LOGICAL FOUNDATION OF REAL ANALYSIS

Least Upper Bound Property from Dedekind Cut. For the existence of limit the *least upper bound property* is needed for the set $\mathbb{R}$ of all real numbers. The least upper bound property of $\mathbb{R}$ is a consequence of the definition of $\mathbb{R}$ by Dedekind cuts.

Definition of Least Upper Bound. A real number A is an *upper bound* of a nonempty subset E of $\mathbb{R}$ if $x \leq A$ for all $x \in E$. A real number a is the *least upper bound* of a nonempty subset E of $\mathbb{R}$ if (i) a is an upper bound of E and (ii) any real number b which is $< a$ cannot be an upper bound of E. The least upper bound of E is denoted by $\sup E$ and is also called the *supremum* of E.

Least Upper Bound Property of $\mathbb{R}$. The set $\mathbb{R}$ of all real numbers satisfies the upper bound property which states that the least upper bound exists for any nonempty subset E of $\mathbb{R}$ which admits an upper bound. This least upper bound property of $\mathbb{R}$ is a consequence of its definition from Dedekind cuts. We recall here the definition of Dedekind cuts and how $\mathbb{R}$ is defined from it. The existence of $\sup E$ is proved by taking the union of all Dedekind cuts which correspond to points of E. The verification of the details of the argument was assigned as a problem in the first homework assignment of last semester's Math 55a.

Definition of Dedekind Cut. Let $\mathbb{Q}_+$ denote the set of all positive rational numbers. A proper subset ξ of $\mathbb{Q}_+$ is called a (Dedekind) *cut* if

(i) $\xi \neq \emptyset$ (and $\xi \neq \mathbb{Q}_+$);

(ii) $x \in \xi$ and $y \in \mathbb{Q}_+ - \xi$ imply $x < y$;

(iii) $\nexists\, x \in \xi$ such that $x \geq y$ for $y \in \xi$;

where $\mathbb{Q}_+ - \xi$ means the complement of the set ξ in $\mathbb{Q}_+$.

Identification of Positive Rational Numbers as Dedekind Cuts. We denote by $\mathbb{R}_+$ the set of all (Dedekind) cuts. An element r of $\mathbb{Q}_+$ is identified with the (Dedekind) cut ξ_r which is defined as the set of all $s \in \mathbb{Q}_+$ such that $s < r$. The map $r \mapsto \xi_r$ identifies $\mathbb{Q}_+$ as a subset of $\mathbb{R}_+$.

Motivation for Dedekind Cut. The motivation of a Dedekind cut ξ is the set $(0,\xi)\cap\mathbb{Q}_+$ when ξ is regarded as a positive real number and $(0,\xi)$ means the open interval whose end-points are 0 and the positive real number ξ.

Definition of Ordering in $\mathbb{R}_+$. Ordering in $\mathbb{R}_+$ is defined as follows. Two (Dedekind) cuts ξ and η satisfy $\xi > \eta$ if as subsets of $\mathbb{Q}_+$ the set ξ contains the set η. Two (Dedekind) cuts ξ and η satisfy $\xi < \eta$ if as subsets of $\mathbb{Q}_+$ the set ξ is contained in the set η.

Definition of Addition in $\mathbb{R}_+$. The sum $\xi + \eta$ of two (Dedekind) cuts ξ and η is defined as the (Dedekind) cut ζ which consists of all $z \in \mathbb{Q}_+$ of the form $z = x + y$ for some $x \in \xi$ and some $y \in \eta$.

Definition of Multiplication in $\mathbb{R}_+$. The product $\xi \cdot \eta$ of two (Dedekind) cuts ξ and η is defined as the (Dedekind) cut ζ which consists of all $z \in \mathbb{Q}_+$ of the form $z = x \cdot y$ for some $x \in \xi$ and some $y \in \eta$.

Construction of the Set of All Real Numbers The set $\mathbb{R}$ of all real numbers is defined as the union of the three disjoint subsets $-\mathbb{R}_+$, $\{0\}$, and $\mathbb{R}_+$, where 0 is a new symbol and $-\mathbb{R}_+$ as a set is bijective to $\mathbb{R}_+$ with the element $-\xi$ of $-\mathbb{R}_+$ corresponding to the element ξ of $\mathbb{R}_+$. The arithmetic operations of addition and multiplication for $\mathbb{R}$ are naturally defined from those for $\mathbb{R}_+$. The notion of ordering in $\mathbb{R}$ is naturally defined from that in $\mathbb{R}_+$.

Greatest Lower Bound. For a subset E of $\mathbb{E}$, by considering the set $\{-a\}_{a\in E}$, we can apply the same argument to lower bounds instead of upper bounds and obtain the corresponding *greatest lower bound property* of $\mathbb{R}$. The greatest lower bound of a subset E of $\mathbb{R}$ is also called its *infimum* and is denoted by $\inf E$.

