

# Solutions to PS 6 (Math 121)

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## Question 1: 5.1/1

The answers are at the back of the book.

## Question 2: 5.1/3b

Find the eigenvalues, corresponding eigenvectors, if possible the eigenbasis and in such case also the diagonal matrix that is equivalent to the following matrix:

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

By evaluating  $\det(A - tI) = -t^3 + 6t^2 - 11t + 6 = -(t-1)(t-2)(t-3)$ , we get that the eigenvalues are 3, 2, 1. Note that since we have three distinct eigenvalues, of which each must have an eigenspace of dimension at least 1, we now that the matrix is diagonalizable with each eigenspace having dimension exactly 1. Then by finding the nullspaces for  $A - \lambda I$  we can identify the eigenspaces:  $E_1 = \text{span}\{(-1, -1, 1)\}$ ,  $E_2 = \text{span}\{(-1, 1, 0)\}$  and  $E_3 = \text{span}\{(-1, 0, 1)\}$ . Therefore the diagonal matrix is:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and the change of basis matrix is:

$$Q = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

## Question 3: 5.1/4e

For the following linear operator find the eigenvalues of  $T$  and an ordered basis such that  $[T]_\beta$  is diagonal. let  $V = P_2(\mathbb{R})$  and  $T(f(x)) = xf'(x) + f(2)x + f(3)$ .

let us start in the standard basis:  $T(1) = x + 1$ ,  $T(x) = 3x + 3$ ,  $T(x^2) = 2x^2 + 4x + 9$ . This corresponds to a matrix:

$$A = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

And  $\det(A - tI) = -t(t-2)(t-4)$ . So our eigenvalues are  $\{0, 2, 4\}$  and the corresponding eigenvectors are obtained by solving for the nullspaces of  $A - \lambda I$ :  $v_1 = (-3, 1, 0)$ ,  $v_2 = (-3, -13, 4)$  and  $v_3 = (1, 1, 0)$  and these define the basis in which  $[T]$  is diagonal.

### Question 4: 5.1/7

Let the determinant of a linear transformation be defined as a determinant of  $[T]_\beta$  in any ordered basis  $\beta$ .

1. Show that it is well defined: that is it is in fact independent of choice of basis.  
We have shown that determinants of two similar matrices are equal. But this means that  $[T]_\beta = [T]_\gamma$  since these two are similar.
2. Show that  $T$  is invertible iff  $\det(T) \neq 0$ . Suppose  $T$  is invertible, then  $[T]_\beta$  is invertible and hence  $\det(T) = \det([T]_\beta) \neq 0$ . Now assume that  $\det(T) \neq 0$ , this means that  $\det[T]_\beta = \det(A) \neq 0$ , which means that  $L_A$  is invertible, but by definition  $L_A = L_{[T]_\beta} = T$  and so  $T$  must be invertible.
3. This equality directly follows from the similar equality for determinants of matrices.
4. This equality again follows from the similar equality for determinants of matrices and the fact that  $[TU]_\beta = [T]_\beta[U]_\beta$ .
5. Again, this time one can use the fact that  $[T + U]_\beta = [T]_\beta + [U]_\beta$ .

### Question 5: 5.1/14

For any square matrix  $A$  prove that  $A$  and  $A^t$  have the same characteristic polynomial and the hence the same eigenvalues.

We are simply trying to show that  $\det(A - \lambda I) = \det(A^t - \lambda I)$ . Note that  $\det(B) = \det(B^t)$  and so  $\det(A - \lambda I) = \det((A - \lambda I)^t) = \det(A^t - \lambda I^t) = \det(A^t - \lambda I)$ .

### Question 6: 5.1/17

Let  $T$  be a linear operator on  $V = M_n(\mathbb{R})$  defined by  $T(A) = A^t$ .

1. Show that the only eigenvalues are  $\pm 1$ .  
Let  $A$  be an eigenvector of  $T$  with an eigenvalue  $\lambda$ . Then  $T(T(A)) = T(\lambda A) = \lambda T(A) = \lambda^2 A$ . However, just looking at the operator in general we can see that for any  $A$ ,  $T(T(A)) = T(A^t) = (A^t)^t = A$ . Therefore  $\lambda^2 A = A$ , which means that  $\lambda^2 = 1$  and so the only option for lambda is  $\lambda = \pm 1$ .
2. In case  $\lambda = 1$ , then  $T(A) = A^t = A$ , and those we know as symmetric matrices. In the other case  $T(A) = A^t = -A$ , we called them skew-symmetric matrices.
3. We have already established these facts in previous exercises:  
The space of square matrices is a direct sum of the spaces of symmetric and skew-symmetric matrices. This means that if we take the union of bases of skew-symmetric matrices and symmetric matrices we will form a basis of  $M_n(\mathbb{R})$  that makes  $T$  diagonal. We have also shown in the past that the basis of symmetric matrices consists of matrices  $A_{ij}$  such that all entries are zero except  $a_{ij} = a_{ji} = 1$ . Similarly we have shown that the basis for the skew-symmetric matrices consist of matrices  $B_{ij}$  such that all entries are zero except  $b_{ij} = -b_{ji} = 1$  and  $b_{ii} = 0$ . Therefore we have a basis that diagonalizes  $T$ .

### Question 7: 5.2/7

Find the expression for  $A^n$  for any positive integer  $n$  if

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$$

it would be extremely useful to diagonalize  $A$  since it is much easier to find powers of diagonal matrices:

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \quad Q = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$$

Then we know that  $A = Q^{-1}DQ$  and so  $A^n = Q^{-1}D^nQ$ . But powers of  $D$  are quite easily computable since  $D$  is diagonal:

$$A^n = Q^{-1}D^nQ = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2(-1)^n & 5^n \\ (-1)^n & 2 \times 5^n \end{pmatrix}$$

### Question 8: 5.2/13

We have shown that  $A$  and  $A^t$  share the same characteristic polynomial and hence they not only have the same eigenvalues but also the same algebraic multiplicities. Then denote the corresponding eigenspaces  $E_\lambda$  and  $E'_\lambda$ .

1. Show by counter example that the eigenspaces don't have to be necessarily the same. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

then we can verify that although  $A(1,0) = (1,0)$  yet  $A^t(1,0) = (1,1)$  and so  $E_1 \neq E'_1$ .

2. Prove that for any eigenvalue  $\lambda$ ,  $\dim(E_\lambda) = \dim(E'_\lambda)$ .  
We know that  $\dim(E_\lambda) = n - \text{rank}(A - \lambda I)$ , where  $n$  is the dimension of our space. From last week's exercise we know that  $\text{rank}(T) = \text{rank}(T^t)$  and since all the matrices are square and defined on the same spaces than  $n$  is the same as well. Therefore  $\dim(E_\lambda) = n - \text{rank}(A - \lambda I) = n - \text{rank}((A - \lambda I)^t) = n - \text{rank}(A^t - \lambda I) = \dim(E'_\lambda)$
3. By our previous exercise we know that the characteristic polynomials are the same, the algebraic multiplicities are the same and by part b) of this exercise we know that the geometric multiplicities are the same. Therefore if one matrix is diagonalizable so is the other.  
(Note: There is a (possibly) simpler argument: if  $A$  is diagonalizable then there exist matrices such that  $D = Q^{-1}AQ$ , where  $D$  is diagonal. Then by transposing the whole equation we get:  $D^t = Q^t A^t (Q^t)^{-1}$ , where  $D^t$  is quite naturally still diagonal.)