\tau_{i+1}}.$$

Then the fact that $\mathcal{M}_1$ is $\mathrm{Gal}(K/K_0)$-stable ensures that for some $\lambda_{t_{i+1}} \in E$ we must have that $u^{j_{\tau_{i+1}}} e_{\tau_{i+1}}$ and $(u^{e-j_{\tau_{i+1}}} f_{\tau_{i+1}} + \lambda_{\tau_{i+1}} u^{i_{\tau_{i+1}}} e_{\tau_{i+1}})$ generate $\mathcal{M}_1^{\tau_{i+1}}$, and of course if $\tau_{i+2} \notin J$ we can take $\lambda_{\tau_{i+1}} = 0$.

So, beginning at k we inductively define f_{τ_i} and λ_{τ_i} for all i, which automatically satisfy all the required properties, except that we do not know that

$$\phi_1(u^{e-j_{\tau_{k-1}}} f_{\tau_{k-1}} + \lambda_{\tau_{k-1}} u^{i_{\tau_{k-1}}} e_{\tau_{k-1}}) = (b^{-1})_{k-1} f_{\tau_k}.$$

However, because $k+1 \notin J$, we may replace f_{τ_k} with $\phi_1(u^{e-j_{\tau_{k-1}}} f_{\tau_{k-1}} + \lambda_{\tau_{k-1}} u^{i_{\tau_{k-1}}} e_{\tau_{k-1}})/(b^{-1})_{k-1}$ without altering the fact that

$$\phi_1(u^e f_{\tau_k}) = (b^{-1})_k f_{\tau_{k+1}},$$

so we are done.

Suppose now that $J = S$. Then we may carry out a similar inductive procedure starting with τ_1, and we again define f_{τ_i} and λ_{τ_i} for all i, satisfying all the required properties, except that we do not know that

$$\phi_1(f_{\tau_r} + \lambda_{\tau_r} u^{i_{\tau_r}} e_{\tau_r}) = f_{\tau_1}.$$

We wish to redefine f_{τ_1} to be $\phi_1(f_{\tau_r} + \lambda_{\tau_r} e_{\tau_r})$, and we claim that doing so does not affect the relation

$$\phi_1(f_{\tau_1} + \lambda_{\tau_1} u^{i_{\tau_1}} e_{\tau_1}) = b f_{\tau_2}.$$

To see this, note that we are modifying f_{τ_1} by a multiple of e_{τ_1} which is in the image of ϕ_1, which by considering the action of $\mathrm{Gal}(K/K_0)$ must in fact be of the form

$\theta u^{pi_{\tau_r}} e_{\tau_1}$, with $\theta \in E$ and $pi_{\tau_r} \equiv i_{\tau_1} \bmod e$. Now, the assumption that ρ is S-regular means that $i_{\tau_1} = e - \sum_{l=1}^{r} p^{r-l}(b_{\tau_{l+1}} - 1) \equiv -b_{\tau_1} \bmod p$, with $2 \le b_{\tau_l} \le p-2$. Now, if we write $pi_{\tau_r} = i_{\tau_1} + me$, we see that $m \equiv i_{\tau_1} \equiv -b_{\tau_1} \bmod p$, and since $2 \le b_{\tau_1} \le p-2$ we see that $m \ge 2$. But then $\phi_1(\theta u^{pi_{\tau_r}} e_{\tau_1}) = \phi_1(\theta u^{i_{\tau_1}+(m-1)e} u^e e_{\tau_1})$ is divisible by $u^{p(m-1)e}$ and is thus 0, as required.

□

Theorem 3.3.3. *Assume that ρ is J-regular and has a model of type J. Then ρ has a lift of type J, which is potentially ordinary if and only if $J = S$ or $J = \emptyset$.*

Proof. It suffices to write down an element $\mathcal{M}_J$ of $W(E) - \mathrm{Mod}_{cris,dd,K_0}$ such that $T_0(\mathcal{M}_J) = \mathcal{M}$, where $\mathcal{M}$ is as in Lemma 3.3.2 (the claim about ordinarity will be immediate from the form of $\mathrm{Fil}^1 \mathcal{M}$). We can write $S_{K,W(E)}$ as $\oplus_{\tau \in S} S_K$, and we then define

$$\mathcal{M}_J^{\tau_i} = S_K \cdot e_{\tau_i} + S_K \cdot f_{\tau_i}$$

$$\hat{g}(e_{\tau_i}) = \left(\left(\widetilde{\psi}_1' \prod_{\sigma \in J} \widetilde{\omega}_\sigma^{-p} \right)(g) \right) e_{\tau_i}$$

$$\hat{g}(f_{\tau_i}) = \left(\left(\widetilde{\psi}_2' \prod_{\sigma \notin J} \widetilde{\omega}_\sigma^{-p} \right)(g) \right) f_{\tau_i}$$

If $\tau_{i+1} \in J$,

$$\mathrm{Fil}^1 \mathcal{M}_J^{\tau_i} = \mathrm{Fil}^1 S_K \cdot \mathcal{M}_J^{\tau_i} + S_K \cdot (f_{\tau_i} + \tilde{\lambda}_{\tau_i} u^{i_{\tau_i}} e_{\tau_i})$$

$$\phi(e_{\tau_i}) = (\tilde{a}^{-1})_i e_{\tau_{i+1}}$$

$$\phi(f_{\tau_i} + \tilde{\lambda}_{\tau_i} u^{i_{\tau_i}} e_{\tau_i}) = (\tilde{b}^{-1})_i p f_{\tau_{i+1}}$$

If $\tau_{i+1} \notin J$,

$$\mathrm{Fil}^1 \mathcal{M}_J^{\tau_i} = \mathrm{Fil}^1 S_K \cdot \mathcal{M}_J^{\tau_i} + S_K \cdot e_{\tau_i}$$

$$\phi(e_{\tau_i}) = (\tilde{a}^{-1})_i p e_{\tau_{i+1}}$$

$$\phi(f_{\tau_i}) = (\tilde{b}^{-1})_i f_{\tau_{i+1}}$$

Here a tilde denotes a Teichmüller lift.

Firstly we verify that this really is an element of $W(E) - \mathrm{Mod}^1_{cris,dd,K_0}$. Of the properties in Definition 3.2.1, the only non-obvious points are that $\mathrm{Fil}^1 \mathcal{M}_J \cap I\mathcal{M}_J = I \,\mathrm{Fil}^1 \mathcal{M}_J$ for all ideals I of $\mathcal{O}_L$, and that $\phi(\mathrm{Fil}^1 \mathcal{M}_J)$ is contained in $p\mathcal{M}_J$ and generates it over $S_{K,W(E)}$. But these are both straightforward; that $\mathrm{Fil}^1 \mathcal{M}_J \cap I\mathcal{M}_J = I \,\mathrm{Fil}^1 \mathcal{M}_J$ follows at once from the definition of $\mathrm{Fil}^1 \mathcal{M}_J$ and the corresponding assertion for S_K, and that $\phi(\mathrm{Fil}^1 \mathcal{M}_J)$ is contained in $p\mathcal{M}_J$ and generates it over $S_{K,W(E)}$ follows by inspection.

It is immediate from the definition of T_0 that $T_0(\mathcal{M}_J) \simeq \mathcal{M}$. To see that $T_{st,2}^{K_0}(\mathcal{M}_J)$ is a lift of ρ of type J, note firstly that the ϕ-action shows that the determinant is a finite order character times the cyclotomic character. That the lift is of type J is then immediate from the form of the $\mathrm{Gal}(K/K_0)$-action, and the definition of the type associated to a potentially semistable Galois representation in terms of the corresponding filtered module (which is obtained from the strongly divisible module by inverting p).

□

3.4. Breuil modules and Fontaine-Laffaille theory. In this section we relate the notion of having a model of type J to that of possessing a certain crystalline lift. Suppose as usual that $\rho \sim \left(\begin{smallmatrix} \psi_1 & * \\ 0 & \psi_2 \end{smallmatrix}\right)$, and that we can write $\psi_1|_{I_{K_0}} = \prod_{\tau \in J} \omega_\tau^{b_\tau}$, $\psi_2|_{I_{K_0}} = \prod_{\tau \notin J} \omega_\tau^{b_\tau}$ with $2 \leq b_\tau \leq p-2$ (note that for a fixed J it is *not* always possible to do this, even after twisting). In this case we define canonical crystalline lifts $\psi_{1,J}$, $\psi_{2,J}$ of ψ_1, ψ_2, as in section 2. That is, we demand that $\psi_{i,J}(\mathrm{Frob}_p)$ is the Teichmüller lift of $\psi_i(\mathrm{Frob}_p)$, and that:

- $\psi_{1,J}$ is crystalline, and the Hodge-Tate weight of $\psi_{1,J}$ with respect to τ is b_τ if $\tau \in J$, and 0 if $\tau \notin J$.
- $\psi_{2,J}$ is crystalline, and the Hodge-Tate weight of $\psi_{2,J}$ with respect to τ is b_τ if $\tau \notin J$, and 0 if $\tau \in J$.

The main result of this section is

Proposition 3.4.1. *Under the above hypotheses, ρ has a model of type J if and only if ρ has a lift to a crystalline representation* $\left(\begin{smallmatrix} \psi_{1,J} & * \\ 0 & \psi_{2,J} \end{smallmatrix}\right)$.

