LOGICAL FOUNDATION OF REAL ANALYSIS

Least Upper Bound Property from Dedekind Cut. For the existence of limit the *least upper bound property* is needed for the set $\mathbb{R}$ of all real numbers. The least upper bound property of $\mathbb{R}$ is a consequence of the definition of $\mathbb{R}$ by Dedekind cuts.

Definition of Least Upper Bound. A real number A is an *upper bound* of a nonempty subset E of $\mathbb{R}$ if $x \leq A$ for all $x \in E$. A real number a is the *least upper bound* of a nonempty subset E of $\mathbb{R}$ if (i) a is an upper bound of E and (ii) any real number b which is $< a$ cannot be an upper bound of E. The least upper bound of E is denoted by $\sup E$ and is also called the *supremum* of E.

Least Upper Bound Property of $\mathbb{R}$*.* The set $\mathbb{R}$ of all real numbers satisfies the upper bound property which states that the least upper bound exists for any nonempty subset E of $\mathbb{R}$ which admits an upper bound. This least upper bound property of $\mathbb{R}$ is a consequence of its definition from Dedekind cuts. We recall here the definition of Dedekind cuts and how $\mathbb{R}$ is defined from it. The existence of $\sup E$ is proved by taking the union of all Dedekind cuts which correspond to points of E. The verification of the details of the argument was assigned as a problem in the first homework assignment of last semester's Math 55a.

Definition of Dedekind Cut. Let $\mathbb{Q}_+$ denote the set of all positive rational numbers. A proper subset ξ of $\mathbb{Q}_+$ is called a (Dedekind) *cut* if

(i) $\xi \neq \emptyset$ (and $\xi \neq \mathbb{Q}_+$);

(ii) $x \in \xi$ and $y \in \mathbb{Q}_+ - \xi$ imply $x < y$;

(iii) $\nexists\, x \in \xi$ such that $x \geq y$ for $y \in \xi$;

where $\mathbb{Q}_+ - \xi$ means the complement of the set ξ in $\mathbb{Q}_+$.

Identification of Positive Rational Numbers as Dedekind Cuts. We denote by $\mathbb{R}_+$ the set of all (Dedekind) cuts. An element r of $\mathbb{Q}_+$ is identified with the (Dedekind) cut ξ_r which is defined as the set of all $s \in \mathbb{Q}_+$ such that $s < r$. The map $r \mapsto \xi_r$ identifies $\mathbb{Q}_+$ as a subset of $\mathbb{R}_+$.

Motivation for Dedekind Cut. The motivation of a Dedekind cut ξ is the set $(0,\xi)\cap\mathbb{Q}_+$ when ξ is regarded as a positive real number and $(0,\xi)$ means the open interval whose end-points are 0 and the positive real number ξ.

Definition of Ordering in $\mathbb{R}_+$. Ordering in $\mathbb{R}_+$ is defined as follows. Two (Dedekind) cuts ξ and η satisfy $\xi > \eta$ if as subsets of $\mathbb{Q}_+$ the set ξ contains the set η. Two (Dedekind) cuts ξ and η satisfy $\xi < \eta$ if as subsets of $\mathbb{Q}_+$ the set ξ is contained in the set η.

Definition of Addition in $\mathbb{R}_+$. The sum $\xi+\eta$ of two (Dedekind) cuts ξ and η is defined as the (Dedekind) cut ζ which consists of all $z\in\mathbb{Q}_+$ of the form $z=x+y$ for some $x\in\xi$ and some $y\in\eta$.

Definition of Multiplication in $\mathbb{R}_+$. The product $\xi\cdot\eta$ of two (Dedekind) cuts ξ and η is defined as the (Dedekind) cut ζ which consists of all $z\in\mathbb{Q}_+$ of the form $z=x\cdot y$ for some $x\in\xi$ and some $y\in\eta$.

Construction of the Set of All Real Numbers The set $\mathbb{R}$ of all real numbers is defined as the union of the three disjoint subsets $-\mathbb{R}_+$, $\{0\}$, and $\mathbb{R}_+$, where 0 is a new symbol and $-\mathbb{R}_+$ as a set is bijective to $\mathbb{R}_+$ with the element $-\xi$ of $-\mathbb{R}_+$ corresponding to the element ξ of $\mathbb{R}_+$. The arithmetic operations of addition and multiplication for $\mathbb{R}$ are naturally defined from those for $\mathbb{R}_+$. The notion of ordering in $\mathbb{R}$ is naturally defined from that in $\mathbb{R}_+$.