Limits of Sequences. We introduce the notion of the limit of a sequence, the relation between the convergence of a series and that of its sequence of partial sums, and the rôle of uniform convergence of a family of sequences in the commutativity of limit-taking, one as the index of each sequence going to infinity and the other as the index of family members going to infinity.

Definition of Limit of Sequence. A sequence x_n in $\mathbb{R}$ approaches a as its limit if it eventually gets inside any prescribed neighborhood of a. More precisely, given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in (a-\varepsilon, a+\varepsilon)$ for $n \geq N$. The notation for the convergence of x_n to a is $\lim_{n\to\infty} x_n = a$. An alternative notation is $x_n \to a$ as $n \to \infty$. The set $(a-\varepsilon, a+\varepsilon)$ is called the ε-neighborhood of a. *For the time being* when we talk about a neighborhood we mean the ε-neighborhood for some $\varepsilon > 0$. When a sequence x_n converges to a as $n \to \infty$, then any subsequence x_{n_k} converges to the same limit a as $k \to \infty$. Here for the definition of a subsequence we require that n_k is a strictly increasing function mapping $k \in \mathbb{N}$ to $n_k \in \mathbb{N}$.

Alternative Definition of Limit of Sequence Without Specifying Limit (Cauchy Sequence). A sequence x_n in $\mathbb{R}$ is a *Cauchy sequence* if for any $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for any $m, n \geq N$.

A sequence in $\mathbb{R}$ converges to some limit if and only if it is a Cauchy sequence. From the triangle inequality, the direction that a sequence convergent to some $a \in \mathbb{R}$ is Cauchy is clear, because if $|x_n - a| < \varepsilon$ for any $n \geq N_\varepsilon$, then

$$|x_m - x_n| \leq |x_m - a| + |a - x_n| < \varepsilon$$

for $m, n \geq N_{\frac{\varepsilon}{2}}$.

For the other direction of showing that every Cauchy sequence converges, the use of $\varepsilon = 1$ implies that the set $\{x_n\}_{n\in\mathbb{N}}$ is contained in $[-A, A]$ for

$$A = \max\left(|x_1|, \cdots, |x_{N_1-1}|, |x_{N_1}| + 1\right)$$

if $N_1 \in \mathbb{N}$ satisfied $|x_m - x_n| < 1$ for $m, n \geq N_1$.

We first introduce the notation of *lim sup* of the sequence x_n, which is the abbreviation for the longer full expression *limit superior*. For m let E_m be the set $\{x_n\}_{n\geq m}$. Let a_m be the supremum of E_m. Then $a_{m+1} \leq a_m$, because E_{m+1} is contained in E_m. The infimum a of $\{a_m\}_{m\in\mathbb{N}}$ is called the *lim sup* of the sequence x_n and is denoted by $\limsup_{n\to\infty} x_n$. Another simpler way to introduce the lim sup of x_n is that

$$\limsup_{n\to\infty} x_n = \inf_{m\in\mathbb{N}} \sup_{n\geq m} x_n.$$

For any $k \in \mathbb{N}$ the real number $a + \frac{1}{k}$ is not a lower bound of $\{a_m\}_{m\in\mathbb{N}}$ and there exists some $a \leq a_{m_k} < a + \frac{1}{k}$. Since $a \leq a_p \leq a_{m_k}$ for $p \geq m_k$, by replacing m_k by $\max(m_k, k)$ we can assume without loss of generality that $m_k \geq k$. Since $a_{m_k} - \frac{1}{k}$ is not an upper bound of E_{m_k} and a_{m_k} is an upper bound of E_{m_k} there exists some $a_{m_k} - \frac{1}{k} \leq x_{n_k} \leq a_{m_k}$ with $n_k \geq m_k$. From $a \leq a_{m_k} < a + \frac{1}{k}$ and $a_{m_k} - \frac{1}{k} \leq x_{n_k} \leq a_{m_k}$ it follows that $a - \frac{1}{k} \leq a_{m_k} \leq a + \frac{1}{k}$ with some $m_k \geq k$. With the use of lim sup, what we have shown is that *any bounded sequence admits a subsequence which converges to its lim sup.*

Since x_n is a Cauchy sequence, for any given $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for $m, n \geq N$. By replacing N by another integer $\geq N$, we can assume without loss of generality that $\frac{1}{N} < \varepsilon$. Then when $n \geq N$, we can choose $m = m_N \geq N$ and get

$$|x_n - a| \leq |x_n - x_{m_N}| + |x_{m_N} - a| < \varepsilon + \frac{1}{N} < 2\varepsilon.$$

This shows that the Cauchy sequence x_n always converges $a = \limsup_{n\to\infty} x_n$.

Relation Between Sequence and Series. A series $\sum_{k=0}^{n} a_k$ with $a_k \in \mathbb{R}$ converges to a limit L if and only if the sequence $s_n = \sum_{k=0}^{n} a_k$ of partial sums converges to L. To a given sequence s_n a series with terms $a_n = s_n - s_{n-1}$ can be constructed whose partial sum is s_n.