Proof. The idea of the proof is to express both the condition of having a model of type J and the condition of having a crystalline lift of the prescribed type in terms of conditions on strongly divisible modules. In fact, we already have a description of the general model of type J in terms of Breuil modules with descent data, and it is easy to write down the general crystalline representation $\left(\begin{smallmatrix} \psi_{1,J} & * \\ 0 & \psi_{2,J} \end{smallmatrix}\right)$ in terms of Fontaine-Laffaille theory. The only difficulty comes in relating the generic fibres of the Breuil modules to the generic fibres of the Fontaine-Laffaille modules, as the image of the functors describing passage to the generic fibre is in general too complicated to describe directly. Fortunately, it is relatively easy to compare the two generic fibres we obtain, without explicitly determining either.

Let $\mathcal{M} \in \mathrm{BrMod}_{dd}^{k-1}$ for some $k \in [2, p-1]$. There is a contravariant functor T_{st}^* from $\mathrm{BrMod}_{dd}^{k-1}$ to the category of E-representations of G_{K_0} given by

$$T_{st}^*(\mathcal{M}) := \mathrm{Hom}_{k[u]/u^{ep},\phi,N,\mathrm{Fil}^{\cdot},G_{K/K_0}}(\mathcal{M}, \hat{A}_{st})$$

(where compatibility with $\mathrm{Fil}^{\cdot}$ means that the image of $\mathcal{M}_{k-1}$ is contained in $\mathrm{Fil}^{k-1}\hat{A}_{st}$). This functor is exact and faithful, and preserves dimension in the obvious sense.

Consider now the objects of $\mathrm{BrMod}_{dd}^{p-2}$ with trivial descent data (i.e. those for which the action of G_{K/K_0} is trivial) and on which $N = 0$. By Breuil's generalisation of Fontaine-Laffaille theory (see [Bre98]) these objects correspond via T_{st}^* to the π-torsion in crystalline representations with Hodge-Tate weights in $[0, p-2]$. In order to compare the generic fibres of these Galois representations to those of finite flat group schemes with descent data, we need to be able to compare elements of BrMod_{dd}^1 and $\mathrm{BrMod}_{dd}^{p-2}$. This is straightforward: it is easy to check that there is a fully faithful functor from BrMod_{dd}^1 to $\mathrm{BrMod}_{dd}^{p-2}$, given by defining (for $\mathcal{M} \in \mathrm{BrMod}_{dd}^1$) $\mathcal{M}_{p-2} := u^{e(p-3)}\mathcal{M}_1$, $\phi_{p-2}(u^{e(p-3)}x) = \phi_1(x)$ for all $x \in \mathcal{M}_1$, and leaving the other structures unchanged. This functor commutes with T_{st}^*.

Because we are now using the functor T_{st}^* rather than $T_{st,2}^{K_0}$, the form of the Breuil modules (and in particular their descent data) corresponding to models of type J under T_{st}^* is slightly different. Explicitly, we see from Lemma 3.3.2 that ρ has a model of type J if and only if there are $\lambda_{\tau_i} \in E$ with $\lambda_{\tau_i} = 0$ if $\tau_{i+1} \notin J$, and elements $a, b \in E^\times$ such that $\rho \cong T_{st}^*(\mathcal{M})$, where

$$\mathcal{M}^{\tau_i} = E[u]/u^{ep} \cdot e_{\tau_i} + E[u]/u^{ep} \cdot f_{\tau_i}$$
$$\mathcal{M}^{\tau_i}_{p-2} = E[u]/u^{ep} \cdot u^{(p-3)e+j_{\tau_i}} e_{\tau_i} + E[u]/u^{ep} \cdot (u^{(p-2)e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{(p-3)e+i_{\tau_i}} e_{\tau_i})$$
$$\phi_{p-2}(u^{(p-3)e+j_{\tau_i}} e_{\tau_i}) = (a^{-1})_i e_{\tau_{i+1}}$$
$$\phi_{p-2}(u^{(p-2)e-j_{\tau_i}} f_{\tau_i} + \lambda_{\tau_i} u^{(p-3)e+i_{\tau_i}} e_{\tau_i}) = (b^{-1})_i f_{\tau_{i+1}}$$
$$\hat{g}(e_{\tau_i}) = \left(\left(\prod_{\sigma \notin J} \omega_\sigma^{p-b_\sigma}\right)(g)\right) e_{\tau_i}$$
$$\hat{g}(f_{\tau_i}) = \left(\left(\prod_{\sigma \in J} \omega_\sigma^{p-b_\sigma}\right)(g)\right) f_{\tau_i}$$
$$N(e_{\tau_i}) = 0$$
$$N(f_{\tau_i}) = -\frac{(a)_{i-1}}{(b)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}}} e_{\tau_i}$$

where $\lambda_{\tau_i} \in E$, with $\lambda_{\tau_i} = 0$ if $\tau_{i+1} \notin J$, the i_{τ_i} are such that $\mathcal{M}_{p-2}$ is Galois-stable and $0 \leq i_{\tau_i} \leq e-1$, and

$$j_{\tau_i} = \begin{cases} e \text{ if } \tau_{i+1} \in J \\ 0 \text{ if } \tau_{i+1} \notin J. \end{cases}$$

It is an easy exercise to write down the reductions mod p of the strongly divisible modules corresponding to crystalline representations $\left(\begin{smallmatrix} \psi_{1,J} & * \\ 0 & \psi_{2,J} \end{smallmatrix}\right)$. One obtains the following general form:

$$\mathcal{N}^{\tau_i} = E[u]/u^{ep} \cdot E_{\tau_i} + E[u]/u^{ep} \cdot F_{\tau_i}$$
$$\mathcal{N}^{\tau_i}_{p-2} = E[u]/u^{ep} \cdot u^{e(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))} E_{\tau_i} + E[u]/u^{ep} \cdot u^{e(p-2-b_{\tau_i}\delta_J(\tau_i))} F_{\tau_i}$$
$$\phi_{p-2}(u^{e(p-2-b_{\tau_i}\delta_{J^c}(\tau_i))} E_{\tau_i}) = (a^{-1})_i E_{\tau_{i+1}}$$
$$\phi_{p-2}(u^{e(p-2-b_{\tau_i}\delta_J(\tau_i))} F_{\tau_i}) = (b^{-1})_i (F_{\tau_{i+1}} - \lambda'_{\tau_i} E_{\tau_{i+1}})$$
$$\hat{g}(E_{\tau_i}) = E_{\tau_i}$$
$$\hat{g}(F_{\tau_i}) = F_{\tau_i}$$
$$N(E_{\tau_i}) = 0$$
$$N(F_{\tau_i}) = 0$$

where $\lambda'_{\tau_i} \in E$, with $\lambda'_{\tau_i} = 0$ if $\tau_{i+1} \notin J$. We claim that if for each i we have

$$\lambda'_{\tau_i}(b)_i = \lambda'_{\tau_i}(a)_i \tag{3.4.1}$$

then $T^*_{st}(\mathcal{M}) \cong T^*_{st}(\mathcal{N})$. This is of course enough to demonstrate the proposition, as given any set of λ_{τ_i} (respectively λ'_{τ_i}) such that $\lambda_{\tau_i} = 0$ (respectively $\lambda'_{\tau_i} = 0$) if $\tau_{i+1} \notin J$, we may choose a set of λ'_{τ_i} (respectively λ_{τ_i}) so that (3.4.1) holds.

Assume now that (3.4.1) holds. Note that we may write both $\mathcal{M}$ and $\mathcal{N}$ as extensions

$$0 \to \mathcal{M}'' \to \mathcal{M} \to \mathcal{M}' \to 0$$
$$0 \to \mathcal{N}'' \to \mathcal{N} \to \mathcal{N}' \to 0$$

with $T^*_{st}(\mathcal{M}'') \cong T^*_{st}(\mathcal{N}'') \cong \psi_2$, $T^*_{st}(\mathcal{M}') \cong T^*_{st}(\mathcal{N}') \cong \psi_1$.

To prove that $T^*_{st}(\mathcal{M}) \cong T^*_{st}(\mathcal{N})$, we will construct a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}'' & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_{\mathcal{M}''}} & & \downarrow{\scriptstyle f_{\mathcal{M}}} & & \downarrow{\scriptstyle f_{\mathcal{M}'}} & & \\
0 & \longrightarrow & \mathcal{P}'' & \longrightarrow & \mathcal{P} & \longrightarrow & \mathcal{P}' & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle f_{\mathcal{N}''}} & & \uparrow{\scriptstyle f_{\mathcal{N}}} & & \uparrow{\scriptstyle f_{\mathcal{N}'}} & & \\
0 & \longrightarrow & \mathcal{N}'' & \longrightarrow & \mathcal{N} & \longrightarrow & \mathcal{N}' & \longrightarrow & 0
\end{array}$$

such that each of $T^*_{st}(f_{\mathcal{M}''})$, $T^*_{st}(f_{\mathcal{M}'})$, $T^*_{st}(f_{\mathcal{N}''})$ and $T^*_{st}(f_{\mathcal{N}'})$ are isomorphisms. From the five lemma it then follows that $T^*_{st}(f_{\mathcal{M}})$ and $T^*_{st}(f_{\mathcal{N}})$ are isomorphisms, as required.