Greatest Lower Bound. For a subset E of $\mathbb{E}$, by considering the set $\{-a\}_{a\in E}$, we can apply the same argument to lower bounds instead of upper bounds and obtain the corresponding *greatest lower bound property* of $\mathbb{R}$. The greatest lower bound of a subset E of $\mathbb{R}$ is also called its *infimum* and is denoted by $\inf E$.

Limits of Sequences. We introduce the notion of the limit of a sequence, the relation between the convergence of a series and that of its sequence of partial sums, and the rôle of uniform convergence of a family of sequences in the commutativity of limit-taking, one as the index of each sequence going to infinity and the other as the index of family members going to infinity.

Definition of Limit of Sequence. A sequence x_n in $\mathbb{R}$ approaches a as its limit if it eventually gets inside any prescribed neighborhood of a. More precisely, given any $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $x_n\in(a-\varepsilon,a+\varepsilon)$ for $n\geq N$. The notation for the convergence of x_n to a is $\lim_{n\to\infty}x_n=a$.

An alternative notation is $x_n \to a$ as $n \to \infty$. The set $(a-\varepsilon, a+\varepsilon)$ is called the ε-neighborhood of a. When a sequence x_n converges to a as $n \to \infty$, then any subsequence x_{n_k} converges to the same limit a as $k \to \infty$. Here for the definition of a subsequence we require that n_k is a strictly increasing function mapping $k \in \mathbb{N}$ to $n_k \in \mathbb{N}$.

Alternative Definition of Limit of Sequence Without Specifying Limit (Cauchy Sequence). A sequence x_n in $\mathbb{R}$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for any $m, n \geq N$.

A sequence in $\mathbb{R}$ converges to some limit if and only if it is a Cauchy sequence. From the triangle inequality, the direction that a sequence convergent to some $a \in \mathbb{R}$ is Cauchy is clear, because if $|x_n - a| < \varepsilon$ for any $n \geq N_\varepsilon$, then

$$|x_m - x_n| \leq |x_m - a| + |a - x_n| < \varepsilon$$

for $m, n \geq N_{\frac{\varepsilon}{2}}$.

For the other direction of showing that every Cauchy sequence converges, the use of $\varepsilon = 1$ implies that the set $\{x_n\}_{n \in \mathbb{N}}$ is contained in $[-A, A]$ for

$$A = \max\left(|x_1|, \cdots, |x_{N_1 - 1}|, |x_{N_1}| + 1\right)$$

if $N_1 \in \mathbb{N}$ satisfied $|x_m - x_n| < 1$ for $m, n \geq N_1$. We first introduce the notation of *lim sup* of the sequence x_n, which is the abbreviation for the longer full expression *limit superior*. For m let E_m be the set $\{x_n\}_{n \geq m}$. Let a_m be the supremum of E_m. Then $a_{m+1} \leq a_m$, because E_{m+1} is contained in E_m. The infimum a of $\{a_m\}_{m \in \mathbb{N}}$ is called the *lim sup* of the sequence x_n and is denoted by $\limsup_{n \to \infty} x_n$. Another simpler way to introduce the lim sup of x_n is that

$$\limsup_{n \to \infty} x_n = \inf_{m \in \mathbb{N}} \sup_{n \geq m} x_n.$$

For any $k \in \mathbb{N}$ the real number $a + \frac{1}{k}$ is not a lower bound of $\{a_m\}_{m \in \mathbb{N}}$ and there exists some $a \leq a_{m_k} < a + \frac{1}{k}$. Since $a \leq a_p \leq a_{m_k}$ for $p \geq m_k$, by replacing m_k by $\max(m_k, k)$ we can assume without loss of generality that $m_k \geq k$. Since $a_{m_k} - \frac{1}{k}$ is not an upper bound of E_{m_k} and a_{m_k} is an upper bound of E_{m_k} there exists some $a_{m_k} - \frac{1}{k} \leq x_{n_k} \leq a_{m_k}$ with $n_k \geq m_k$. From $a \leq a_{m_k} < a + \frac{1}{k}$ and $a_{m_k} - \frac{1}{k} \leq x_{n_k} \leq a_{m_k}$ it follows that $a - \frac{1}{k} \leq a_{m_k} \leq a + \frac{1}{k}$ with some $m_k \geq k$.

Since x_n is a Cauchy sequence, for any given $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for $m, n \geq N$. By replacing N by another integer $\geq N$, we can assume without loss of generality that $\frac{1}{N} < \varepsilon$. Then when $n \geq N$, we can choose $m = m_N \geq N$ and get

$$|x_n - a| \leq |x_n - x_{m_N}| + |x_{m_N} - a| < \varepsilon + \frac{1}{N} < 2\varepsilon.$$

This shows that the Cauchy sequence x_n always converges $a = \limsup_{n\to\infty} x_n$.

TO BE CONTINUED ...