LOGICAL FOUNDATION OF REAL ANALYSIS

Least Upper Bound Property from Dedekind Cut. For the existence of limit the *least upper bound property* is needed for the set $\mathbb{R}$ of all real numbers. The least upper bound property of $\mathbb{R}$ is a consequence of the definition of $\mathbb{R}$ by Dedekind cuts.

Definition of Least Upper Bound. A real number A is an *upper bound* of a nonempty subset E of $\mathbb{R}$ if $x \leq A$ for all $x \in E$. A real number a is the *least upper bound* of a nonempty subset E of $\mathbb{R}$ if (i) a is an upper bound of E and (ii) any real number b which is $< a$ cannot be an upper bound of E. The least upper bound of E is denoted by $\sup E$ and is also called the *supremum* of E.

Least Upper Bound Property of $\mathbb{R}$. The set $\mathbb{R}$ of all real numbers satisfies the upper bound property which states that the least upper bound exists for any nonempty subset E of $\mathbb{R}$ which admits an upper bound. This least upper bound property of $\mathbb{R}$ is a consequence of its definition from Dedekind cuts. We recall here the definition of Dedekind cuts and how $\mathbb{R}$ is defined from it. The existence of $\sup E$ is proved by taking the union of all Dedekind cuts which correspond to points of E. The verification of the details of the argument was assigned as a problem in the first homework assignment of last semester's Math 55a.

Definition of Dedekind Cut. Let $\mathbb{Q}_+$ denote the set of all positive rational numbers. A proper subset ξ of $\mathbb{Q}_+$ is called a (Dedekind) *cut* if

(i) $\xi \neq \emptyset$ (and $\xi \neq \mathbb{Q}_+$);

(ii) $x \in \xi$ and $y \in \mathbb{Q}_+ - \xi$ imply $x < y$;

(iii) $\nexists\, x \in \xi$ such that $x \geq y$ for $y \in \xi$;

where $\mathbb{Q}_+ - \xi$ means the complement of the set ξ in $\mathbb{Q}_+$.

Identification of Positive Rational Numbers as Dedekind Cuts. We denote by $\mathbb{R}_+$ the set of all (Dedekind) cuts. An element r of $\mathbb{Q}_+$ is identified with the (Dedekind) cut ξ_r which is defined as the set of all $s \in \mathbb{Q}_+$ such that $s < r$. The map $r \mapsto \xi_r$ identifies $\mathbb{Q}_+$ as a subset of $\mathbb{R}_+$.

Motivation for Dedekind Cut. The motivation of a Dedekind cut ξ is the set $(0,\xi)\cap\mathbb{Q}_+$ when ξ is regarded as a positive real number and $(0,\xi)$ means the open interval whose end-points are 0 and the positive real number ξ.

Definition of Ordering in $\mathbb{R}_+$*.* Ordering in $\mathbb{R}_+$ is defined as follows. Two (Dedekind) cuts ξ and η satisfy $\xi > \eta$ if as subsets of $\mathbb{Q}_+$ the set ξ contains the set η. Two (Dedekind) cuts ξ and η satisfy $\xi < \eta$ if as subsets of $\mathbb{Q}_+$ the set ξ is contained in the set η.

Definition of Addition in $\mathbb{R}_+$*.* The sum $\xi+\eta$ of two (Dedekind) cuts ξ and η is defined as the (Dedekind) cut ζ which consists of all $z\in\mathbb{Q}_+$ of the form $z=x+y$ for some $x\in\xi$ and some $y\in\eta$.

Definition of Multiplication in $\mathbb{R}_+$*.* The product $\xi\cdot\eta$ of two (Dedekind) cuts ξ and η is defined as the (Dedekind) cut ζ which consists of all $z\in\mathbb{Q}_+$ of the form $z=x\cdot y$ for some $x\in\xi$ and some $y\in\eta$.

Construction of the Set of All Real Numbers The set $\mathbb{R}$ of all real numbers is defined as the union of the three disjoint subsets $-\mathbb{R}_+$, $\{0\}$, and $\mathbb{R}_+$, where 0 is a new symbol and $-\mathbb{R}_+$ as a set is bijective to $\mathbb{R}_+$ with the element $-\xi$ of $-\mathbb{R}_+$ corresponding to the element ξ of $\mathbb{R}_+$. The arithmetic operations of addition and multiplication for $\mathbb{R}$ are naturally defined from those for $\mathbb{R}_+$. The notion of ordering in $\mathbb{R}$ is naturally defined from that in $\mathbb{R}_+$.

Greatest Lower Bound. For a subset E of $\mathbb{E}$, by considering the set $\{-a\}_{a\in E}$, we can apply the same argument to lower bounds instead of upper bounds and obtain the corresponding *greatest lower bound property* of $\mathbb{R}$. The greatest lower bound of a subset E of $\mathbb{R}$ is also called its *infimum* and is denoted by $\inf E$.

