Problem Set 2 for Math 250, Fall 2006.

Due on Wednesday, October 4.

1. If E and K are two extension fields of F, then a F-map from E to K is a field homomorphism from E to K which is the identity when restricted to F.

   (a) Suppose that E is generated over F by (some of the) roots of $f \in F[x]$. Show that there exists at most $[E : F]$ and at least one F-maps $\eta : E \to K$, from E to a splitting field K of f over F, with equality when f is separable.

   (b) Let E and K be two finite extension fields of F. Show that the number of F-maps from E to K is at most $[E : F]$. (In class we used this with $E = K$.)

2. Let E be a finite extension of F. Prove that the set of elements $u \in E$ which are separable over F forms a subfield of E.

3. Let E and E' be two Galois extensions of F which are subfields of an ambient field K. Prove that

$$[EE' : F] = \frac{[E : F][E' : F]}{[E \cap E' : F]}$$

   where EE' is the smallest subfield of K containing E and E'. (Hint: use the fact that for a Galois extension we have $|\mathrm{Gal}(E/F)| = [E : F]$). Can the conditions on E and E' be weakened?

4. Let F be a field of characteristic p, and let a be an element not of the form $b^p - b$ for $b \in F$. Determine the Galois group over F of the splitting field of $x^p - x - a$.

5. Let E be the splitting field of $x^5 - 2$ over $\mathbb{Q}$. Determine the Galois group of $E/\mathbb{Q}$. Determine the subgroups of $\mathrm{Gal}(E/\mathbb{Q})$ and the corresponding subfields under the fundamental theorem of Galois theory.

6. Define $\mathbb{C}$ to be the splitting field of $x^2 + 1$ over $\mathbb{R}$.

   (a) Suppose that $f(x) \in \mathbb{R}[x]$. Let E be the splitting field of $f(x)(x^2 + 1)$ over $\mathbb{R}$. Show that $E = \mathbb{C}$. (Hint: show that E is Galois over $\mathbb{R}$ and apply the Galois correspondence to a Sylow 2-subgroup of $\mathrm{Gal}(E/\mathbb{R})$.) You may use the intermediate value theorem from real analysis, but no theorems from complex analysis!

   (b) Conclude (with an argument) from (a) that $\mathbb{C}$ is algebraically closed.

7. Let F be a field of characteristic p. A polynomial $f(x) \in F[x]$ is a p-polynomial if it has the form $x^{p^m} + a_1 x^{p^{m-1}} + \cdots + a_m x$. Show that a monic polynomial of positive degree is a p-polynomial if and only if its roots in a splitting field form a vector space over $\mathbb{F}_p$, and each root has the same multiplicity p^e.

8. Let E be the splitting field of $f(x) \in F[x]$ over F. Show that E is normal over F.