Midterm 1 for Math 121, Fall 2006.

Wednesday October 25
Time allowed: 53 minutes

Apart from the first problem you must fully justify your answers. You may assume all vector spaces are finite-dimensional unless otherwise stated. Recall that for a linear operator $T : V \to V$, the notation $N(T)$ denotes the null space and $R(T)$ denotes the range.

1. (20 points) Mark the following statements true or false. No justification is needed.

   (a) Let W_1, W_2 be two vector subspaces of V and β_1, β_2 be bases of W_1, W_2 respectively. Then $\beta_1 \cap \beta_2$ is a basis of $W_1 \cap W_2$.

   (b) Let $\mathcal{P}(\mathbb{R})$ denote the (infinite-dimensional) vector space consisting of all polynomials with real coefficients. Let $g(x) \in \mathcal{P}(\mathbb{R})$ be some fixed polynomial. Then the function $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ given by $T(p(x)) = g(x)p(x)$ is linear.

   (c) Let V be a vector space and $T, U : V \to V$ be two linear operators. Then $N(T) \subset N(TU)$.

   (d) Let V be a vector space and $T, U : V \to V$ be two linear operators. Then $R(TU) \subset R(T)$.

   (e) Let S_1 and S_2 be two subsets of a vector space V. Let

   $$S = S_1 + S_2 = \{s_1 + s_2 \mid s_1 \in S_1 \text{ and } s_2 \in S_2\}.$$

   (In other words, S is the set of vectors which are expressible as the sum of a vector in S_1 with a vector in S_2.) Then $\text{span}(S_1 + S_2) = \text{span}(S_1 \cup S_2)$.

2. (30 points) Let $V = \mathcal{P}_3(\mathbb{R})$ denote the vector space of polynomials with real coefficients with degree at most 3. Let $\beta = \{1, x, x^2, x^3\}$ be the standard basis of V. Let $T : V \to V$ be given by $T(p(x)) = 2\frac{d}{dx}p(x) - \frac{d^2}{dx^2}p(x)$. You may assume T is linear.

   (a) Compute the matrix $[T]_\beta$.

   (b) Compute the matrix $[T^t]_{\beta^*}$, where $\beta^* = \{f_0^*, f_1^*, f_2^*, f_3^*\} \subset V^*$ is the dual basis of β.

   (c) Let $f \in V^*$ be given by $f(p(x)) = p(2)$ (you do not need to check that $f \in V^*$). Calculate $[f]_{\beta^*}$ and $T^t(f)$. The element $T^t(f)$ should be described by giving a formula for its value on a polynomial $p(x) \in V$.

3. (50 points) Let $V = M_{2\times 2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries.

   (a) Let W denote the set of matrices $\{A = (a_{ij}) \in V : a_{11} + a_{12} = a_{21}\}$. Prove that W is a vector subspace of V.

   (b) Let $U : V \to \mathbb{R}$ be the linear transformation defined by $U(A) = a_{21} - a_{11} - a_{12}$, where $A = (a_{ij}) \in V$. Prove that U is linear.

   (c) State the Dimension Theorem (also called the rank-nullity theorem).

   (d) Use the Dimension Theorem to calculate the dimension of W.

4. (15 **bonus** points) A linear transformation $T : V \to V$ is called *nice* if V is the direct sum of $N(T)$ and $R(T)$. Prove that if T satisfies $T(T(v)) = T(v)$ for every $v \in V$ then T is *nice*. Is the converse true?