k-Shapes

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Motivation: combinatorial and geometric branching

Let $k \geq 1$ be fixed.

Combinatorial branching:

1. k-Schur functions $s_\lambda^{(k)}(X)$ (or $s_\lambda^{(k)}(X;t)$) are symmetric functions indexed by partitions whose parts are bounded by k. Introduced to study Macdonald polynomials ([LLM]). We have $s_\lambda^{(\infty)} = s_\lambda$.

   How does $s_\lambda^{(k)}$ expand in terms of $\{s_\mu^{(k+1)}\}$?

2. affine Stanley symmetric functions $\tilde{F}_w$ are symmetric functions indexed by affine permutations $w \in \tilde{S}_{k+1}$. They encode information about reduced decompositions of w.

   How can one write $\tilde{F}_w$ in terms of $\{\tilde{F}_v \mid v \in \tilde{S}_k\}$?

Geometric branching: Consider the natural inclusion

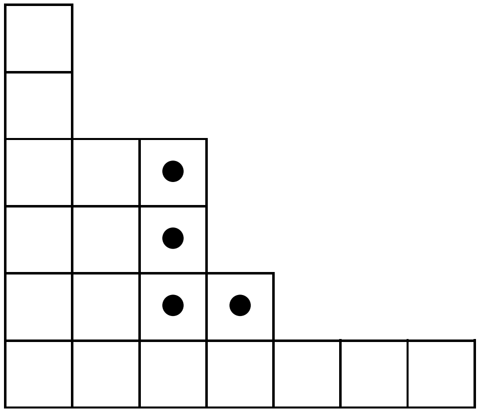
$$\Omega SU(k) \longrightarrow \Omega SU(k+1)$$

How can one expand the Schubert classes $\xi_w \in H_*(\Omega SU(k))$ in terms of the Schubert classes $\xi_v \in H_*(\Omega SU(k+1))$?

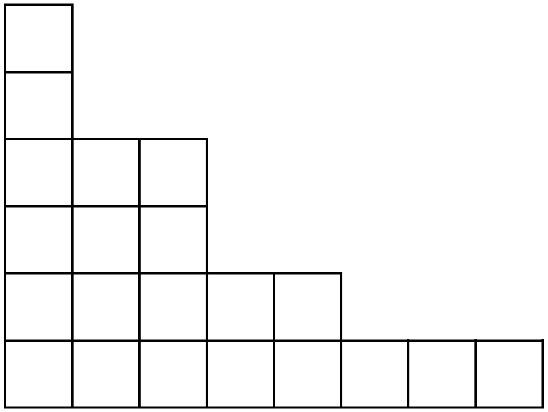
How can one expand the Schubert classes $\xi^v \in H^*(\Omega SU(k+1))$ in terms of the Schubert classes $\xi^w \in H^*(\Omega SU(k))$?

Cores

A n-core is a partition from which a n-ribbon cannot be removed. Example with $n = 4$:



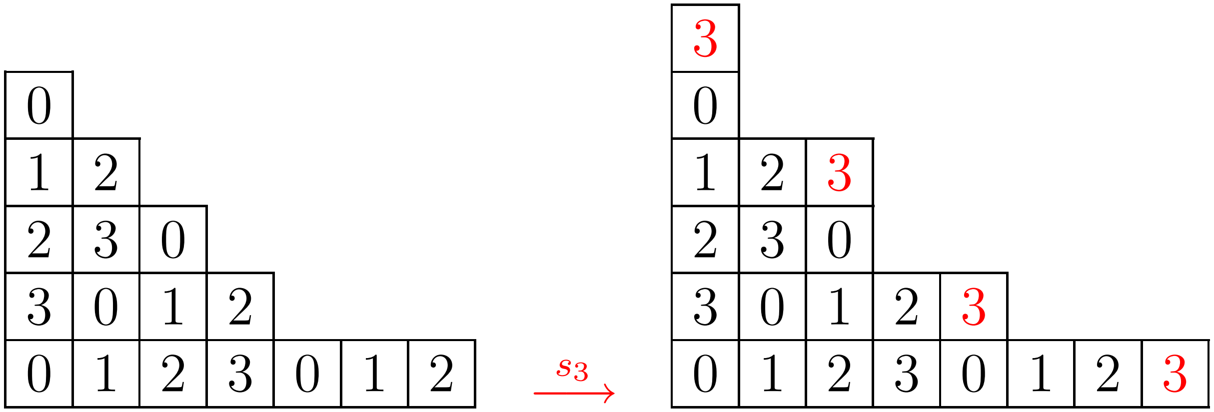
Not a 4-Core



4-Core

Affine symmetric group action

The affine symmetric group acts on cores. The generator s_i adds/removes all boxes on diagonals with residue i:



Lemma. There are no gaps between the boxes added. That is, they occur on diagonals $j, j+n, j+2n, \ldots, j+rn$.