Limits of Sequences. We introduce the notion of the limit of a sequence, the relation between the convergence of a series and that of its sequence of partial sums, and the rôle of uniform convergence of a family of sequences in the commutativity of limit-taking, one as the index of each sequence going to infinity and the other as the index of family members going to infinity.

Definition of Limit of Sequence. A sequence x_n in $\mathbb{R}$ approaches a as its limit if it eventually gets inside any prescribed neighborhood of a. More precisely, given any $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $x_n\in(a-\varepsilon,a+\varepsilon)$ for $n\geq N$. The notation for the convergence of x_n to a is $\lim_{n\to\infty}x_n=a$. An alternative notation is $x_n \to a$ as $n \to \infty$. The set $(a - \varepsilon, a + \varepsilon)$ is called the ε-neighborhood of a. *For the time being* when we talk about a neighborhood we mean the ε-neighborhood for some $\varepsilon > 0$. When a sequence x_n converges to a as $n \to \infty$, then any subsequence x_{n_k} converges to the same limit a as $k \to \infty$. Here for the definition of a subsequence we require that n_k is a strictly increasing function mapping $k \in \mathbb{N}$ to $n_k \in \mathbb{N}$.

Alternative Definition of Limit of Sequence Without Specifying Limit (Cauchy Sequence). A sequence x_n in $\mathbb{R}$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for any $m, n \geq N$.

A sequence in $\mathbb{R}$ converges to some limit if and only if it is a Cauchy sequence. From the triangle inequality, the direction that a sequence convergent to some $a \in \mathbb{R}$ is Cauchy is clear, because if $|x_n - a| < \varepsilon$ for any $n \geq N_\varepsilon$, then

$$|x_m - x_n| \leq |x_m - a| + |a - x_n| < \varepsilon$$

for $m, n \geq N_{\frac{\varepsilon}{2}}$.

For the other direction of showing that every Cauchy sequence converges, the use of $\varepsilon = 1$ implies that the set $\{x_n\}_{n \in \mathbb{N}}$ is contained in $[-A, A]$ for

$$A = \max\left(|x_1|, \cdots, |x_{N_1 - 1}|, |x_{N_1}| + 1\right)$$

if $N_1 \in \mathbb{N}$ satisfied $|x_m - x_n| < 1$ for $m, n \geq N_1$.

We first introduce the notation of *lim sup* of the sequence x_n, which is the abbreviation for the longer full expression *limit superior*. For m let E_m be the set $\{x_n\}_{n \geq m}$. Let a_m be the supremum of E_m. Then $a_{m+1} \leq a_m$, because E_{m+1} is contained in E_m. The infimum a of $\{a_m\}_{m \in \mathbb{N}}$ is called the *lim sup* of the sequence x_n and is denoted by $\text{lim sup}_{n \to \infty} x_n$. Another simpler way to introduce the lim sup of x_n is that

$$\limsup_{n \to \infty} x_n = \inf_{m \in \mathbb{N}} \sup_{n \geq m} x_n.$$

For any $k \in \mathbb{N}$ the real number $a + \frac{1}{k}$ is not a lower bound of $\{a_m\}_{m \in \mathbb{N}}$ and there exists some $a \leq a_{m_k} < a + \frac{1}{k}$. Since $a \leq a_p \leq a_{m_k}$ for $p \geq m_k$, by replacing m_k by $\max(m_k, k)$ we can assume without loss of generality that $m_k \geq k$. Since $a_{m_k} - \frac{1}{k}$ is not an upper bound of E_{m_k} and a_{m_k} is an upper bound of E_{m_k} there exists some $a_{m_k} - \frac{1}{k} \leq x_{n_k} \leq a_{m_k}$ with $n_k \geq m_k$. From $a \leq a_{m_k} < a + \frac{1}{k}$ and $a_{m_k} - \frac{1}{k} \leq x_{n_k} \leq a_{m_k}$ it follows that $a - \frac{1}{k} \leq a_{m_k} \leq a + \frac{1}{k}$ with some $m_k \geq k$. With the use of lim sup, what we have shown is that *any bounded sequence admits a subsequence which converges to its lim sup.*

Since x_n is a Cauchy sequence, for any given $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for $m, n \geq N$. By replacing N by another integer $\geq N$, we can assume without loss of generality that $\frac{1}{N} < \varepsilon$. Then when $n \geq N$, we can choose $m = m_N \geq N$ and get

$$|x_n - a| \leq |x_n - x_{m_N}| + |x_{m_N} - a| < \varepsilon + \frac{1}{N} < 2\varepsilon.$$

This shows that the Cauchy sequence x_n always converges $a = \limsup_{n\to\infty} x_n$.