Uniform Convergence of a Sequence of Sequences and Commutativity of Limit-Taking. For every fixed $m \in \mathbb{N}$ suppose a sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is given which converges to $a^{(m)}$. The convergence of $x_n^{(m)}$ to $a^{(m)}$ is said to be *uniform* in m as $n \to \infty$ if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ independent of m such that $\left|x_n^{(m)} - a^{(m)}\right| < \varepsilon$ for $n \geq N$. Assume now the convergence of $x_n^{(m)}$ to $a^{(m)}$ is *uniform* in m as $n \to \infty$. Assume also that for every fixed $n \in \mathbb{N}$ the sequence $\left\{x_n^{(m)}\right\}_{m\in\mathbb{N}}$ converges to some a_n as $m \to \infty$. Then the sequence a_n converges to some a as $n \to \infty$. Moreover, the sequence $a^{(m)}$ converges to the same a as $m \to \infty$ so that

$$\lim_{m\to\infty} \lim_{n\to\infty} x_n^{(m)} = \lim_{n\to\infty} \lim_{m\to\infty} x_n^{(m)}.$$

In other words, the limiting process for $m \to \infty$ commutes with the limiting process for $n \to \infty$, because

$$\lim_{m\to\infty} \lim_{n\to\infty} x_n^{(m)} = \lim_{m\to\infty} a^{(m)} = a.$$
$$\lim_{n\to\infty} \lim_{m\to\infty} x_n^{(m)} = \lim_{m\to\infty} a_n = a.$$

An alternative way to describe the condition of the uniform convergence of $x_n^{(m)}$ to $a^{(m)}$ as $n \to \infty$ is that the Cauchy property of the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is uniform in m as $n \to \infty$, which means that given any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ independent of m such that $\left|x_n^{(m)} - x_k^{(m)}\right| < \varepsilon$ for $n, k \geq N$. The one direction of the equivalence of being uniformly Cauchy and uniformly convergent is from the triangle inequality

$$\left|x_n^{(m)} - x_k^{(m)}\right| \leq \left|x_n^{(m)} - a^{(m)}\right| + \left|a^{(m)} - x_k^{(m)}\right|$$

and the other direction is from letting $k \to \infty$ in

$$\left|x_n^{(m)} - x_k^{(m)}\right| < \varepsilon \quad \text{for} \quad n, k \geq N$$

to get

$$\left|x_n^{(m)} - a^{(m)}\right| \leq \varepsilon \quad \text{for} \quad m \geq N.$$

(Note that the difference between $\leq \varepsilon$ and $< \varepsilon$ can be handled by starting with $\frac{\varepsilon}{2}$ rather than ε.)

When the formulation in terms of the Cauchy property is used, the statement about commutativity of limit-taking with assumption of uniformity can be stated as follows.

Theorem. If the family of sequences $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ (with $m \in \mathbb{N}$ as index for the member of the family) is uniformly Cauchy in m as $n \to \infty$ and if for every fixed $n \in \mathbb{N}$ the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is Cauchy as $m \to \infty$, then both sides of the equation

$$\lim_{m\to\infty} \lim_{n\to\infty} x_n^{(m)} = \lim_{n\to\infty} \lim_{m\to\infty} x_n^{(m)}$$

exist and are equal.

(Note that $\lim_{n\to\infty} x_n^{(m)}$ exists because the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is Cauchy as $n \to \infty$ by assumption and also $\lim_{m\to\infty} x_n^{(m)}$ exists because the sequence $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ is Cauchy as $m \to \infty$ by assumption.) The proof of the theorem is as follows. For fixed $m \in \mathbb{N}$ let $a^{(m)}$ be the limit of $\left\{x_n^{(m)}\right\}_{n\in\mathbb{N}}$ as $n \to \infty$. For $\varepsilon > 0$ by the uniform Cauchy property in m there exists $N \in \mathbb{N}$ independent of m such that

$$(*) \qquad \left|x_n^{(m)} - x_k^{(m)}\right| < \varepsilon \quad \text{for} \quad n, k \geq N.$$

The key technique of the verification is to separately let $k \to \infty$ and let $m \to \infty$. It is possible to take both limits when the N in $(*)$ is independent of m.