In fact, we take

$$\begin{aligned}
\mathcal{P}^{\tau_i} &= E[u]/u^{ep}\cdot e'_{\tau_i} + E[u]/u^{ep}\cdot f'_{\tau_i} \\
\mathcal{P}^{\tau_i}_{p-2} &= E[u]/u^{ep}\cdot u^{n_{\tau_i}} e'_{\tau_i} + E[u]/u^{ep}\cdot (u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}) \\
\phi_{p-2}(u^{n_{\tau_i}} e'_{\tau_i}) &= (a^{-1})_i e'_{\tau_{i+1}} \\
\phi_{p-2}(u^{n'_{\tau_i}} f'_{\tau_i} + \lambda_{\tau_i} u^{n_{\tau_i}-\beta_{\tau_{i+1}}} e'_{\tau_i}) &= (b^{-1})_i f'_{\tau_{i+1}} \\
\hat{g}(e'_{\tau_i}) &= \nu_{1,\tau_i}(g) e'_{\tau_i} \\
\hat{g}(f'_{\tau_i}) &= \nu_{2,\tau_i}(g) f'_{\tau_i} \\
N(e'_{\tau_i}) &= 0 \\
N(f'_{\tau_i}) &= -\frac{(a)_{i-1}}{(b)_{i-1}} i_{\tau_{i-1}} \lambda_{\tau_{i-1}} u^{p i_{\tau_{i-1}} - p\alpha_{\tau_i}} e'_{\tau_i}
\end{aligned}$$

where

$$\begin{aligned}
\alpha_{\tau_i} &= \sum_{j=0}^{r-1} p^{r-1-j} \left(b_{\tau_{i+j}} \delta_{J^c}(\tau_{i+j}) - \delta_{J^c}(\tau_{i+j+1}) \right) \\
\beta_{\tau_i} &= \sum_{j=0}^{r-1} p^{r-1-j} \left(b_{\tau_{i+j}} \delta_{J}(\tau_{i+j}) - \delta_{J}(\tau_{i+j+1}) \right) \\
\nu_{1,\tau_i}(g) &= \begin{cases} \prod_{\sigma\notin J} \omega_\sigma^{p-b_\sigma}(g) & \text{if } \tau_i \notin J \\ 1 & \text{if } \tau_i \in J \end{cases} \\
\nu_{2,\tau_i}(g) &= \begin{cases} \prod_{\sigma\in J} \omega_\sigma^{p-b_\sigma}(g) & \text{if } \tau_i \in J \\ 1 & \text{if } \tau_i \notin J \end{cases} \\
n_{\tau_i} &= (p-2-\delta_{J^c}(\tau_i) b_{\tau_i}) e + p\delta_{J^c}(\tau_i)\alpha_{\tau_i} - \delta_{J^c}(\tau_{i+1})\alpha_{\tau_{i+1}} \\
n'_{\tau_i} &= (p-2-\delta_{J}(\tau_i) b_{\tau_i}) e + p\delta_{J}(\tau_i)\beta_{\tau_i} - \delta_{J}(\tau_{i+1})\beta_{\tau_{i+1}}
\end{aligned}$$

We then define $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ by

$$\begin{aligned}
f_{\mathcal{M}}(e_{\tau_i}) &= u^{-p\alpha_{\tau_i}\delta_J(\tau_i)} e'_{\tau_i} \\
f_{\mathcal{M}}(f_{\tau_i}) &= u^{-p\beta_{\tau_i}\delta_{J^c}(\tau_i)} f'_{\tau_i} \\
f_{\mathcal{N}}(E_{\tau_i}) &= u^{p\alpha_{\tau_i}\delta_{J^c}(\tau_i)} e'_{\tau_i} \\
f_{\mathcal{N}}(E_{\tau_i}) &= u^{p\beta_{\tau_i}\delta_{J}(\tau_i)} f'_{\tau_i}
\end{aligned}$$

We then define $\mathcal{P}'$ to be the submodule generated by the e'_{τ_i}, and $\mathcal{P}''$ to be the quotient obtained by $e'_{\tau_i} \mapsto 0$. The remaining maps are then defined by the commutativity of the diagram.

The verification that this choice of $\mathcal{P}$, $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ work is then mostly the matter of lengthy but straightforward calculation, remembering at all times that if $\tau_{i+1} \notin J$ then $\lambda_{\tau_i} = \lambda'_{\tau_i} = 0$, and that we are assuming that (3.4.1) holds. We sketch the non-trivial steps, which are those involving inequalities, as opposed to equalities.

Firstly, in order that the maps $f_{\mathcal{M}}$ and $f_{\mathcal{N}}$ be defined, it is necessary that the exponents of u in their definition be non-negative. In fact, we have that $\tau_i \in J$ if and only if $\beta_{\tau_i} > 0$ if and only if $\alpha_{\tau_i} \leq 0$. To see this, note that from the definition, the sign of α_{τ_i} is determined by the sign of the first non-zero term in the sum (this uses that $2 \leq b_{\tau_j} \leq p-2$). If $\tau_i \notin J$ then the first term is positive, and thus so is the whole sum. If $\tau_i \in J$ then either every term in the sum is zero, or the first non-zero term must be negative. A similar analysis applies to the sign of β_{τ_i}.

In order to check that $\mathcal{P}^{\tau_i}_{p-2}$ is defined and contains $u^{e(p-2)}\mathcal{P}^{\tau_i}$, we need to know that $0 \leq n_{\tau_i}, n'_{\tau_i} \leq e(p-2)$, and furthermore that if $\tau_{i+1} \in J$ then $0 \leq n_{\tau_i} - \beta_{\tau_{i+1}} \leq e(p-2)$. Firstly, note that by definition (and the fact that $2 \leq b_{\tau_j} \leq p-2$ for all j) we have

$$-e/(p-1) \leq \alpha_{\tau_i},\ \beta_{\tau_i} \leq e(p-2)/(p-1).$$

Secondly, by definition one has

$$p\alpha_{\tau_i} - \alpha_{\tau_{i+1}} = e(b_{\tau_i}\delta_{J^c}(\tau_i) - \delta_{J^c}(\tau_{i+1})),$$

$$p\beta_{\tau_i} - \beta_{\tau_{i+1}} = e(b_{\tau_i}\delta_{J}(\tau_i) - \delta_{J}(\tau_{i+1})).$$

Using these facts, together with the fact that $\tau_i \in J$ if and only if $\beta_{\tau_i} > 0$ if and only if $\alpha_{\tau_i} \leq 0$, one easily checks each of these inequalities case by case (that is, in each of the 4 possibilities as to whether τ_i, τ_{i+1} is in J or J^c).

Finally, in order that $N(f'_{\tau_i})$ be defined, we need to check that if $\tau_i \in J$ then $i_{\tau_i} \geq \alpha_{\tau_{i+1}}$. But if $\tau_i \in J$ then $i_{\tau_i} - \alpha_{\tau_{i+1}} = e - \beta_{\tau_{i+1}} > 0$, as required. □

3.5. Weights and types. We recall some definitions and results from [Dia05]. Fix, as ever, $\rho \sim \left(\begin{smallmatrix}\psi_1 & * \\ 0 & \psi_2\end{smallmatrix}\right)$. We make the following definitions:

Definition 3.5.1. A weight $\sigma_{\vec{a},\vec{b}}$ is *compatible* with ρ (via J) if and only if there exists a subset $J \in S$ so that

$$\psi_1|_{I_{K_0}} = \prod_{\tau\in S}\omega_\tau^{a_\tau}\prod_{\tau\in J}\omega_\tau^{b_\tau},\ \psi_2|_{I_{K_0}} = \prod_{\tau\in S}\omega_\tau^{a_\tau}\prod_{\tau\notin J}\omega_\tau^{b_\tau}$$

Suppose that these equations hold. We define

$$c_{\tau_i} = \begin{cases} b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \in J \\ p - b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \notin J \end{cases}$$

where δ_J is the characteristic function of J. Define a character $\chi_{\vec{a},\vec{b},J}$ by

$$\chi_{\vec{a},\vec{b},J} = \prod_{\tau_i\in S}\omega_{\tau_i}^{a_{\tau_i}}\prod_{\tau_i\notin J}\omega_{\tau_i}^{b_{\tau_i}-p}.$$

Then we define a representation $I_{\vec{a},\vec{b},J}$ of $\mathrm{GL}_2(k)$ and a type $\tau_{\vec{a},\vec{b},J}$ by

$$I_{\vec{a},\vec{b},J} = I\left(\tilde{\chi}_{\vec{a},\vec{b},J}, \tilde{\chi}_{\vec{a},\vec{b},J}\prod_{\tau\in S}\tilde{\omega}_\tau^{c_\tau}\right)$$

$$\tau_{\vec{a},\vec{b},J} = \tilde{\chi}_{\vec{a},\vec{b},J} \oplus \tilde{\chi}_{\vec{a},\vec{b},J}\prod_{\tau\in S}\tilde{\omega}_\tau^{c_\tau}$$

Note that if $\overline{\rho}$ is compatible with $\sigma_{\vec{a},\vec{b}}$, then a lift of type J is precisely a lift of type $\tau_{\vec{a},\vec{b},J}$.