Proposition. Every core can be obtained from the empty partition in this way, identifying n-cores with $\tilde{S}_n/S_n$.

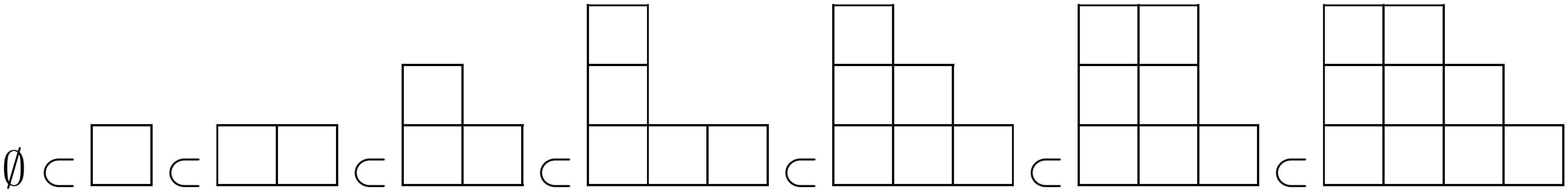
Weak tableaux

A standard weak tableau is a sequence of cores

$$\emptyset \subsetneq \lambda^{(1)} \subsetneq \lambda^{(2)} \subsetneq \cdots \subsetneq \lambda^{(r)}$$

such that each $\lambda^{(i)}$ is obtained from $\lambda^{(i-1)}$ by the action of some s_j.

4	6			
3	5	7		
1	2	4	6	7



Wear Schur functions

Definition ([LM]). A semistandard weak tableau is a usual semistandard Young tableau which can be obtained from a standard weak tableau by relabeling all boxes with an i to j_i, for each i.

$$\begin{array}{|c|c|c|c|c|}
\cline{1-2}
3 & 5 & \multicolumn{3}{c}{} \\
\cline{1-3}
2 & 4 & 5 & \multicolumn{2}{c}{} \\
\hline
1 & 1 & 3 & 5 & 5 \\
\hline
\end{array}$$

Let λ be a n-core.

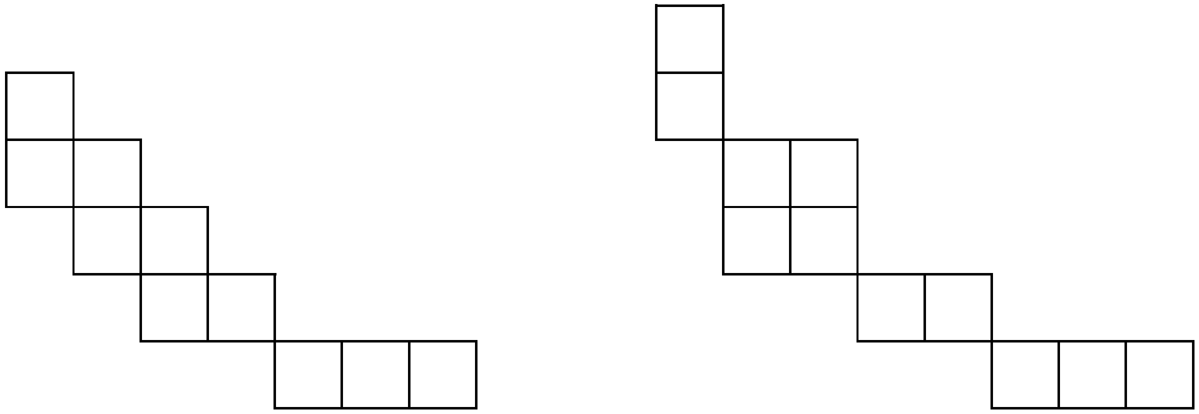
Definition ([LM]). The weak Schur function (or affine Schur function, or dual k-Schur function) F_λ is the generating function of weak tableaux of shape λ.

k-skew

Let $k = n - 1$.

Definition. The degree of a core is the minimal number of simple generators needed to obtain it from the empty partition.

