Midterm 2 for Math 121, Fall 2006.

Monday October 20.
Time allowed: 53 minutes.

You may assume all vector spaces are finite-dimensional unless otherwise stated. There are 110 points on this exam, and full score is 100 points.

1. Let $A \in M_3(\mathbb{C})$ be the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

(a) (40 points) Find the determinant, rank, characteristic polynomial, the eigenvalues and the eigenvectors of A. Find an invertible matrix Q such that $D = Q^{-1}AQ$ is diagonal, and compute D. (Hint: all the eigenvalues are integers, and all the eigenvectors can be chosen to have integer coordinates.)

(b) (15 points) Solve the following system of linear differential equations.

$$\begin{aligned} x_1'(t) &= x_1(t) + 3x_3(t) \\ x_2'(t) &= 2x_1(t) + 4x_2(t) + 2x_3(t) \\ x_3'(t) &= 3x_1(t) + 2x_2(t) + x_3(t) \end{aligned}$$

(c) (15 points) Let B be the matrix

$$\begin{pmatrix} 1/6 & 0 & 1/2 \\ 1/3 & 2/3 & 1/3 \\ 1/2 & 1/3 & 1/6 \end{pmatrix}.$$

Compute $\lim_{m\to\infty} B^m$.

(d) (10 points) Students at the three schools Harvard, MIT and Princeton can choose every year which school they will attend next year (they also never graduate). Students at Harvard will remain at Harvard with 1/6 probability, go to MIT with probability 1/3 and to Princeton with probability 1/2. Students at MIT will remain at MIT with 2/3 probability, go to Princeton with probability 1/3, and never pick Harvard. Students at Princeton go to Harvard with 1/2 probability, go to MIT with probability 1/3 and stay at Princeton with probability 1/6.

Suppose in year 2003 there are 405 students at Harvard, 270 students at MIT and 135 students at Princeton. As time approaches the apocalypse (year ∞) how many students are at Harvard, MIT and Princeton?

All three schools are happy to accept fractional students should the situation arise. (For a bonus 5 points, when does this situation first arise? Are there any schools which never have fractional students?)

2. (25 points) Let V be a vector space with dimension n. Suppose $T : V \to V$ is a linear transformation with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding eigenvectors $v_1, \ldots v_n$. Prove that the T-cyclic subspace W generated by $v = v_1 + v_2 + \ldots v_n$ is equal to V. (Recall that W is the span of the vectors $\{v, T(v), T^2(v), \ldots\}$.)