Proposition 3.5.2. *Suppose that $\sigma_{\vec{a},\vec{b}}$ is regular. If ρ is compatible with $\sigma_{\vec{a},\vec{b}}$ via J, then ρ is compatible with precisely one of the Jordan-Hölder factors of the reduction mod p of $I_{\vec{a},\vec{b},J}$, and that factor is isomorphic to $\sigma_{\vec{a},\vec{b}}$.*

Proof. We use the explicit computations of [Dia05]. Firstly, note that reduction mod p and the notion of compatibility both commute with twisting, so we may replace ρ by $\rho\otimes\chi^{-1}_{\vec{a},\vec{b},J}$. By Proposition 1.1 of [Dia05], we have $\overline{I}_{\vec{a},\vec{b},J} \sim \oplus_{K\subset S}\sigma_{\vec{a}_K,\vec{b}_K}$ where a_K and b_K are defined as follows:

$$a_{K,\tau_i} = \begin{cases} 0 & \text{if } \tau_i \in K \\ c_{\tau_i} + \delta_K(\tau_{i+1}) & \text{if } \tau_i \notin K \end{cases}$$

$$b_{K,\tau_i} = \begin{cases} c_{\tau_i} + \delta_K(\tau_{i+1}) & \text{if } \tau_i \in K \\ p - c_{\tau_i} - \delta_K(\tau_{i+1}) & \text{if } \tau_i \notin K \end{cases}$$

By the definition of the c_τ, we see at once that $\sigma_{\vec{a}_J,\vec{b}_J} = \sigma_{\vec{a},\vec{b}}$, and in fact

$$\psi_1|_{I_{K_0}} = \prod_{\tau\in S}\omega_\tau^{a_{J,\tau}}\prod_{\tau\in J}\omega_\tau^{b_{J,\tau}},\ \psi_2|_{I_{K_0}} = \prod_{\tau\in S}\omega_\tau^{a_{J,\tau}}\prod_{\tau\notin J}\omega_\tau^{b_{J,\tau}}.$$

If ρ is compatible with another Jordan-Hölder factor, there are subsets J', $K' \subset S$, $J' \neq J$ such that

$$\psi_1|_{I_{K_0}} = \prod_{\tau\in S}\omega_\tau^{a_{J,\tau}}\prod_{\tau\in J}\omega_\tau^{b_{J,\tau}} = \prod_{\tau\in S}\omega_\tau^{a_{J',\tau}}\prod_{\tau\in K'}\omega_\tau^{b_{J',\tau}},$$

$$\psi_2|_{I_{K_0}} = \prod_{\tau\in S}\omega_\tau^{a_{J,\tau}}\prod_{\tau\notin J}\omega_\tau^{b_{J,\tau}} = \prod_{\tau\in S}\omega_\tau^{a_{J',\tau}}\prod_{\tau\notin K'}\omega_\tau^{b_{J',\tau}}.$$

Using the formulae above, the first equation simplifies to

$$\prod_{\tau_i\in S}\omega_{\tau_i}^{c_{\tau_i}+\delta_J(\tau_{i+1})} = \prod_{\tau_i\in(J'\cap K')\cup(J'^c\cap K'^c)}\omega_{\tau_i}^{c_{\tau_i}+\delta_{J'}(\tau_{i+1})}\prod_{\tau_{i+1}\in K'\cap J'^c}\omega_{\tau_i}.$$

By the assumption that $\sigma_{\vec{a},\vec{b}}$ is regular, we have $1 \le c_{\tau_i} \le p-2$ and $2 \le c_{\tau_i} + \delta_J(\tau_{i+1}) \le p-2$ for each i. Then we see that we can equate the exponents of ω_{τ_i} on each side of each equation, and we easily obtain $(J'\cap K')\cup(J'^c\cap K'^c) = S$, whence $J' = K'$. But then the equation becomes

$$\prod_{\tau_i\in S}\omega_{\tau_i}^{\delta_J(\tau_{i+1})} = \prod_{\tau_i\in S}\omega_{\tau_i}^{\delta_{J'}(\tau_{i+1})},$$

whence $J = J'$, a contradiction. □

Remark 3.5.3. Note that it follows from the formulae in the proof of Proposition 3.5.2 that if $\sigma_{\vec{a},\vec{b}}$ is regular, then all the Jordan-Hölder factors of the reduction mod p of $I_{\vec{a},\vec{b},J}$ are weakly regular.

Proposition 3.5.4. *Let θ_1, θ_2 be two tamely ramified characters of I_{K_0} which extend to G_{K_0}. If ρ has a potentially Barsotti-Tate lift (with determinant equal to a finite order character times the p-adic cyclotomic character) of type $\theta_1\oplus\theta_2$, where θ_1, θ_2 are tame characters of I_{K_0} which extend to G_{K_0}, then ρ is compatible with some weight occurring in the mod p reduction of $I(\theta_1,\theta_2)$.*

Proof. This follows easily from consideration of the possible Breuil modules corresponding to the π_L-torsion in the p-divisible group of such a lift (where the corresponding Galois representation is valued in $\mathcal{O}_L$, and π_L is a uniformiser of $\mathcal{O}_L$). The case $\theta_1 = \theta_2$ is easy, so from now on we assume that $\theta_1 \neq \theta_2$. The π-torsion must contain a closed sub-group-scheme (with descent data) with generic fibre ψ_1. Suppose that this group scheme corresponds to a Breuil module with descent data $\mathcal{M}$. Then we can choose a basis so that $\mathcal{M}$ takes the following form:

$$\mathcal{M}^{\tau_i} = E[u]/u^{ep} \cdot x_{\tau_i}$$
$$\mathcal{M}_1^{\tau_i} = E[u]/u^{ep} \cdot u^{r_i} x_{\tau_i}$$
$$\phi_1(u^{r_i} x_{\tau_i}) = (a^{-1})_i x_{\tau_{i+1}}$$
$$\hat{g}(x_{\tau_i}) = \theta^i(g) x_{\tau_i}$$

Here $0 \leq r_i \leq e$ is an integer. Now, by Proposition B.3.1 of [CDT99], because the lift is of type $\theta_1 \oplus \theta_2$, for each i we must have $\theta^i = \theta_1$ or θ_2 (here and below we denote the reduction mod p of the θ_i by the same symbol). Define subsets Y, Z by

$$Y = \{\tau_i \in S | \theta^i \neq \theta^{i+1}\},$$
$$Z = \{\tau_i \in S | \theta^i = \theta_1\}.$$

Because $\theta_1 \neq \theta_2$, if $i \in Y$ then the compatibility of the ϕ_1- and $\mathrm{Gal}(K/K_0)$-actions determines r_i uniquely, and if $i \in Y^c$ then we can take either $r_i = 0$ or $r_i = e$. Having written down all possible $\mathcal{M}$, we now need to determine their generic fibres. This is a straightforward calculation using Example 3.7 of [Sav08]. Without loss of generality, we may twist and assume that $\theta_1 = \prod_{\tau_i \in S} \omega_{\tau_i}^{c_i}$, $\theta_2 = 1$, with $0 \leq c_i \leq p-1$. Then one easily obtains

$$\psi_1|_{I_{K_0}} = \omega_{\tau_1}^{m_1+n_1} \prod_{\tau_i \in \{Y^c | r_i = e\}} \omega_{\tau_i} \prod_{\tau_i \in Y \cap Z} \omega_{\tau_i},$$

where

$$m_1 = \begin{cases} 0 & \text{if } \tau_1 \notin Z \\ c_1 + pc_r + \cdots + p^{r-1}c_2 & \text{if } \tau_1 \in Z \end{cases}$$

$$n_1 = \frac{1}{e} \sum_{\tau_i \in Y \cap Z^c} p^{r-i}(p^i c_1 + p^{i+1} c_r + \cdots + c_i + \cdots + p^{i-1} c_2) - \frac{1}{e} \sum_{\tau_i \in Y \cap Z} p^{r-i}(p^i c_1 + p^{i+1} c_r + \cdots + c_i + \cdots + p^{i-1} c_2).$$

Now, consider the coefficient of c_1 in n_1. The sets $Y \cap Z^c$ and $Y \cap Z$ have equal cardinality, so this coefficient is in fact zero. Thus the coefficient of c_1 in $m_1 + n_1$ is 1 if $\tau_1 \in Z$, and 0 otherwise. By cyclic symmetry, we obtain

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i \in Z} \omega_{\tau_i}^{c_i} \prod_{\tau_i \in X} \omega_{\tau_i},$$

where

$$X = \{\tau_i \in Y^c | r_i = e\} \cup (Y \cap Z)$$

We wish to show that ρ is compatible with some weight in the reduction mod p of $I(\theta_1, \theta_2)$. It is easy to check that the determinant of ρ is correct, so it suffices to examine ψ_1; in the notation of Proposition 3.5.2, we see that ρ is compatible with $\sigma_{\vec{a}_K, \vec{b}_K}$ via L if and only if

$$\psi_1|_{I_{K_0}} = \prod_{\tau_i \in (K^c \cap L) \cup (K \cap L^c)} \omega_{\tau_i}^{c_i + \delta_{K^c}(\tau_{i+1})} \prod_{\tau_i \in S} \omega_{\tau_i}^{\delta_{K \cap L}(\tau_{i+1})}$$

(note that our convention that $\theta_2 = 1$ causes K^c to appear in this formula rather than K).