Proposition. It can be calculated by removing all boxes with hooklength greater than k:

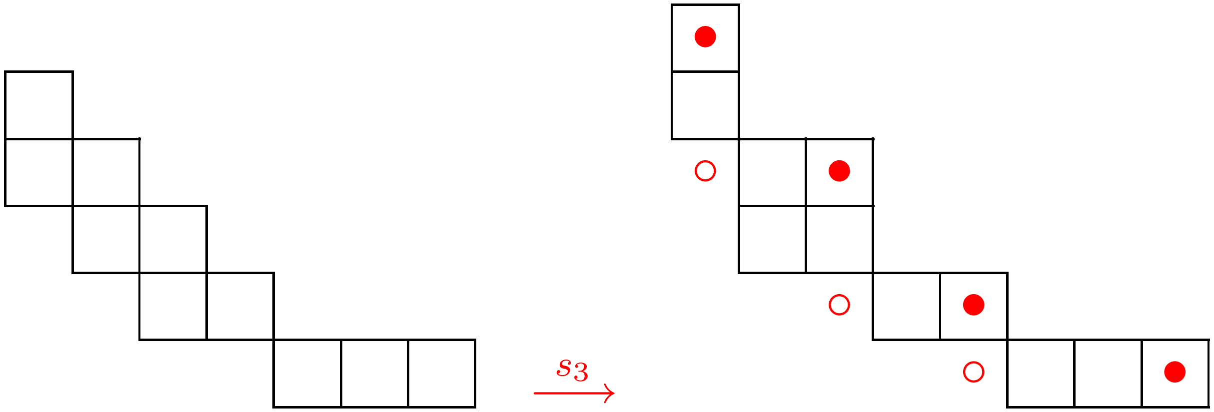


So degree $= 10$ and degree $= 11$.

Observation: You get the same skew shape if you only keep all boxes with hooklength greater than $k + 1$.

k-strings

What happens to the k-skew when you act with s_i?



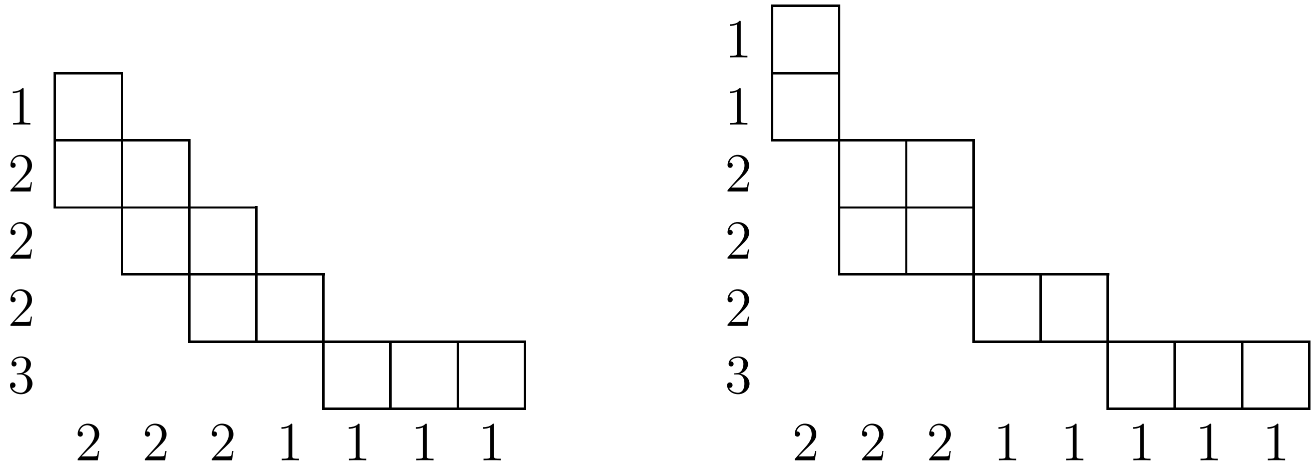
This is an example of a string (which we will define in a couple of slides).

k-boundary

To tackle k-branching, we want to be able to deal with k-cores and $(k+1)$-cores at the same time.

Let λ be any partition. The k-boundary $\partial\lambda$ of λ is the skew shape you get by removing all boxes with hooklength greater than k.

Definition. The row shape $\mathrm{rs}(\lambda)$ (resp. column shape $\mathrm{cs}(\lambda)$) of λ is the composition obtained by reading the number of boxes in each row (resp. column) of $\partial\lambda$ from bottom to top.

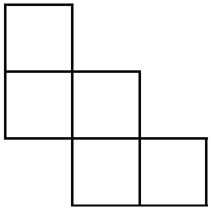


k-shapes

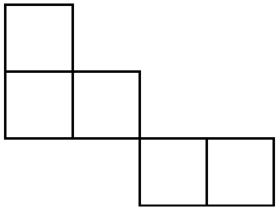
Main Definition. A partition λ is a k-shape if $\mathrm{rs}(\lambda)$ and $\mathrm{cs}(\lambda)$ are both partitions.