We now do the first of the two limit-taking processes in $(*)$. The first one is for $k \to \infty$. By letting $k \to \infty$ in $(*)$, we obtain

$$(\#) \qquad \left|x_n^{(m)} - a^{(m)}\right| < \varepsilon \quad \text{for} \quad n \geq N.$$

Before we do the second limit-taking process in $(*)$ for $m \to \infty$, we first label the limit of the Cauchy sequence $x_n^{(m)}$ in m for fixed n. For fixed $n \in \mathbb{N}$ let a_n be the limit of $\left\{x_n^{(m)}\right\}_{n \in \mathbb{N}}$ as $m \to \infty$. For $\varepsilon > 0$ there exists M_n such that

$$(\flat) \qquad \left|x_n^{(m)} - a_n\right| < \varepsilon \quad \text{for} \quad m \geq M_n.$$

We now do the second limit-taking process in $(*)$ for $m \to \infty$. By letting $m \to \infty$ in $(*)$ (and using the notation a_n which we have just introduced), we obtain

$$(\dagger) \qquad |a_n - a_k| < \varepsilon \quad \text{for} \quad n, k \geq N.$$

Thus the sequence $\{a_n\}_{n \in \mathbb{N}}$ is Cauchy as $n \to \infty$ and its limit a exists. Setting $n = N$ and letting $k \to \infty$ in $(\dagger)$, we obtain

$$(\natural) \qquad |a_N - a| < \varepsilon.$$

We now verify that the limit of $a^{(m)}$ is a as $m \to \infty$ by using the so-called 3ε technique. For $m \geq M_N$ we have

$$(**) \qquad \left|a^{(m)} - a\right| \leq \left|a^{(m)} - x_N^{(m)}\right| + \left|x_N^{(m)} - a_N\right| + |a_N - a| < 3\varepsilon,$$

because each of the three expressions

$$\left|a^{(m)} - x_N^{(m)}\right|, \qquad \left|x_N^{(m)} - a_N\right|, \qquad |a_N - a|$$

is $< \varepsilon$ from $(\#)$, $(\flat)$, and $(\natural)$. From $(**)$ it follows that

$$\lim_{m \to \infty} a^{(m)} = a.$$

Thus we have

$$\begin{aligned} \lim_{m \to \infty} \lim_{n \to \infty} x_n^{(m)} &= \lim_{m \to \infty} a^{(m)} = a \\ &= \lim_{n \to \infty} a_n = \lim_{n \to \infty} \lim_{m \to \infty} x_n^{(m)}. \end{aligned}$$

The two key steps in the argument are (i) letting $k \to \infty$ and $n \to \infty$ separately in the inequality

$$\left|x_n^{(m)} - x_n^{(m)}\right| < \varepsilon \quad \text{for} \quad n, k \geq N$$

from the uniform Cauchy property of convergence and (ii) the 3ε argument involving the triangle inequality.

Properties of Sets Coming From Limits. Given a set E in $\mathbb{R}$. We are concerned about several properties of E coming from limits.

Closed Subsets. Given a subset E of $\mathbb{R}$, if the limit of every convergent sequence in E also belongs to E, we say that E is *closed*.

This means that E is closed under the operation of taking limits of sequences in E.

Open Subsets. A subset E of $\mathbb{R}$ is said to be open if its complement $\mathbb{R} - E$ is a closed subset of $\mathbb{R}$.

Interior, Closure, and Boundary. For a subset E of $\mathbb{R}$ the smallest closed subset $\bar{E}$ of $\mathbb{R}$ is called the *closure* of E. For a subset E of $\mathbb{R}$ the largest open subset E^o of $\mathbb{R}$ is called the *interior* of E. (The interior of E is also denoted by $\operatorname{Int} E$.) For a subset E of $\mathbb{R}$ the complement $\bar{E} - E^o$ of E^o in $\bar{E}$ is called the boundary bE of E. (The boundary of E is also denoted by ∂E.)

Compact Subsets. Given a subset E of $\mathbb{R}$, if every sequence in E contains a subsequence which converges to some point in E, we say that E is *compact*.

The adjective "compact" here is in the sense of "occupying little space compared with others of its type, as in a compact camera or a compact car." Actually a subset E of $\mathbb{R}$ is compact if and only if it is bounded and closed. Here E "bounded" means that E is contained in some finite interval $[-A, A]$ in $\mathbb{R}$ with A being some positive real number. The verification of the characterization of a compact subset of $\mathbb{R}$ by boundedness and closedness is as follows.

Since any subsequence of a convergent sequence has the same limit, it follows any compact set E must be closed. A compact set E must also be bounded, otherwise there exists in E either an unbounded strictly increasing sequence or an unbounded strictly decreasing sequence which cannot have any limit. Conversely, if E is bounded and closed, every sequence in E is bounded and (as shown above when we verify that any Cauchy sequence converges to some limit) admits a subsequence whose limit is its lim sup which must be in E due to the closedness of E.

Connected Subsets. Given a subset E of $\mathbb{R}$, if for any two distinct points $a < b$ of E the interval $[a, b]$ is contained in E, we say that E is connected.

Limit of Function, Continuity, and Differentiability. We now introduce the notions of the limit of a function, the definition of continuity for a function, and the definition of the derivative of a function.