The result now follows upon taking, for example,

$$K = \{\tau_i \in Z | \tau_{i-1} \notin X\} \cup \{\tau_i \in Z^c | \tau_{i-1} \in X\}$$

and

$$L = \{\tau_i | \tau_{i-1} \in X\}.$$

□

Proposition 3.5.5. *Suppose that $\sigma_{\vec{a},\vec{b}}$ is regular. If ρ is compatible with $\sigma_{\vec{a},\vec{b}}$ via J, and ρ has a lift of type J, then ρ has a model of type J.*

Proof. This follows from similar considerations to those involved in the proof of Proposition 3.5.4. Consider the π_L-torsion in the p-divisible group corresponding to the lift of type J. It contains a closed sub-group-scheme (with descent data) with generic fibre ψ_1. Suppose that this group scheme corresponds to a Breuil module with descent data $\mathcal{M}$. Then we can choose a basis so that $\mathcal{M}$ takes the following form:

$$\mathcal{M}^{\tau_i} = E[u]/u^{ep} \cdot x_{\tau_i}$$
$$\mathcal{M}_1^{\tau_i} = E[u]/u^{ep} \cdot u^{r_i} x_{\tau_i}$$
$$\phi_1(u^{r_i} x_{\tau_i}) = (a^{-1})_i x_{\tau_{i+1}}$$
$$\hat{g}(x_{\tau_i}) = \theta^i(g) x_{\tau_i}$$

Again, by Proposition B.3.1 of [CDT99] and the definition of a lift of type J, for each i we must have $\theta^i = \theta_1$ or $\theta^i = \theta_2$ where

$$\theta_1 = \prod_{\tau \in S} \omega_\tau^{a_\tau} \prod_{\tau \in J} \omega_\tau^{b_\tau - p},$$

$$\theta_2 = \prod_{\tau \in S} \omega_\tau^{a_\tau} \prod_{\tau \in J^c} \omega_\tau^{b_\tau - p}.$$

Note that $\psi_1 = \theta_1 \prod_{\tau_i \in S} \omega_{\tau_i}^{\delta_J(\tau_{i+1})}$. Without loss of generality, we can twist so that $\theta_1 = \prod_{\tau_i \in S} \omega_{\tau_i}^{c_i}$, $\theta_2 = 1$, with $0 \le c_i \le p-1$. Then we obtain

$$\theta_1 = \theta_1 \theta_2^{-1} = \prod_{\tau_i \in J} \omega_{\tau_i}^{b_{\tau_j} - \delta_J(\tau_{i+1})} \prod_{\tau_i \in J^c} \omega_{\tau_i}^{p - b_{\tau_i} - \delta_J(\tau_{i+1})}.$$

Since $0 \le c_i \le p-1$ and $2 \le b_{\tau_i} \le p-2$, we obtain

$$c_i = \begin{cases} b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \in J \\ p - b_{\tau_i} - \delta_J(\tau_{i+1}) & \text{if } \tau_i \notin J \end{cases}$$

Note that this implies that $2 \le c_i + \delta_J(\tau_{i+1}) \le p-2$. Now, using the same definitions of X, Y and Z as in the proof of Proposition 3.5.4, we can compare the two expressions we have for ψ_1 to obtain

$$\prod_{\tau_i \in S} \omega_{\tau_i}^{c_i + \delta_J(\tau_{i+1})} = \prod_{\tau_i \in Z} \omega_{\tau_i}^{c_i} \prod_{\tau_i \in X} \omega_{\tau_i}.$$

Since $2 \le c_i + \delta_J(\tau_{i+1}) \le p-2$, this gives $Z = S$, and $X = \{\tau_i | \tau_{i+1} \in J\}$. Since $Z = S$, we have $Y = \emptyset$, and thus the fact that $X = \{\tau_i | \tau_{i+1} \in J\}$ means that $\mathcal{M}$ is in fact of class J. It is then clear that the π_L-torsion is a model of ρ of type J, as required.

□

4. Local analysis - the irreducible case

We now prove the analogues of some of the results of section 3 in the case where ρ is irreducible.

We assume that ρ is irreducible from now on. In addition to the assumptions made at the beginning of section 3, we now also assume that $\mathbb{F}_{p^2} \subset E$, where $\rho : G_{K_0} \to \mathrm{GL}_2(E)$. Let k' be the (unique) quadratic extension of k.

Label the embeddings $k' \hookrightarrow \overline{\mathbb{F}}_p$ as $S' = \{\sigma_i\}$, $0 \le i \le 2r-1$, so that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$, and $\sigma_i|_k = \tau_{\pi(i)}$, where $\pi : \mathbb{Z}/2r\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$ is the natural surjection. For simplicity of notation we will sometimes refer to the elements of S' as elements of $\mathbb{Z}/2r\mathbb{Z}$, and the elements of S as elements of $\mathbb{Z}/r\mathbb{Z}$.

Recall that we say that a subset $H \subset S'$ is a *full subset* if $|H| = |\pi(H)| = r$.

Definition 4.0.6. We say that ρ is *compatible* with a weight $\sigma_{\vec{a},\vec{b}}$ (via J) if there exists a full subset $J \subset S'$ such that

$$\rho|_{I_{K_0'}} \sim \prod_{\sigma \in S'} \omega_\sigma^{a_\sigma} \begin{pmatrix} \prod_{\sigma \in J} \omega_\sigma^{b_\sigma} & 0 \\ 0 & \prod_{\sigma \notin J} \omega_\sigma^{b_\sigma} \end{pmatrix},$$

where we write a_σ, b_σ for $a_{\pi(\sigma)}$, $b_{\pi(\sigma)}$ respectively.

Note that the predicted set of weights $W(\overline{\rho})$ is just the set of compatible weights; this is one way in which the irreducible case is simpler than the reducible one.

Given a regular weight $\sigma_{\vec{a},\vec{b}}$ and a full subset $J \subset S'$, we wish to define a representation and a type. Let $K_J = \pi(J \cap \{1, \dots, r\})$. Then let

$$c_i = \begin{cases} b_i + \delta_{K_J}(1) - 1 & \text{if } 0 = i \in K_J \\ p - b_i + \delta_{K_J}(1) - 1 & \text{if } 0 = i \notin K_J \\ b_i - \delta_{K_J}(i+1) & \text{if } 0 \ne i \in K_J \\ p - b_i - \delta_{K_J}(i+1) & \text{if } 0 \ne i \notin K_J \end{cases}$$

Define a character

$$\psi_{\vec{a},\vec{b},J} = \omega_{\tau_0}^{-\delta_{K_J}(1)} \prod_{\tau \in S} \omega_\tau^{a_\tau} \prod_{\tau \notin K_J} \omega_\tau^{b_\tau - p}.$$

Then we define

$$I'_{\vec{a},\vec{b},J} = \Theta\left(\tilde{\psi}_{\vec{a},\vec{b},J} \tilde{\omega}_{\sigma_r} \prod_{i=1}^{r} \tilde{\omega}_{\sigma_i}^{c_i}\right)$$

$$\tau'_{\vec{a},\vec{b},J} = \tilde{\psi}_{\vec{a},\vec{b},J} \tilde{\omega}_{\sigma_r} \prod_{i=1}^{r} \tilde{\omega}_{\sigma_i}^{c_i} \oplus \left(\tilde{\psi}_{\vec{a},\vec{b},J} \tilde{\omega}_{\sigma_r} \prod_{i=1}^{r} \tilde{\omega}_{\sigma_i}^{c_i}\right)^{p^r}$$

Proposition 4.0.7. *Recall that $\sigma_{\vec{a},\vec{b}}$ is regular. If ρ is compatible with $\sigma_{\vec{a},\vec{b}}$ via J, then ρ is compatible with precisely one of the Jordan-Hölder factors of the reduction mod p of $I'_{\vec{a},\vec{b},J}$, and that factor is isomorphic to $\sigma_{\vec{a},\vec{b}}$.*

Proof. We may twist and assume without loss of generality that $\psi_{\vec{a},\vec{b},J} = 1$. Then by Proposition 1.3 of [Dia05] (note here that Diamond's conventions on the numbering of the elements of S' are the opposite of our's, so that his σ_i is our σ_{-i}), the Jordan-Hölder factors of the reduction mod p of $I'_{\vec{a},\vec{b},J}$ are $\{\sigma_{\vec{a}_K,\vec{b}_K}\}_{K\subset S}$, where

$$a_{K,\tau_i}=\begin{cases}\delta_K(1) & \text{if } 0=i\in K\\ c_i+1 & \text{if } 0=i\notin K\\ 0 & \text{if } 0\neq i\in K\\ c_i+\delta_K(i+1) & \text{if } 0\neq i\notin K\end{cases}$$

$$b_{K,\tau_i}=\begin{cases}c_i+1-\delta_K(1) & \text{if } 0=i\in K\\ p-c_i+\delta_K(1)-1 & \text{if } 0=i\notin K\\ c_i+\delta_K(i+1) & \text{if } 0\neq i\in K\\ p-c_i-\delta_K(i+1) & \text{if } 0\neq i\notin K\end{cases}$$

From the definition of the c_i and of $\psi_{\vec{a},\vec{b},J}$, we have $\sigma_{\vec{a}_{K_J},\vec{b}_{K_J}}=\sigma_{\vec{a},\vec{b}}$. Suppose that ρ is compatible with $\sigma_{\vec{a}_{K'},\vec{b}_{K'}}$ via J'. Then, replacing J' by $(J')^c$ if necessary, we must have

$$\prod_{i\in S'}\omega_{\sigma_i}^{a_{K_J,i}}\prod_{i\in J}\omega_{\sigma_i}^{b_{K_J,i}}=\prod_{i\in S'}\omega_{\sigma_i}^{a_{K',i}}\prod_{i\in J'}\omega_{\sigma_i}^{b_{K',i}}.$$