Proposition. k-cores and $(k+1)$-cores are both k-shapes.

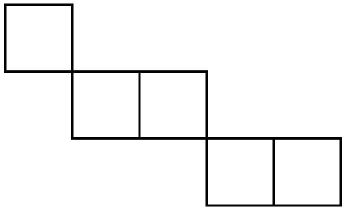
Let $k = 3$.



4-core

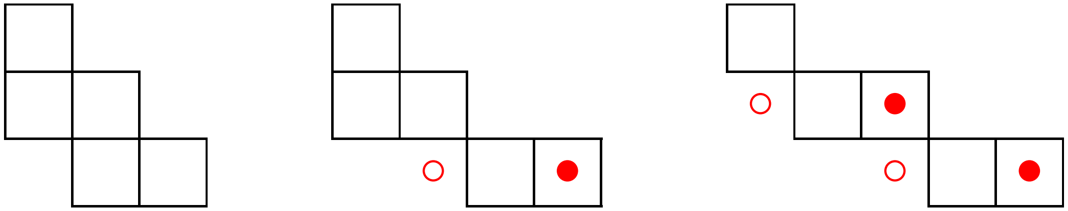


Not a (3 or 4)-core



3-core

Note how the rows appear to be getting pushed out.

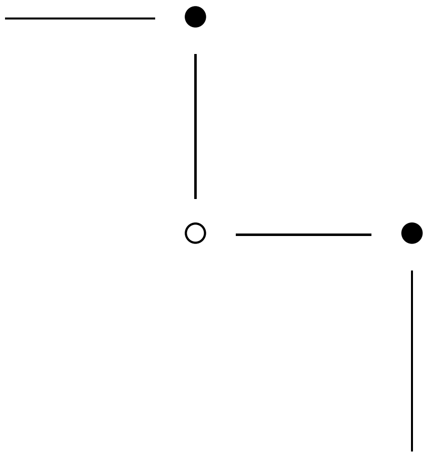


Strings

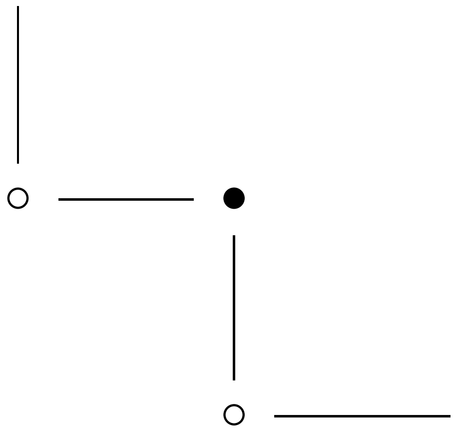
Let λ be a partition.

Definition. A string is a collection of λ addable-corners which are distance k or $k+1$ apart. A pair of addable corners in a string are connected by a removable corner of $\partial\lambda$.

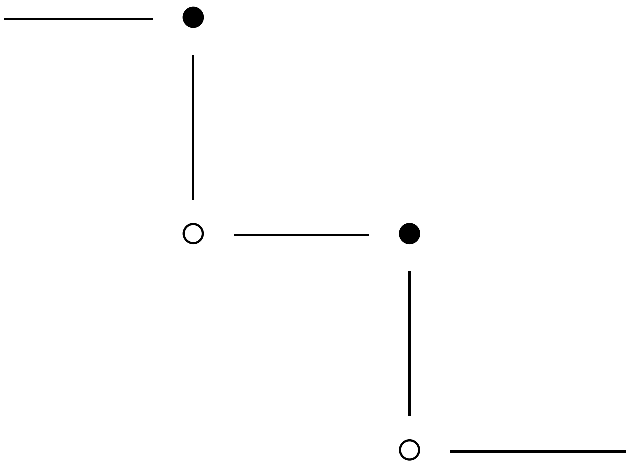
Strings come in four types.