Limit of Function. Let $f(x)$ be a real-valued function defined on (α, β) for some $\alpha < \beta$ and let $a \in (\alpha, \beta)$. A real number L is the *limit* of $f(x)$ as $x \to a$ if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for $0 < |x - a| < \delta$. In other words, the value of f is in any prescribed neighborhood of L when the variable x is in a sufficiently small deleted neighborhood of a. A deleted neighborhood of a point means the neighborhood of a point minus the point itself. The notation for the statement that L is the limit of $f(x)$ as $x \to a$ is

$$L = \lim_{x \to a} f(x).$$

One-Sided Limit of Function. Let $f(x)$ be a real-valued function defined on (a, b) for some $a < b$. A real number L is the *limit of $f(x)$ as $x \to a$ from above (or from the right)* if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for $0 < x - a < \delta$. The notation for this statement is

$$L = \lim_{x \to a+} f(x).$$

A real number L is the *limit of $f(x)$ as $x \to b$ from below (or from the left)* if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for $0 < b - x < \delta$. The notation for this statement is

$$L = \lim_{x \to b-} f(x).$$

Relation Between Limit of Function and Limit of Sequence. Let $f(x)$ be a real-valued function defined on (α, β) for some $\alpha < \beta$ and let $a \in (\alpha, \beta)$. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{n \to \infty} f(x_n) = L$ for any sequence x_n in $(\alpha, \beta) - \{a\}$ which converges to a as $n \to \infty$.

The verification is as follows. Suppose $\lim_{x\to a} f(x) = L$. Then for any given $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for $0 < |x-a| < \delta$. Suppose x_n is a sequence in $(\alpha, \beta) - \{a\}$ which converges to a as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $|x_n - a| < \delta$ for $n \geq N$. Putting the two inequalities together, we conclude that $|f(x_n) - L| < \varepsilon$ for $n \geq N$, which means that $\lim_{n\to\infty} f(x_n) = L$.

On the other hand, suppose $\lim_{n\to\infty} f(x_n) = L$ for any sequence x_n in $(\alpha, \beta) - \{a\}$ which converges to a as $n \to \infty$. We are going to prove by *reductio ad absurdum* that $\lim_{x\to a} f(x) = L$. Suppose the contrary. Then there exists some $\varepsilon > 0$ such that the statement that $|f(x) - L| < \varepsilon$ for any $0 < |x-a| < \delta$ fails to hold no matter what $\delta > 0$ we choose. In particular, when we choose $\delta = \frac{1}{n}$, the statement that $|f(x) - L| < \varepsilon$ for any $0 < |x-a| < \frac{1}{n}$ fails to hold. It means that for any $n \in \mathbb{N}$ there exists some x_n with $0 < |x_n - a| < \frac{1}{n}$ such that $|f(x_n) - L| \geq \varepsilon$. Now from $0 < |x_n - a| < \frac{1}{n}$ for any $n \in \mathbb{N}$ it follows that the sequence x_n in $(\alpha, \beta) - \{a\}$ converges to a as $n \to \infty$. By assumption $\lim_{n\to\infty} f(x_n) = L$, which means that there exists some $N \in \mathbb{N}$ such that $|f(x_n) - L| < \varepsilon$ for any $n \geq N$, which contradicts $|f(x_n) - L| \geq \varepsilon$ for any $n \in \mathbb{N}$.

Definition of Continuity at Interior Point. Let $f(x)$ be a real-valued function defined on (α, β) for some $\alpha < \beta$ and let $a \in (\alpha, \beta)$. The function $f(x)$ is said to be *continuous* at a if

$$\lim_{x\to a} f(x) = f(a).$$

Definition of Continuity at End-Point. Let $f(x)$ be a real-valued function defined on $[a, b)$ for some $a < b$. The function $f(x)$ is said to be *continuous* at the left end-point a if

$$\lim_{x\to a+} f(x) = f(a).$$

Let $f(x)$ be a real-valued function defined on $(a, b]$ for some $a < b$. The function $f(x)$ is said to be *continuous* at the right end-point b if

$$\lim_{x\to b-} f(x) = f(b).$$

Definition of Continuity on Interval. Let $a < b$ be real numbers and I be one of the intervals (a, b), $(a, b]$, $[a, b)$, $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), or $[a, \infty)$. A real-valued function $f(x)$ defined on I. The function $f(x)$ is said to be continuous on I if it is continuous at every point of I.

Achievement of Supremum of Continuous Function on Bounded Closed Interval. Let $a < b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a, b]$. Let M be the supremum of $f(x)$ for $a \leq x \leq b$. Then M is achieved at some point of $[a, b]$, that is, there exists $\xi \in [a, b]$ such that $M = f(\xi)$.

The verification is as follows. There exists $x_n \in [a, b]$ with $f(x_\nu) \to M$ as x_n. By replacing x_n by a subsequence, we can assume without loss of generality that $x_\nu \to \xi$ for some $\xi \in [a, b]$. It follows from the continuity of $f(x)$ at ξ that $M = f(\xi)$.

By applying this to $-f(x)$ on $[a, b]$, we conclude that the infimum of $f(x)$ on $[a, b]$ is also achieved at some point η of $[a, b]$, that is, $\inf_{a \leq x \leq b} f(x) = f(\eta)$.