Relation Between Sequence and Series. A series $\sum_{k=0}^n a_k$ with $a_k \in \mathbb{R}$ converges to a limit L if and only if the sequence $s_n = \sum_{k=0}^n a_k$ of partial sums converges to L. To a given sequence s_n a series with terms $a_n = s_n - s_{n-1}$ can be constructed whose partial sum is s_n.

Uniform Convergence of a Sequence of Sequences and Commutativity of Limit-Taking. For every fixed $m \in \mathbb{N}$ suppose a sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is given which converges to $a^{(m)}$. The convergence of $x_n^{(m)}$ to $a^{(m)}$ is said to be *uniform* in m as $n \to \infty$ if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ independent of m such that $\left|x_n^{(m)} - a^{(m)}\right| < \varepsilon$ for $n \geq N$. Assume now the convergence of $x_n^{(m)}$ to $a^{(m)}$ is *uniform* in m as $n \to \infty$. Assume also that for every fixed $n \in \mathbb{N}$ the sequence $\left\{x_n^{(m)}\right\}_{m\in\mathbb{N}}$ converges to some a_n as $m \to \infty$. Then the sequence a_n converges to some a as $n \to \infty$. Moreover, the sequence $a^{(m)}$ converges to the same a as $m \to \infty$ so that

$$\lim_{m\to\infty} \lim_{n\to\infty} x_n^{(m)} = \lim_{n\to\infty} \lim_{m\to\infty} x_n^{(m)}.$$

In other words, the limiting process for $m \to \infty$ commutes with the limiting process for $n \to \infty$, because

$$\lim_{m\to\infty} \lim_{n\to\infty} x_n^{(m)} = \lim_{m\to\infty} a^{(m)} = a.$$

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An alternative way to describe the condition of the uniform convergence of $x_n^{(m)}$ to $a^{(m)}$ as $n \to \infty$ is that the Cauchy property of the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is uniform in m as $n \to \infty$, which means that given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ independent of m such that $\left|x_n^{(m)} - x_k^{(m)}\right| < \varepsilon$ for $n, k \geq N$. The one direction of the equivalence of being uniformly Cauchy and uniformly convergent is from the triangle inequality

$$\left|x_n^{(m)} - x_k^{(m)}\right| \leq \left|x_n^{(m)} - a^{(m)}\right| + \left|a^{(m)} - x_k^{(m)}\right|$$

and the other direction is from letting $k \to \infty$ in

$$\left|x_n^{(m)} - x_k^{(m)}\right| < \varepsilon \quad \text{for} \quad n, k \geq N$$

to get

$$\left|x_n^{(m)} - a^{(m)}\right| \leq \varepsilon \quad \text{for} \quad m \geq N.$$

(Note that the difference between $\leq \varepsilon$ and $< \varepsilon$ can be handled by starting with $\frac{\varepsilon}{2}$ rather than ε.)

When the formulation in terms of the Cauchy property is used, the statement about commutativity of limit-taking with assumption of uniformity can be stated as follows.

Theorem. If the family of sequences $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ (with $m \in \mathbb{N}$ as index for the member of the family) is uniformly Cauchy in m as $n \to \infty$ and if for every fixed $n \in \mathbb{N}$ the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is Cauchy as $m \to \infty$, then both sides of the equation

$$\lim_{m\to\infty} \lim_{n\to\infty} x_n^{(m)} = \lim_{n\to\infty} \lim_{m\to\infty} x_n^{(m)}$$

exist and are equal.

(Note that $\lim_{n\to\infty} x_n^{(m)}$ exists because the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is Cauchy as $n \to \infty$ by assumption and also $\lim_{m\to\infty} x_n^{(m)}$ exists because the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is Cauchy as $m \to \infty$ by assumption.) The proof of the theorem is as follows. For fixed $m \in \mathbb{N}$ let $a^{(m)}$ be the limit of $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ as $n \to \infty$. For $\varepsilon > 0$ by the uniform Cauchy property in m there exists $N \in \mathbb{N}$ independent of m such that

$$(*) \qquad \left|x_n^{(m)} - x_k^{(m)}\right| < \varepsilon \quad \text{for} \quad n, k \geq N.$$

The key technique of the verification is to separately let $k \to \infty$ and let $m \to \infty$. It is possible to take both limits when the N in $(*)$ is independent of m.