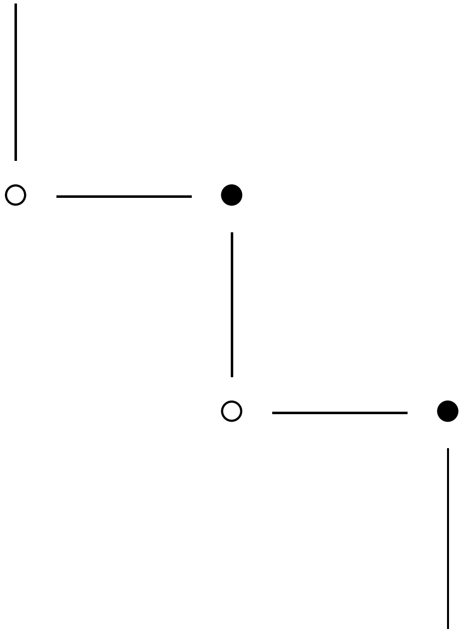
Cover-type



Cocover-type

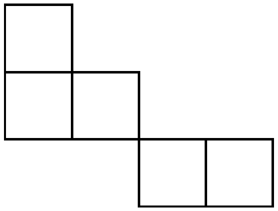


Column-type

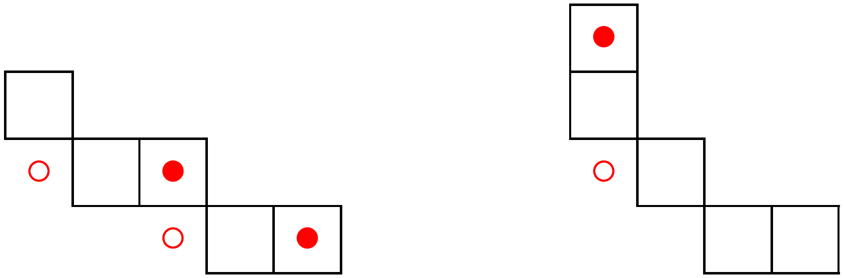


Row-type

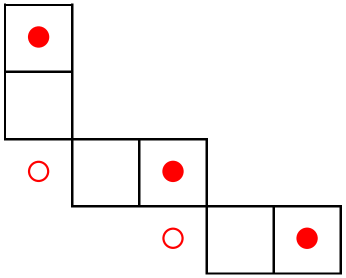
Let $k = 3$ and $\lambda = (5, 2, 1)$.



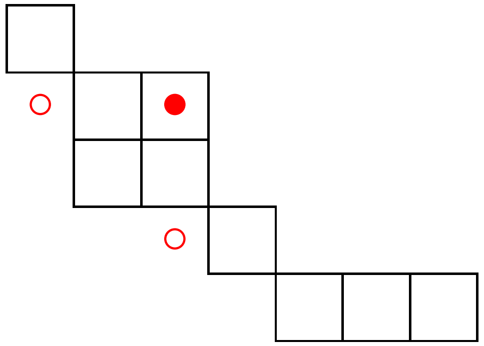
Row and column type:



Cover type:



Co-cover type (for another shape):

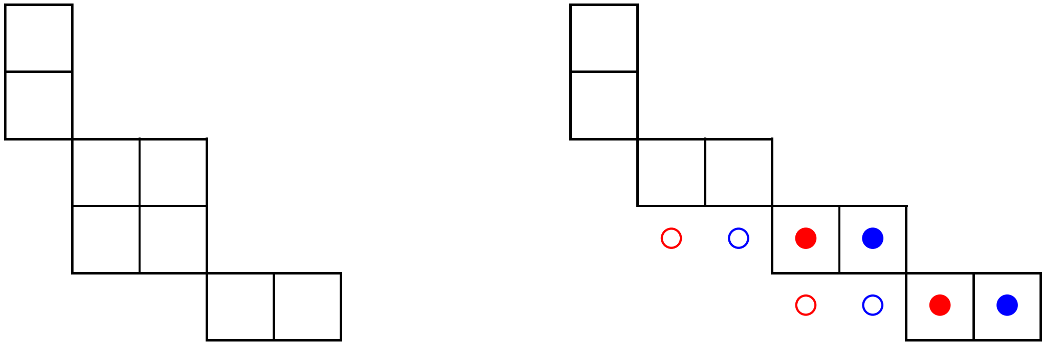


Note that in this final example we do not obtain a k-shape. We do not ask that adding strings give a k-shape.

Moves

Definition. A row move (resp. column move) is a sequence of row-type strings (resp. column-type strings) which

1. Have the same diagram.
2. Occur in adjacent columns (resp. rows).
3. Start and end at a k-shape.



Observation: Row (column) moves do not change row (column) shapes.