Uniform Continuity of Continuous Function on Bounded Closed Interval. Let $a < b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a, b]$. Then $f(x)$ is *uniformly continuous* on $[a, b]$ in the sense that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for $x, y \in [a, b]$ with $|x - y| < \delta$.

The verification is by *reductio ad absurdum.* Assume that there exist some ε and $x_n, y_n \in [a, b]$ for $n \in \mathbb{N}$ such that $|x_n - y_n| < \frac{1}{n}$ and yet $|f(x_n) - f(y_n)| \geq \varepsilon$ for $n \in \mathbb{N}$. By replacing the sequences x_n, y_n by their subsequences, we can assume without loss of generality that x_n converges to some $\xi \in [a, b]$ and y_n converges to some $\eta \in [a, b]$ as $n \to \infty$. It follows from $|x_n - y_n| < \frac{1}{n}$ that the two points ξ and η must be the same. By the continuity of f at ξ there exists some $\delta > 0$ such that $|f(x) - f(\xi)| < \frac{\varepsilon}{2}$ when $|x - \xi| < \frac{\varepsilon}{2}$. There exists some $N \in \mathbb{N}$ such that $|x_n - \xi| < \delta$ and $|y_n - \xi| < \delta$ for $n \geq N$. It follows that for $n \geq N$,

$$|f(x_n) - f(y_n)| \leq |f(x_n) - f(\xi)| + |f(y_n) - f(\xi)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which contradicts $|f(x_n) - f(y_n)| \geq \varepsilon$ for $n \in \mathbb{N}$.

Remark. This result on uniform continuity of continuous functions on bounded closed intervals will be used in the definition given below of Riemann integrals of continuous functions.

Definition of Derivative at Interior Point. Let $f(x)$ be a real-valued function defined on (α, β) for some $\alpha < \beta$ and let $a \in (\alpha, \beta)$. The derivative $f'(a)$ of the function $f(x)$ at a is defined by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(if it exists). The function $f(x)$ is said to be *differentiable* at a if $f'(a)$ exists.

Definition of Derivative at End-Point. Let $f(x)$ be a real-valued function defined on $[a, b)$ for some $a < b$. The derivative $f'(a)$ of the function $f(x)$ at the left end-point a is defined by

$$f'(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

(if it exists). In order to highlight that the derivative is for a left end-point, the more precise notation $f'_+(a)$ is also used instead of $f'(a)$.

Let $f(x)$ be a real-valued function defined on $(a, b]$ for some $a < b$. The derivative $f'(b)$ of the function $f(x)$ at the right end-point b is defined by

$$f'(b) = \lim_{x \to b-} \frac{f(x) - f(b)}{x - b}$$

(if it exists). In order to highlight that the derivative is for a right end-point, the more precise notation $f'_-(b)$ is also used instead of $f'(b)$.

Derivative as Higher-Order Approximation by Polynomial of Degree at Most One. Let $f(x)$ be a real-valued function defined on (a, b) for some $a < b$ and let $x_0 \in (a, b)$ and $y_0 = f(x_0)$. An alternative description of the differentiability of $f(x)$ at x_0 is that the function $y = f(x)$ can be approximated by a polynomial $y = Ax + B$ of degree ≤ 1 at the point (x_0, y_0) to an order higher than the first in the sense that

$$f(x) = Ax + B + E(x)$$

with

$$\frac{E(x)}{x - x_0} \to 0 \quad \text{as} \quad x \to x_0.$$

In such a case the derivative of $f(x)$ at x_0 is given by A. This alternative description is justified as follows. Assume that we have such a higher-order approximation a polynomial of degree at most one. By letting $x \to x_0$ in $f(x) = Ax + B + E(x)$, we obtain $B = f(x_0) - Ax_0$ and

$$\frac{f(x) - f(x_0)}{x - x_0} = A + \frac{E(x)}{x - x_0}$$

which has limit A as $x \to x_0$. On the other hand, if $f'(x_0)$ exists, we can set $A = f'(x_0)$ and $B = f(x_0) - Ax_0$ and

$$E(x) = f(x) - Ax - B = f(x) - f(x_0) - f'(x_0)(x - x_0)$$

so that

$$\frac{E(x)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)$$

has limit 0 as $x \to x_0$.

Vanishing of Derivative at Interior Extremal Point. Let $a < b$ be real numbers and $f(x)$ be a real-valued function defined on (a, b). Assume that there exists $\xi \in (a, b)$ such that the supremum of f on (a, b) is achieved at ξ, that is, $f(\xi) \geq f(x)$ for $x \in (a, b)$. If $f'(\xi)$ exists, then it must be zero.