Using the formulae above, this becomes

$$\begin{aligned}&\omega_{\sigma_0}^{\delta_{J',K'}(1)}\omega_{\sigma_r}^{\delta_{J',K'}(r+1)}\prod_{i\in T'}\omega_{\sigma_i}^{c_i+\delta_{K'}(i+1)}\prod_{i\in S'}\omega_{\sigma_i}^{\delta_{J'\cap\pi^{-1}((K')^c)}(i+1)}\\ &=\omega_{\sigma_0}^{\delta_{J,K_J}(1)}\omega_{\sigma_r}^{\delta_{J,K_J}(r+1)}\prod_{i\in T_J}\omega_{\sigma_i}^{c_i+\delta_{K_J}(i+1)}\prod_{i\in S'}\omega_{\sigma_i}^{\delta_{J\cap\pi^{-1}(K_J^c)}(i+1)},\end{aligned}\tag{4.0.1}$$

where

$$\begin{aligned}T_J&=(J\cap\pi^{-1}(K_J))\cup(J^c\cap\pi^{-1}(K_J^c))=\{1,\dots,r\},\\ T'&=(J'\cap\pi^{-1}(K'))\cup((J')^c\cap\pi^{-1}((K')^c)),\\ \delta_{J,K_J}(i+1)&=\begin{cases}1-\delta_{K_J}(i+1) & \text{if } i\in T_J\\ \delta_{K_J}(i+1) & \text{if } i\notin T_J,\end{cases}\\ \delta_{J',K'}(i+1)&=\begin{cases}1-\delta_{K'}(i+1) & \text{if } i\in T'\\ \delta_{K'}(i+1) & \text{if } i\notin T'.\end{cases}\end{aligned}$$

Note that (since $\sigma_{\vec{a},\vec{b}}$ is regular) all the exponents on the right hand side of (4.0.1) are in the range $[0,p-1]$. On the left hand side, this is true except possibly for the exponents of ω_{σ_0}, ω_{σ_r}. Since $T_J=\{1,\dots,r\}$, it is easy to see that the only opportunity for this not to hold is for the exponent of ω_{σ_0} to be p on the left hand side and 0 on the right hand side. However, in order for the exponent of ω_{σ_0} to be p on the left hand side we require $c_0=p-2$, which requires that $1\in K_J$. But then the exponent of ω_{σ_0} on the right hand side is 1, a contradiction.

Thus we may equate exponents on each side of (4.0.1). In particular, if $i\neq 0$, we have (again because $\sigma_{\vec{a},\vec{b}}$ is regular) $c_i+\delta_{K_J}(i+1)\in[2,p-2]$, so that we must have $\{1,\dots,r-1\}\subset T'$. We also have $c_0\in[1,p-2]$. If $0\in T'$, we see that the exponent of ω_{σ_0} on the left hand side of (4.0.1) is $c_0+1+\delta_{J'\cap(K')^c}(1)=c_0+1$ (because $1\in T'$), which is at least 2. However the exponent of ω_{σ_0} on the right hand side of (4.0.1) is 0 or 1, as $0\notin T_J$, which is a contradiction. Thus $T'=T_J=\{1,\dots,r\}$.

Then (4.0.1) simplifies to

$$\prod_{i=0}^{r-1}\omega_{\sigma_i}^{\delta_{K'}(i+1)}\prod_{i=r}^{2r-1}\omega_{\sigma_i}^{\delta_{(K')^c}(i+1)}=\prod_{i=0}^{r-1}\omega_{\sigma_i}^{\delta_{K_J}(i+1)}\prod_{i=r}^{2r-1}\omega_{\sigma_i}^{\delta_{K_J^c}(i+1)},$$

whence $K' = K_J$, as required. □

Remark 4.0.8. Note that it follows easily from the formulae in the proof of Proposition 4.0.7 that if $\sigma_{\vec{a},\vec{b}}$ is regular, then all the Jordan-Hölder factors of the reduction mod p of $I'_{\vec{a},\vec{b},J}$ are weakly regular.

Theorem 4.0.9. *Assume that $\sigma_{\vec{a},\vec{b}}$ is regular and that ρ is compatible with $\sigma_{\vec{a},\vec{b}}$ via J. Then ρ has a lift of type $\tau'_{\vec{a},\vec{b},J}$, and this lift is not potentially ordinary.*

Proof. A simple computation shows that we in fact have

$$\tau'_{\vec{a},\vec{b},J} = \prod_{\tau\in S}\omega_\tau^{a_\tau}\prod_{\sigma\in J}\omega_\sigma^{b_\sigma-p}\oplus\prod_{\tau\in S}\omega_\tau^{a_\tau}\prod_{\sigma\notin J}\omega_\sigma^{b_\sigma-p}.$$

This means that we only need to make a very minor modification to the proof of Theorem 3.3.3. Let $K_0' = W(k')[1/p]$. Fix $\pi' = (-p)^{1/(p^{2r}-1)}$, and let $K' = K_0'(\pi')$. Let g_ϕ be the nontrivial element of $\mathrm{Gal}(K'/K_0)$ which fixes π'. It is clear from the proof of Theorem 3.3.3 that for some choice of $a\in W(E)^\times$ the following object of $W(E)-\mathrm{Mod}^1_{cris,dd,K_0}$ provides us with the required lift.

$$\mathcal{M}_J^{\sigma_i} = S_K\cdot e_{\sigma_i} + S_K\cdot f_{\sigma_i}$$

$$\hat{g}_\phi(e_{\sigma_i}) = f_{\sigma_{i+r}}$$

$$\hat{g}_\phi(f_{\sigma_i}) = e_{\sigma_{i+r}}$$

If $g\in\mathrm{Gal}(K'/K_0')$,

$$\hat{g}(e_{\sigma_i}) = \left(\left(\prod_{\tau\in S}\widetilde{\omega}_\tau^{a_\tau}\prod_{\sigma\in J}\widetilde{\omega}_\sigma^{b_\sigma-p}\right)(g)\right)e_{\sigma_i}$$

$$\hat{g}(f_{\sigma_i}) = \left(\left(\prod_{\tau\in S}\widetilde{\omega}_\tau^{a_\tau}\prod_{\sigma\notin J}\widetilde{\omega}_\sigma^{b_\sigma-p}\right)(g)\right)f_{\sigma_i}$$

If $\sigma_{i+1}\in J$,

$$\mathrm{Fil}^1\mathcal{M}_J^{\sigma_i} = \mathrm{Fil}^1 S_K\cdot\mathcal{M}_J^{\sigma_i} + S_K\cdot f_{\sigma_i}$$

$$\phi(e_{\sigma_i}) = (a^{-1})_i e_{\sigma_{i+1}}$$

$$\phi(f_{\sigma_i}) = (a^{-1})'_i p f_{\sigma_{i+1}}$$

If $\sigma_{i+1}\notin J$,

$$\mathrm{Fil}^1\mathcal{M}_J^{\sigma_i} = \mathrm{Fil}^1 S_K\cdot\mathcal{M}_J^{\sigma_i} + S_K\cdot e_{\sigma_i}$$

$$\phi(e_{\sigma_i}) = (a^{-1})_i p e_{\sigma_{i+1}}$$

$$\phi(f_{\sigma_i}) = (a^{-1})'_i f_{\sigma_{i+1}}$$

Here the notation $(x)'_i$ means x if $i = r+1$ and 1 otherwise.

□

5. Global Results

5.1. We now show how the local results obtained in the previous sections can be combined with lifting theorems to prove results about the possible weights of mod p Hilbert modular forms. Firstly, we show that if $\overline{\rho}$ is modular of some regular weight, then $\overline{\rho}$ is compatible with that weight, by making use of Lemma 2.1.3 and Proposition 3.5.4. We then turn this analysis around. We take a conjectural regular weight σ for $\overline{\rho}$, and using modularity lifting theorems we produce a modular lift of $\overline{\rho}$ of a specific type, which is enough to prove that $\overline{\rho}$ is modular of weight σ by Propositions 3.5.2 and 4.0.7.

Assume now that F is a totally real field in which $p > 2$ is unramified, and that $\overline{\rho} : G_F \to \mathrm{GL}_2(E)$ is a continuous representation, known to be modular, where E is a finite extension of $\mathbb{F}_p$.

Let $W(\overline{\rho})$ be the conjectural set of Serre weights for $\overline{\rho}$, as defined in Section 2. Recall that the elements of $W(\overline{\rho})$ are just the tensor products of elements of $W_v(\overline{\rho})$, for $v|p$, and that such elements are of the form $\sigma_{\vec{a},\vec{b}}$, as described above. We say that a weight is (weakly) regular if and only if it is a tensor product of (weakly) regular weights.

The following argument is based on an argument of Michael Schein (c.f. Proposition 5.11 of [Sch08]), and is due to him in the case that $\overline{\rho}|_{G_{F_v}}$ is irreducible.

Lemma 5.1.1. *Suppose that $p \geq 5$, that $\overline{\rho}$ is modular of weight $\sigma = \otimes_v \sigma^v_{\vec{a},\vec{b}}$, and that σ is weakly regular. Then for each v, either $\overline{\rho}|_{G_{F_v}}$ is compatible with $\sigma^v_{\vec{a},\vec{b}}$, or $\sigma^v_{\vec{a},\vec{b}}$ is not regular, and $\overline{\rho}|_{G_{F_v}}$ is not compatible with any regular weight.*

Proof. Suppose firstly that $\overline{\rho}|_{G_{F_v}}$ is reducible. We will assume for the rest of this section that $F_v \neq \mathbb{Q}_p$; the argument needed when $F = \mathbb{Q}_p$ is slightly different, although much simpler, and the result follows from Lemma 4.4.6 of [Gee08]. Then for any nontrivial type $\tau = \chi_1 \oplus \chi_2$ (with χ_1, χ_2 tame characters of I_{F_v} which extend to G_{F_v}) such that $\sigma^v_{\vec{a},\vec{b}}$ occurs in the reduction of $I(\chi_1, \chi_2)$, it follows from Lemma 2.1.3 and Proposition 3.5.4 that there must be a weight $\sigma^v_{\vec{a'},\vec{b'}}$ in the reduction of $I(\chi_1, \chi_2)$ which is compatible with $\overline{\rho}|_{G_{F_v}}$. Since we are working purely locally, we return to the notation of section 3.5.