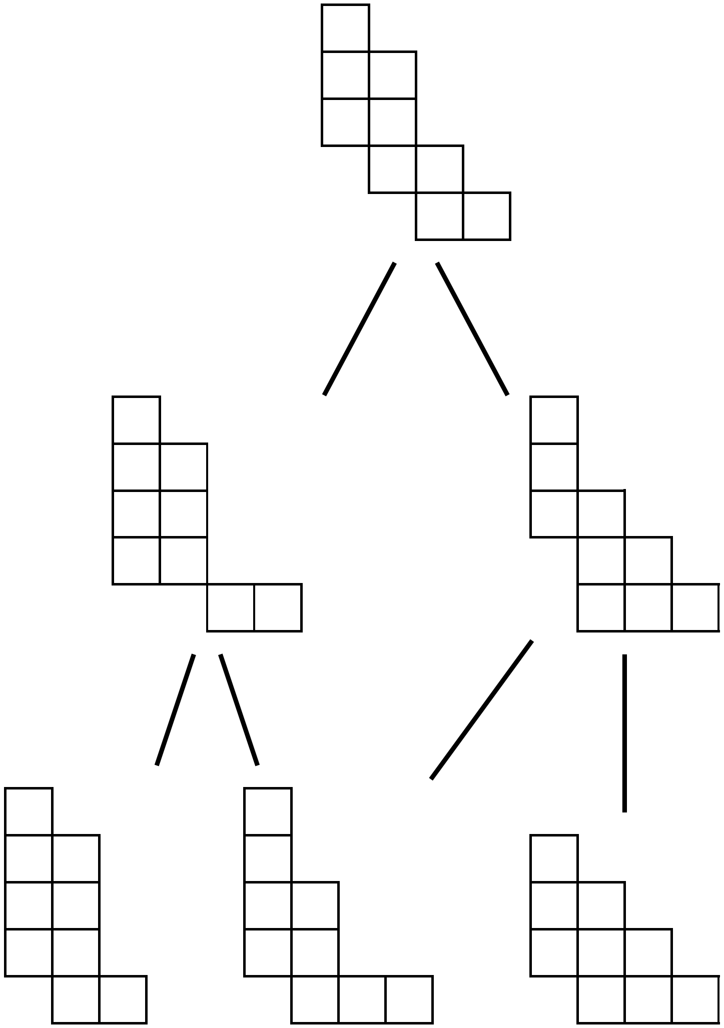
k-shape poset

If two k-shapes are connected by moves, then they have the same number of boxes, and can be considered comparable.

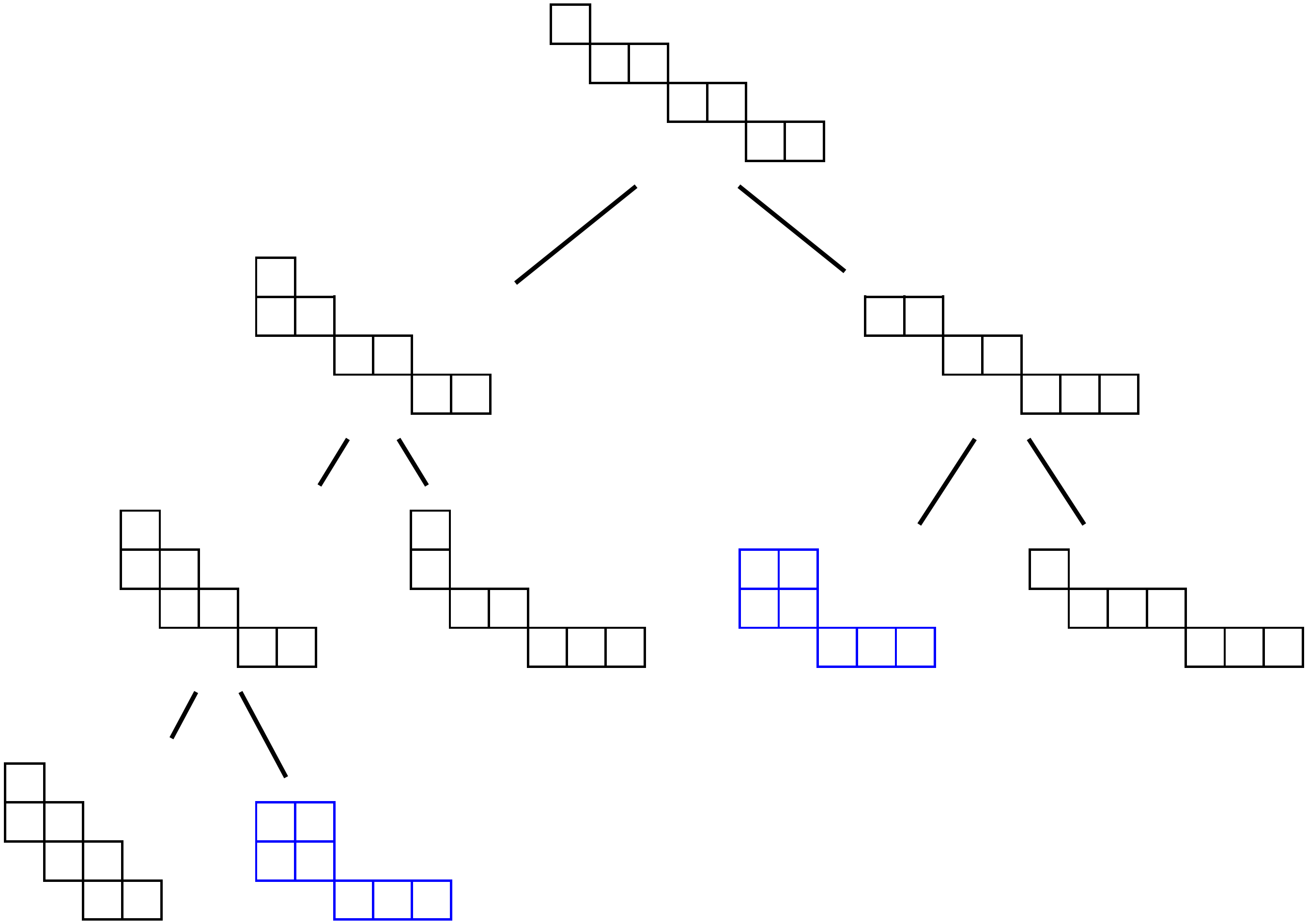
Definition of k-shape poset. $\lambda \prec \mu$ if μ can be obtained from λ by a sequence of row/column moves