The verification is as follows. The derivative $f'(\xi)$ is the limit of the difference quotient

$$\frac{f(x) - f(\xi)}{x - \xi}$$

as $x \to \xi$. When $x > \xi$, the difference quotient

$$\frac{f(x) - f(\xi)}{x - \xi}$$

is nonnegative. Hence $f'(\xi)$ must be nonnegative. On the other hand, when $x < \xi$, the difference quotient

$$\frac{f(x) - f(\xi)}{x - \xi}$$

is nonpositive, implying that $f'(\xi)$ must be nonpositive. Hence $f'(\xi)$ must be zero.

Remark. This alternative definition of derivative by higher-order approximation by polynomial of degree at most one will be used later in the definition of differentiability of functions of several independent variables.

Mean-Value Theorem. Let $a < b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a,b]$ such that $f'(x)$ exists at every point of the open interval (a,b). Then there exists some $\xi \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(\xi).$$

The verification is as follows. First verify the special case where $f(a) = f(b) = 0$. If both $\sup_{[a,b]} f(x)$ and $\inf_{[a,b]} f(x)$ are both zero, then $f(x)$ is identically zero and ξ can be chosen to be any point of (a,b), because $f'(x) = 0$ for every $x \in (a,b)$. For the general case let

$$g(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$

which is the polynomial of degree ≤ 1 which assumes the value $f(a)$ at the point $x = a$ and assumes the value $f(b)$ at the point $x = b$. (This actually is the special case of Lagrange's interpolation formula for a polynomial of degree $\leq n$ which assumes prescribed values at $n+1$ distinct points.) We now replace $f(x)$ by $F(x) = f(x) - g(x)$. Then $F'(\xi) = 0$ for some $\xi \in (a,b)$. From

$$g'(x) = \frac{f(b)-f(a)}{b-a}$$

it follow that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.

Constancy of Function with Zero Derivative. As a direct consequence of the mean-value theorem, if $a < b$ be real numbers and if $f(x)$ is a continuous real-valued function on the bounded closed interval $[a,b]$ whose derivative exists and vanishes at every point of the open interval (a,b), then $f(a) = f(b)$.

Remark. This result on the constancy of functions with zero derivatives will be used below to prove the second part of the Fundamental Theorem of Calculus.

Riemann Integral of Continuous Function. We now introduce the definition of the integral of a function by using Riemann sums and partitions of intervals.

Partition. A partition P of a closed interval $[a,b]$ with $a<b$ being real numbers is a finite sequence of points

$$a=x_0\leq x_1\leq\cdots\leq x_{n-1}\leq x_n=b$$

for some $n\in\mathbb{N}$. A partition $\hat{P}$

$$a=\hat{x}_0\leq \hat{x}_1\leq\cdots\leq \hat{x}_{\hat{n}-1}\leq \hat{x}_{\hat{n}}=b$$

of $[a,b]$ is called a *refinement* of P if

$$\left\{x_j\right\}_{j=0}^{n}\subset\left\{\hat{x}_j\right\}_{j=0}^{\hat{n}}.$$

Riemann Sum. Let $a<b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a,b]$. Let P be a partition

$$a=x_0\leq x_1\leq\cdots\leq x_{n-1}\leq x_n=b$$

of $[a,b]$. A *Riemann sum* for the function f and the partition P means

$$\sum_{j=0}^{n-1}f\left(\xi_j\right)\left(x_{j+1}-x_j\right)$$

for some $\xi_j\in[x_j,x_{j+1}]$ for $0\leq j\leq n$, which we denote by $R(f,P,\xi)$ where ξ means the n-tuple $(\xi_0,\cdots,\xi_{n-1})$. The *upper Riemann sum* for the function f and the partition P means

$$\sum_{j=0}^{n-1}\left(\sup_{[x_j,x_{j+1}]}f\right)\left(x_{j+1}-x_j\right)$$

which we denote by $U(f,P)$. The *lower Riemann sum* for the function f and the partition P means

$$\sum_{j=0}^{n-1}\left(\inf_{[x_j,x_{j+1}]}f\right)\left(x_{j+1}-x_j\right)$$

which we denote by $U(f,P)$. By definition we always have

$$L(f,P)\leq R(f,P,\xi)\leq U(f,P)$$

and, moreover, for any partition $\hat{P}$ of P we always have

$$L(f,P) \leq L(f,\hat{P}) \leq U(f,\hat{P}) \leq U(f,P).$$

Definition of Riemann Integral of Continuous Function. Let $a < b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a,b]$. The integral of $f(x)$ over $[a,b]$ is defined as

$$\inf_P U(f,P)$$

over all partitions of $[a,b]$, which denote by $\int_a^b f$ or by $\int_a^b f(x)dx$ when we want to highlight the rôle of the independent variable x.