Twisting, we may without loss of generality suppose that $a_\tau = 0$ for all τ. By Proposition 1.1 of [Dia05] (and the fact that σ is weakly regular) there is for each $J \subset S$ a unique pair of characters $\prod_{\tau \in S} \tilde{\omega}^{c^J_\tau}$, $\prod_{\tau \in S} \tilde{\omega}^{d^J_\tau}$ (with $0 \leq c^J_\tau, d^J_\tau \leq p-1$) such that if we define

$$\sigma^J = I\left(1, \prod_{\tau \in S} \tilde{\omega}^{d^J_\tau}\right) \otimes \prod_{\tau \in S} \tilde{\omega}_\tau^{c^J_\tau} \circ \det$$

then, with the same notation for reductions as in [Dia05], extended to be compatible with twisting, $\sigma^J_J \sim \sigma_{\vec{a},\vec{b}}$. Then there must (by the argument above) be some subset $K_J \subset S$, such that $\sigma^J_{K_J}$ is compatible with ρ. If $\sigma^J_{K_J} \sim \sigma_{\vec{m}^J_{K_J}, \vec{n}^J_{K_J}}$ this means that there must be a subset $L_J \subset S$ such that

$$\psi_1|_{I_{K_0}} = \prod_{\tau \in S} \omega_\tau^{m^J_{K_J,\tau}} \prod_{\tau \in L_J} \omega_\tau^{n^J_{K_J,\tau}}.$$

By Proposition 1.1 of [Dia05], this is equal to

$$\prod_{\tau_i\in S}\omega_{\tau_i}^{c^J_{\tau_i}}\prod_{\tau_i\in L_J\cap K_J^c}\omega_{\tau_i}^p\prod_{\tau_i\in(L_J\cap K_J)\cup(L_J^c\cap K_J^c)}\omega_{\tau_i}^{d^J_{\tau_i}+\delta_{K_J}(\tau_{i+1})}.$$

Now, since $\sigma_J^J\sim\sigma_{\vec{a},\vec{b}}$, we have

$$\prod_{\tau_i\in S}\omega_{\tau_i}^{c^J_{\tau_i}}\prod_{\tau_i\notin J}\omega_{\tau_i}^{d^J_{\tau_i}+\delta_J(\tau_{i+1})}=\prod_{\tau_i\in S}\omega_{\tau_i}^{a_{\tau_i}}=1,$$

by the assumption that $a_\tau=0$ for all τ, so that in fact

$$\psi_1|_{I_{K_0}}=\prod_{\tau_i\in J^c}\omega_{\tau_i}^{-(d^J_{\tau_i}+\delta_J(\tau_{i+1}))}\prod_{\tau_i\in L_J\cap K_J^c}\omega_{\tau_i}^p\prod_{\tau_i\in(L_J\cap K_J)\cup(L_J^c\cap K_J^c)}\omega_{\tau_i}^{d^J_{\tau_i}+\delta_{K_J}(\tau_{i+1})}.$$

Since $\sigma_J^J\sim\sigma_{\vec{a},\vec{b}}$, we have

$$d^J_{\tau_i}=\begin{cases}b_{\tau_i}-\delta_J(\tau_{i+1}) & \text{if } \tau_i\in J\\ p-b_{\tau_i}-\delta_J(\tau_{i+1}) & \text{if } \tau_i\notin J\end{cases}$$

Substituting, we see that

$$\psi_1|_{I_{K_0}}=\prod_{\tau_i\in(T_J\cap J)\cup(T_J^c\cap J^c)}\omega_{\tau_i}^{b_{\tau_i}}\prod_{\tau_i\in S}\omega_{\tau_i}^{\delta_{L_J\cap K_J^c}(\tau_{i+1})-\delta_{T_J^c\cap J^c}(\tau_{i+1})}\prod_{\tau_i\in T_J}\omega_{\tau_i}^{\delta_{K_J}(\tau_{i+1})-\delta_J(\tau_{i+1})},$$

where we write $T_J=(K_J\cap L_J)\cup(K_J^c\cap L_J^c)$.

Putting $J=S$, we obtain

$$\begin{aligned}\psi_1|_{I_{K_0}}&=\prod_{\tau_i\in T_S}\omega_{\tau_i}^{b_{\tau_i}}\prod_{\tau_i\in S}\omega_{\tau_i}^{\delta_{L_S\cap K_S^c}(\tau_{i+1})}\prod_{\tau_i\in T_S}\omega_{\tau_i}^{\delta_{K_S}(\tau_{i+1})-1}\\ &=\prod_{\tau_i\in T_S}\omega_{\tau_i}^{b_{\tau_i}-\delta_{K_S^c\cap L_S^c}(\tau_{i+1})}\prod_{\tau_i\in T_S^c}\omega_{\tau_i}^{\delta_{L_S\cap K_S^c}(\tau_{i+1})}.\end{aligned}\tag{5.1.1}$$

Now, suppose that $\sigma_{\vec{a},\vec{b}}$ is *not* compatible with ρ, so that for all J we have $K_J\neq J$. We can uniquely write

$$\psi_1|_{I_{K_0}}=\prod_{\tau_i\in S}\omega_{\tau_i}^{c_{\tau_i}}$$

with $0\leq c_{\tau_i}\leq p-1$ not all equal to $p-1$ (in fact, an examination of the product just written shows that the exponents are already in this range). Examining the formula just established, we see that after possibly exchanging ψ_1 and ψ_2 (which we can do, as the definition of "compatible" is unchanged by this exchange), there must be some j such that $b_{\tau_j}\neq 1$, and $c_{\tau_j}=b_{\tau_j}-1$ (else ρ would be compatible with $\sigma_{\vec{a},\vec{b}}$).

Now take $J=\{\tau_j\}$, so that

$$\begin{aligned}\psi_1|_{I_{K_0}}=&\prod_{\tau_i\in(T_{\{\tau_j\}}\cap\{\tau_j\})\cup(T_{\{\tau_j\}}^c\cap\{\tau_j\}^c)}\omega_{\tau_i}^{b_{\tau_i}}\prod_{\tau_i\in S}\omega_{\tau_i}^{\delta_{L_{\{\tau_j\}}\cap K_{\{\tau_j\}}^c}(\tau_{i+1})-\delta_{T_{\{\tau_j\}}^c\cap\{\tau_j\}^c}(\tau_{i+1})}.\\ &\prod_{\tau_i\in T_{\{\tau_j\}}}\omega_{\tau_i}^{\delta_{K_{\{\tau_j\}}}(\tau_{i+1})-\delta_{\{\tau_j\}}(\tau_{i+1})}.\end{aligned}\tag{5.1.2}$$

It is easy to see that the exponent of ω_{τ_i} in this product is always between 0 and $p-1$, unless $i = j-1$ or $i = j$. If the exponent is always between 0 and $p-1$, then we have a contradiction, because we already know that $c_{\tau_j} = b_{\tau_j} - 1$, but from (5.1.2) we see that the exponent of ω_{τ_j} can only be 0, b_{τ_j} or $b_{\tau_j}+1$.

So, at least one of the exponents of $\omega_{\tau_{j-1}}$ and ω_{τ_j} must be -1 or p. We now analyse when this can occur. It's easy to see that the exponent of ω_{τ_j} is -1 if and only if $\tau_j \notin T_{\{\tau_j\}}$ and $\tau_{j+1} \in L^c_{\{\tau_j\}} \cap K_{\{\tau_j\}}$, and it is p if and only if $b_{\tau_j} = p-1$, $\tau_j \in T_{\{\tau_j\}}$ and $\tau_{j+1} \in L_{\{\tau_j\}} \cap K_{\{\tau_j\}}$. Similarly, the exponent of ω_{τ_j} is -1 if and only if $\tau_{j-1} \in T_{\{\tau_j\}}$ and $\tau_j \in L^c_{\{\tau_j\}} \cap K^c_{\{\tau_j\}}$, and it is p if and only if $b_{\tau_{j-1}} = p-1$, $\tau_{j-1} \in T^c_{\{\tau_j\}}$ and $\tau_j \in L_{\{\tau_j\}} \cap K^c_{\{\tau_j\}}$.

Suppose now that the exponent of ω_{τ_j} in (5.1.2) is -1. If we multiply each of the expressions (5.1.1), (5.1.2) by ω_{τ_j}, write each side as a product $\prod_\tau \omega_\tau^{n_\tau}$ with $0 \leq n_\tau \leq p-1$ and equate coefficients of ω_{τ_j} in the resulting expression, we see obtain $b_{\tau_j} = 0$ or 1 (the second case only a possibility when the exponent of $\omega_{\tau_{j-1}}$ in (5.1.2) is p), a contradiction.

Suppose that the exponent of ω_{τ_j} in (5.1.2) is p. Then we again easily see that $p-2 = b_{\tau_j} - 1 = 0$ or 1, which is a contradiction, as we assume $p \geq 5$.