We now do the first of the two limit-taking processes in $(*)$. The first one is for $k \to \infty$. By letting $k \to \infty$ in $(*)$, we obtain

$$(\#) \qquad \left|x_n^{(m)} - a^{(m)}\right| < \varepsilon \quad \text{for} \quad n \geq N.$$

Before we do the second limit-taking process in $(*)$ for $m \to \infty$, we first label the limit of the Cauchy sequence $x_n^{(m)}$ in m for fixed n. For fixed $n \in \mathbb{N}$ let a_n be the limit of $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ as $m \to \infty$. For $\varepsilon > 0$ there exists M_n such that

$$(\flat) \qquad \left|x_n^{(m)} - a_n\right| < \varepsilon \quad \text{for} \quad m \geq M_n.$$

We now do the second limit-taking process in $(*)$ for $m \to \infty$. By letting $m \to \infty$ in $(*)$ (and using the notation a_n which we have just introduced), we obtain

$$(\dagger) \qquad |a_n - a_k| < \varepsilon \quad \text{for} \quad n, k \geq N.$$

Thus the sequence $\{a_n\}_{n\in\mathbb{N}}$ is Cauchy as $n \to \infty$ and its limit a exists. Setting $n = N$ and letting $k \to \infty$ in $(\dagger)$, we obtain

$$(\natural) \qquad |a_N - a| < \varepsilon.$$

We now verify that the limit of $a^{(m)}$ is a as $m \to \infty$ by using the so-called 3ε technique. For $m \geq M_N$ we have

$$(**) \qquad \left|a^{(m)} - a\right| \leq \left|a^{(m)} - x_N^{(m)}\right| + \left|x_N^{(m)} - a_N\right| + |a_N - a| < 3\varepsilon,$$

because each of the three expressions

$$\left|a^{(m)} - x_N^{(m)}\right|, \qquad \left|x_N^{(m)} - a_N\right|, \qquad |a_N - a|$$

is $< \varepsilon$ from $(\#)$, $(\flat)$, and $(\natural)$. From $(**)$ it follows that

$$\lim_{m\to\infty} a^{(m)} = a.$$

Thus we have

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} x_n^{(m)} &= \lim_{m\to\infty} a^{(m)} = a \\ &= \lim_{n\to\infty} a_n = \lim_{n\to\infty}\lim_{m\to\infty} x_n^{(m)}.\end{aligned}$$

The two key steps in the argument are (i) letting $k \to \infty$ and $n \to \infty$ separately in the inequality

$$\left|x_n^{(m)} - x_n^{(m)}\right| < \varepsilon \quad \text{for} \quad n, k \geq N$$

from the uniform Cauchy property of convergence and (ii) the 3ε argument involving the triangle inequality.

Properties of Sets Coming From Limits. Given a set E in $\mathbb{R}$. We are concerned about several properties of E coming from limits.

Closed Subsets. Given a subset E of $\mathbb{R}$, if the limit of every convergent sequence in E also belongs to E, we say that E is *closed*.

This means that E is closed under the operation of taking limits of sequences in E.

Open Subsets. A subset E of $\mathbb{R}$ is said to be open if its complement $\mathbb{R} - E$ is a closed subset of $\mathbb{R}$.

Interior, Closure, and Boundary. For a subset E of $\mathbb{R}$ the smallest closed subset $\bar{E}$ of $\mathbb{R}$ is called the *closure* of E. For a subset E of $\mathbb{R}$ the largest open subset E^o of $\mathbb{R}$ is called the *interior* of E. (The interior of E is also denoted by $\operatorname{Int} E$.) For a subset E of $\mathbb{R}$ the complement $\bar{E} - E^o$ of E^o in $\bar{E}$ is called the boundary bE of E. (The boundary of E is also denoted by ∂E.)

Compact Subsets. Given a subset E of $\mathbb{R}$, if every sequence in E contains a subsequence which converges to some point in E, we say that E is *compact*.

The adjective "compact" here is in the sense of "occupying little space compared with others of its type, as in a compact camera or a compact car." Actually a subset E of $\mathbb{R}$ is compact if and only if it is bounded and closed. Here E "bounded" means that E is contained in some finite interval $[-A, A]$ in $\mathbb{R}$ with A being some positive real number. The verification of the characterization of a compact subset of $\mathbb{R}$ by boundedness and closedness is as follows.

Since any subsequence of a convergent sequence has the same limit, it follows any compact set E must be closed. A compact set E must also be bounded, otherwise there exists in E either an unbounded strictly increasing sequence or an unbounded strictly decreasing sequence which cannot have any limit. Conversely, if E is bounded and closed, every sequence in E is bounded and (as shown above when we verify that any Cauchy sequence converges to some limit) admits a subsequence whose limit is its lim sup which must be in E due to the closedness of E.

Connected Subsets. Given a subset E of $\mathbb{R}$, if for any two distinct points $a < b$ of E the interval $[a, b]$ is contained in E, we say that E is connected.

TO BE CONTINUED ...