Lemma. k-cores are maximal elements of the k-shape poset. $(k+1)$-cores are minimal elements.

$k = 5$



$k = 3$



Main theorem

$$F_\lambda^{(k+1)} = \sum_\mu d_\lambda^\mu F_\mu^{(k)}$$

Nearly a theorem (weak version). The coefficient d_λ^μ is non-zero if and only if $\lambda \prec \mu$.

Nearly a theorem (strong version). The coefficient d_λ^μ is equal to the number of ways to obtain μ from λ by moves, up to an equivalence relation (k-equivalence) generated by:

(1) any two sequences of moves (starting and ending at the same place) which only involves row (resp. column) moves are equivalent;

(2) a particular class of equivalences $\tilde{n}m = \tilde{m}n$ where m is a row move, n is a column move, and each string of m or n either lies in $m \cap n$, or does not intersect $m \cap n$.

Condition (2) is probably the same as a charge preservation condition.

Weak strips for k-shapes

Definition. A weak strip of k-shapes is a pair λ, μ satisfying

(1) μ/λ is a horizontal strip

(2) $\mathrm{rs}(\mu)/\mathrm{rs}(\lambda)$ is a horizontal strip

(3) $\mathrm{cs}(\mu)/\mathrm{cs}(\lambda)$ is a vertical strip

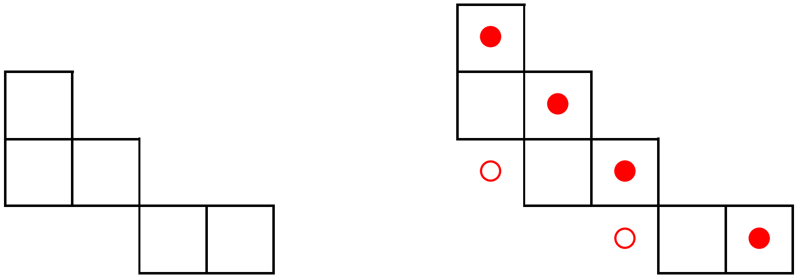
Remarks:

(a) A weak tableaux of k-cores (resp. $(k+1)$-cores) is a sequence of weak strips of k-cores (resp. $(k+1)$-cores).

(b) If μ/λ is a weak strip, and λ is a k-core, μ is not necessarily also a k-core.

(c) Lemma. A weak strip can be expressed (not uniquely) as a sequence of cover-type k-strings, such that all intermediate shapes are k-shapes.

$k = 3$



Row shape changes from $(2, 2, 1)$ to $(2, 2, 2, 1)$

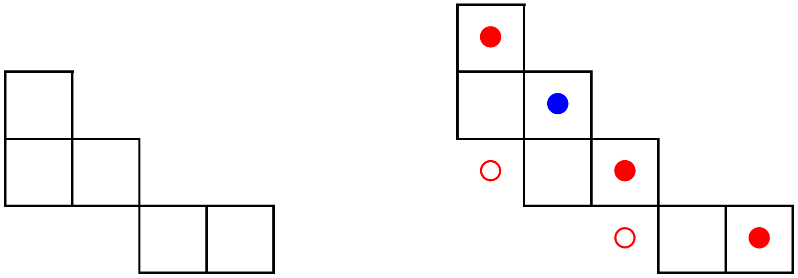
Column shape changes from $(2, 1, 1, 1)$ to $(2, 2, 1, 1, 1)$

Augmentation

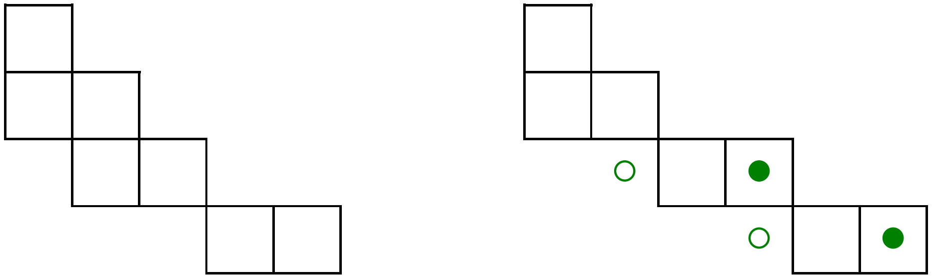
Let $S = \mu/\lambda$ be a strip.

Definition. An augmentation move $m * \mu = \nu$ is a row or column move on μ such that ν/λ is still a strip.

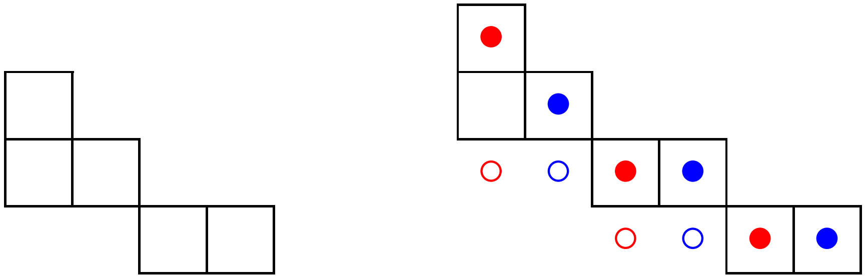
Strip:



Move:



Augmented strip:



Maximal strips

Definition. A strip S is maximal if no non-trivial augmentation moves can be performed on it.

Proposition. Any strip S can be brought to a unique maximal strip $S_{\max}$ by augmentation, and any such sequence of augmentation moves is k-equivalent.

Lemma. If μ/λ is a maximal strip such that λ is a k-core, then μ is automatically a k-core.

There is a similar statement for $(k+1)$-cores, and reverse maximal strips.

Pushouts

Suppose $S = \mu/\lambda$ and $m = \nu/\lambda$ are a strip and a move. In some cases we can construct a diagram

$$\begin{tikzcd}
\lambda \arrow[r, "m"] \arrow[d, "S"'] & \nu \arrow[d, "S'"] \\
\mu \arrow[r, "m'"'] & \eta
\end{tikzcd} \tag{1}$$

and we call $(S', m') = \text{push}(S, m)$ the pushout of (S, m). (I don't know if every η with such a diagram is actually a pushout.) The pair (S', m') are specified by some very rigid conditions.

Proposition. If S is maximal, then (S, m) can always be pushed out.

Augmentations are a special case of (reverse) pushouts.

The bijection

We algorithmically construct

$$(\text{strip } \mu/\lambda,\ \text{path } \lambda \to \nu) \longmapsto (\text{strip } \rho/\nu,\ \text{path } \mu \to \rho)$$

where μ, λ are $(k+1)$-cores and ν, ρ are k-cores. The algorithm is: augment the strip to a maximal strip, pushout the maximal strip with the first move of the path, repeat.

Main claim: This algorithm is well-defined.

Interpret a weak tableau of $(k+1)$-cores as a sequence of weak strips of $(k+1)$-cores. Then the above bijection induces a (weight-preserving) bijection

$$\{\text{weak tableau on } (k+1)\text{-cores with fixed shape } \lambda\}$$

$$\downarrow$$

$$\{\text{pairs } (\text{weak tableau with shape } \nu \text{ on } k\text{-cores}, \text{path from } \lambda \text{ to } \nu)\}$$

More directions

To each k-shape one can associate a weak Schur function $F_\lambda^{(k)}$, which can be defined by either

1. its expansion in terms of F_ν for k-cores ν.
2. generating functions of equivalence classes of tableaux = sequences of strips

Conjecture. For each λ, there is a subvariety of the affine (or infinite) Grassmannian with cohomology class $F_\lambda^{(k)}$.