Suppose that the exponent of $\omega_{\tau_{j-1}}$ in (5.1.2) is p. Then in the same fashion we obtain $b_{\tau_j} - 1$=0, 1, b_{τ_j}, $b_{\tau_j}+1$ or $b_{\tau_j}+2$. The only possibility is that $b_{\tau_j} = 2$, when we in addition (in order that the necessary carrying should occur) require that $b_{\tau_i} = p-1$ for all $i \neq j$.

Finally, suppose that the exponent of $\omega_{\tau_{j-1}}$ in (5.1.2) is -1. Multiply each of (5.1.1), (5.1.2) by $\omega_{\tau_{j-1}}$. Then we see that the only way for equality to hold is again if $b_{\tau_i} = p-1$ for all $i \neq j$.

So, we have deduced that $b_{\tau_i} = p-1$ for all $i \neq j$, so that $\sigma_{\vec{a},\vec{b}}$ is certainly not regular. It now remains to show that ρ is not compatible with any regular weight. Examining the above argument, we see that we have in fact deduced that (again, after possibly exchanging ψ_1, ψ_2)

$$\psi_1|_{I_K} = \omega_{\tau_j}^{b_{\tau_j}-1} \prod_{i \neq j} \omega_{\tau_i}^{p-1},$$

$$\psi_2|_{I_K} = \omega_{\tau_j},$$

with $2 \leq b_{\tau_j} \leq p-1$.

If ρ is compatible with some regular weight, then we have by definition that

$$\psi_1|_{I_K} \psi_2|_{I_K}^{-1} = \prod_{\tau \in J} \omega_\tau^{n_\tau} \prod_{\tau \in J^c} \omega_\tau^{-n_\tau}$$

for some $J \subset S$ and $2 \leq n_\tau \leq p-2$. Substituting, we obtain

$$\omega_{\tau_{j-1}} \prod_{\tau \in J} \omega_\tau^{n_\tau} = \omega_{\tau_j}^{b_{\tau_j}-1} \prod_{\tau \in J^c} \omega_\tau^{n_\tau}.$$

If $\tau_j \in J$ then we can immediately equate coefficients of $\omega_{\tau_{j-1}}$ and deduce a contradiction. If not, then because $n_{\tau_j} + b_{\tau_j} < 2p$ we see that we can still equate coefficients of $\omega_{\tau_{j-1}}$ to obtain a contradiction.

The proof in the irreducible case is very similar, and rather simpler, as less "carrying" is possible. In fact, the argument gives the stronger result that $\overline{\rho}|_{G_{F_v}}$ is compatible with $\sigma^v_{\vec{a},\vec{b}}$ for all v. A proof is given in the proof of Proposition 5.11 of

[Sch08]; note that [Sch08] works in the setting of [BDJ05] (using indefinite quaternion algebras), but the proof of Proposition 5.11 is purely local (using Raynaud's theory of finite flat group schemes of type $(p, \dots, p)$ in place of the Breuil module calculations used in this paper), and applies equally well in our setting. □

Theorem 5.1.2. *If $\overline{\rho}$ is modular of weight σ, and σ is regular, then $\sigma \in W(\overline{\rho})$.*

Proof. Suppose that $\sigma = \otimes_v \sigma^v_{\vec{a},\vec{b}}$, so that we need to show that $\sigma^v_{\vec{a},\vec{b}} \in W_v(\overline{\rho})$ for all $v|p$. By Lemma 5.1.1, $\sigma^v_{\vec{a},\vec{b}}$ is compatible with $\overline{\rho}|_{G_{F_v}}$, via J, say. If $\overline{\rho}|_{G_{F_v}}$ is irreducible, we are done, so assume that it is reducible.By Lemma 2.1.3, $\overline{\rho}|_{G_{F_v}}$ has a lift to a potentially Barsotti-Tate representation of type $\tau_{\vec{a},\vec{b},J}$. By definition, this is a lift of type J. By Proposition 3.5.5, $\overline{\rho}|_{G_{F_v}}$ has a model of type J. Twisting, we may without loss of generality suppose that each $a_\tau = 0$. Then by Proposition 3.4.1, and the definition of $W_v(\overline{\rho})$, we see that $\sigma^v_{\vec{a},\vec{b}} \in W_v(\overline{\rho})$, as required.

□

Theorem 5.1.3. *If $\sigma \in W(\overline{\rho})$ is a regular weight, and σ is non-ordinary, then $\overline{\rho}$ is modular of weight σ. If $\sigma \in W(\overline{\rho})$ is regular, and σ is partially ordinary of type I and $\overline{\rho}$ has a partially ordinary modular lift of type I then $\overline{\rho}$ is modular of weight σ.*

Proof. Suppose that $\sigma = \otimes_v \sigma^v_{\vec{a},\vec{b}}$, so that $\sigma^v_{\vec{a},\vec{b}} \in W_v(\overline{\rho})$ for all $v|p$. Firstly, we note that (by the definition of $W_v(\overline{\rho})$) $\sigma^v_{\vec{a},\vec{b}}$ is compatible with $\overline{\rho}|_{G_{F_v}}$, via J_v, say.

Consider firstly the case where $\overline{\rho}|_{G_{F_v}}$ is reducible. We claim that $\overline{\rho}|_{G_{F_v}}$ has a model of type J_v. To see this, we may twist, and without loss of generality suppose that $a_\tau = 0$ for all τ, so that $\overline{\rho}|_{G_{F_v}} \sim \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$, with $\psi_1|_{I_{F_v}} = \prod_{\tau \in J_v} \omega_\tau^{b_\tau}$, $\psi_2|_{I_{F_v}} = \prod_{\tau \notin J_v} \omega_\tau^{b_\tau}$. Now, by Proposition 3.4.1 (and the definition of $W(\overline{\rho}_v)$) $\overline{\rho}|_{G_{F_v}}$ has a model of type J_v, as required. Then Theorem 3.3.3 shows that $\overline{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate deformation of type $\tau_{\vec{a},\vec{b},J_v}$.

If $\overline{\rho}|_{G_{F_v}}$ is irreducible, then Theorem 4.0.9 shows that shows that $\overline{\rho}|_{G_{F_v}}$ has a potentially Barsotti-Tate deformation of type $\tau'_{\vec{a},\vec{b},J_v}$.

By Corollary 3.1.7 of [Gee08] there is a modular lift $\rho : G_F \to W(E)$ of $\overline{\rho}$ which is potentially Barsotti-Tate of type $\tau_{\vec{a},\vec{b},J_v}$ for each $v|p$ for which $\overline{\rho}|_{G_{F_v}}$ is reducible, and of type $\tau'_{\vec{a},\vec{b},J_v}$ for each $v|p$ for which $\overline{\rho}|_{G_{F_v}}$ is irreducible. Then by Lemma 2.1.3, $\overline{\rho}$ is modular of weight σ' for some Jordan-Hölder constituent σ' of the reduction modulo p of $\otimes_v I_v$, where $I_v = I_{\vec{a},\vec{b},J_v}$ if $\overline{\rho}|_{G_{F_v}}$ is reducible, and $I_v = I'_{\vec{a},\vec{b},J_v}$ otherwise. The result then follows from Propositions 3.5.2 and 4.0.7, Remarks 3.5.3 and 4.0.8, and Lemma 5.1.1.

□

References

[AS86] Avner Ash and Glenn Stevens, *Modular forms in characteristic l and special values of their L-functions*, Duke Math. J. **53** (1986), no. 3, 849–868.

[BCDT01] Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor, *On the modularity of elliptic curves over* **Q***: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939 (electronic).

[BDJ05] Kevin Buzzard, Fred Diamond, and Frazer Jarvis, *On Serre's conjecture for mod l Galois representations over totally real fields*, in preparation, 2005.

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart.

[Bre98] Christophe Breuil, *Construction de représentations p-adiques semi-stables*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 281–327. MR MR1621389 (99k:14034)

[Bre00] ———, *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549. MR MR1804530 (2001k:14087)

[CDT99] Brian Conrad, Fred Diamond, and Richard Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), no. 2, 521–567.

[Dia05] Fred Diamond, *A correspondence between representations of local Galois groups and Lie-type groups*, to appear in L-functions and Galois representations (Durham 2004), 2005.

[Gee06] Toby Gee, *A modularity lifting theorem for weight two Hilbert modular forms*, Math. Res. Lett. **13** (2006), no. 5-6, 805–811.

[Gee07] ———, *Companion forms over totally real fields. II*, Duke Math. J. **136** (2007), no. 2, 275–284.

[Gee08] ———, *Automorphic lifts of prescribed types*, preprint, 2008.

[Kis07] Mark Kisin, *Modularity of 2-adic Barsotti-Tate representations*, preprint, 2007.

[Kis08a] ———, *Moduli of finite flat group schemes, and modularity*, to appear in Annals of Mathematics (2008).

[Kis08b] ———, *Potentially semi-stable deformation rings*, to appear in the Journal of the AMS (2008), preprint.

[Sav05] David Savitt, *On a conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197.

[Sav08] ———, *Breuil modules for Raynaud schemes*, to appear in the Journal of Number Theory (2008).

[Sch08] Michael Schein, *Weights of Galois representations associated to Hilbert modular forms*, Journal Reine Angew. Math. **622** (2008), 57–94.

[Tay06] Richard Taylor, *On the meromorphic continuation of degree two L-functions*, Doc. Math. (2006), no. Extra Vol., 729–779 (electronic).

E-mail address: `tgee@math.harvard.edu`

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY