Solutions to PS 10 (Math 121)

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Question 1: 6.3/18

Let A be a $n \times n$ matrix. Then $\det(A^*) = \det(\overline{A^t}) = \overline{\det(A^t)} = \overline{\det(A)}$.

Question 2: 6.3/22c

Since this a computational exercise I won't show the solution, but please do not hesitate to ask me in case you are unsure about anything.

Question 3: 6.3/24

Let us use our satndard definition of 'finite membered' sequences and define the operator:

$$T(\sigma)(k) = \sum_{i=k}^{\infty} \sigma(i)$$

1. This operator is linear in sums since $T(\sigma + \tau) = \sum(\sigma + \tau) = \sum \sigma + \sum \tau = T(\sigma) + T(\tau)$.
2. $T(e_n)(k) = \sum_{i=k}^{\infty} e_n(i)$ But this is a series such that it is equal to 1 for any position such that $k \leq n$. Hence $T(e_n) = \sum_1^n e_i$.
3. Suppose that T^* exists, then $\langle e_n, T(e_j)\rangle = 1$ if $n \leq j$. On the other hand $\langle e_n, T(e_j)\rangle = \langle T^* e_n, e_j\rangle$. But this means that $T^* e_n$ contains e_j as long as $j \geq n$ and hence $T^* e_n$ is not in our vectorspace. Contradiction.

Question 4: 6.4/8

Let T be a normal operator on a finite dimensional complex vector-space and W be its subspace. Show that if W is T-invariant then it is also T^*-invariant:
Let's use the hint: We know that since T is a normal operator then it is diagonalizable and hence by the hint exercise its restriction on W must be diagonalizable as well. The there exist a basis of W consisting of eigenvectors of T and so also eigenvectors of T^* (T is normal - they share eigenvectors). Hence W has a basis of eigenvectors of T^* and so it is T^*-invariant.

Question 5: 6.4/17

I am going to do this exercise for semidefinite versions since the positive definite are just stricter.

Let T and U be selfadjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove the following:

1. T is positive definite iff its eigenvalues are positive.
   Let us have a look at $\langle Tx, x\rangle = \langle \sum Ta_i v_i, \sum a_i v_i\rangle = \sum \lambda_i |a_i|^2$. Hence this will be guranteed to be positive iff all the $\lambda_i > 0$. Similarly it is a necessary condition.

2. By the choice of basis there exists an isomorphism between F^n and V and so if it is necessary that $\langle Tx, x\rangle > 0$, then in terms of matrices $x^*Ax > 0$ and so $\sum A_{ij} a_i \bar{a_j} > 0$ for all n-tuples.

3. Let us do the easier way first: If $A = B^*B$, then $T = L_A$ is positive definite. This means that $T = U^*U$, where $U = L_B$. Hence $\langle Tx, x\rangle = \langle U^*Ux, x\rangle = \langle Ux, Ux\rangle beq 0$. And so T is positive definite.
   I would like to apologize for this complicated proof, but I cannot find any simpler one. (Feel free to point out any other solution):
   Take the eigenvalues of T and mark them λ_i. Then form a diagonal matrix G with the same multiplicities but all the eigenvalues square rooted (possible since they are non-negative). So now we know that $A = QDQ^{-1} = QG^2Q^{-1} = QG^*GQ^{-1}$. But since T is selfadjoint then its eigen basis is orthogonal and hence the matrix Q is unitary. This means that $QQ^{-1} = QQ^* = I$ and so $A = QG^*GQ^{-1} = QG^*IGQ^{-1} = QG^*Q^{-1}QGQ^{-1}$ and hence $B = QGQ^{-1}$.

4. Suppose that $T$and U are positive semidefinite operators such that $T^2 = U^2$. Since the squares of the operators are equalt then they share the same eigenvectors and eignvalues. In particular this means that T and U also share the same eigenvectors. Also we know that if λ is an eigenvalue of T^2 then $|\sqrt{\lambda}|$ is the associated eigenvalue of T (the absolute value is there because T is positive definite). Therefore U, T share the same eigenvectors with the same eigenvalues which means that $U = T$.

5. Suppose v is an eigenvalue of UT and $UT = TU$. By exercise 14 we can simultaneously diagonalize U and T which means that the eigenvalues of UT are the products of the coresponding eigenvalues of U and T and hence they are non-negative and so UT is positive semidefinite.

6. We know that A is positive definite iff L_A is positive definite, but $L_A = T$ so we are set.

Question 6: 6.5/15

Let U be unitary on a finite dimensional inner product space W and let W be a finite dimensional U-invariant subspace of V.

1. $U(W) = W$ by U-invariance we know that $U(W) \subset W$. But we know that since U is normal, then it is one-to-one and maps subspaces onto equidimensional subspaces and hence $U(W) = W$.

2. We know that $V = W \oplus W^\perp$ and so $V = T(V) = T(W \oplus W^\perp) = T(W) + T(W^\perp) = W + T(W^\perp)$. But this is only possible if $T(W^\perp) \subset W^\perp$.

Question 7: 6.6/2

Since this a computational exercise I won't show the solution, but please do not hesitate to ask me in case you are unsure about anything.