From the condition of the continuity of f on the bounded closed interval $[a,b]$ we can verify as follows that the integral $\int_a^b f$ can alternatively defined as

$$\sup_P U(f,P).$$

By the uniform continuity of f on $[a,b]$, given any $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a,b]$ with $|x - y| < \delta$. Choose a partition P

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$$

of $[a,b]$ such that $x_{j+1} - x_j < \delta$ for $0 \leq j \leq n-1$. Then

$$U(f,P) - L(f,P) \leq \varepsilon(b-a)$$

from

$$\left(\sup_{[x_j,x_{j+1}]} f\right) - \left(\inf_{[x_j,x_{j+1}]} f\right) \leq \varepsilon \quad \text{for} \quad 0 \leq j \leq n-1.$$

Since $\varepsilon > 0$ can be arbitrarily chosen, it follows that

$$\sup_P U(f,P) = \inf_P U(f,P).$$

Estimate of Riemann Integral. Let $a < b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a,b]$. Then

$$\left(\inf_{[a,b]} f\right)(b-a) \leq \int_a^b f \leq \left(\sup_{[a,b]} f\right)(b-a)$$

and

$$\int_a^b |f| \leq \left(\sup_{[a,b]} |f|\right)(b-a).$$

These estimates are clear from the definitions of upper and lower Riemann sums and the Riemann integral.

Additivity of Riemann Integral with Dividing Up of Interval of Integration. Let $a < c < b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a, b]$. Then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

The verification is as follows. Given any $\varepsilon > 0$, as above, from the uniform continuity of f on $[a, c]$ and on $[c, b]$ there exists a partition P_1

$$a = x_0^{(1)} \leq x_1^{(1)} \leq \cdots \leq x_{n_1-1}^{(1)} \leq x_{n_1}^{(1)} = c$$

of $[a, c]$ and there exists a partition P_2

$$c = x_0^{(2)} \leq x_1^{(2)} \leq \cdots \leq x_{n_2-1}^{(2)} \leq x_{n_2}^{(2)} = b$$

of $[c, b]$ such that

$$U(f, P_j) - L(f, P_j) < \varepsilon \quad \text{for} \quad j = 1, 2.$$

In particular, from

$$L(f, P_1) \int_a^c f \leq U(f, P_1) \quad \text{and} \quad L(f, P_2) \int_c^b f \leq U(f, P_2)$$

that

$$-\varepsilon + \int_a^c f \leq L(f, P_1) \quad \text{and} \quad U(f, P_1) \leq \varepsilon + \int_a^c f,$$
$$-\varepsilon + \int_c^b f \leq L(f, P_2) \quad \text{and} \quad U(f, P_2) \leq \varepsilon + \int_c^b f.$$

Let P be the partition of $[a, b]$ whose set of partition

$$a = x_0^{(1)} \leq x_1^{(1)} \leq \cdots \leq x_{n_1-1}^{(1)} \leq x_{n_1}^{(1)} \leq x_0^{(2)} \leq x_1^{(2)} \leq \cdots \leq x_{n_2-1}^{(2)} \leq x_{n_2}^{(2)} = b$$

of $[a, b]$ obtained by putting together the partition P_1 of $[a, c]$ and the partition P_2 of $[c, b]$. Since

$$U(f, P) = U(f, P_1) + U(f, P_2) \quad \text{and} \quad L(f, P) = L(f, P_1) + L(f, P_2),$$

it follows from

$$\begin{aligned} & -2\varepsilon + \int_a^c f + \int_c^b f \\ & \leq L(f, P_1) + L(f, P_2) \\ & = L(f, P) \leq \int_a^b f \leq U(f, P) \\ & = U(f, P_1) + U(f, P_2) \\ & \leq 2\varepsilon + \int_a^c f + \int_c^b f, \end{aligned}$$

that

$$\left| \int_a^b f + \int_a^b f - \int_a^b f \right| \leq 2\varepsilon.$$

The arbitrariness of $\varepsilon > 0$ implies that

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus states that the process of differentiation and the process of integration are reverse processes of each other. There are two parts in the Fundamental Theorem of Calculus. One part starts with integration and then uses differentiation to get back to the original function. Another part starts with differentiation and then uses integration to get back to the original function.

First Part of the Fundamental Theorem of Calculus. Let $a < b$ be real numbers and $f(x)$ be a continuous real-valued function on the bounded closed interval $[a, b]$. Let

$$F(x) = \int_a^x f$$

for $a \leq x \leq b$. Then $F'(x) = f(x)$ for $x \in (a, b)$.

Its proof is as follows. Fix $x_0 \in (a, b)$. Given any $\varepsilon > 0$, from the continuity of f at x_0 there exists some $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for $|x - x_0| < \delta$. This implies that

$$(\ddagger) \qquad \sup_{x_0 - \delta \le x \le x_0 + \delta} |f(x) - f(x_0)| \le \varepsilon.$$

Then for $0 < |x - x_0| < \delta$,

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \begin{cases} \frac{1}{x - x_0} \int_{x_0}^{x} (f - f(x_0)) & \text{if } x_0 < x \\ -\frac{1}{x - x_0} \int_{x}^{x_0} (f - f(x_0)) & \text{if } x < x_0 \end{cases}$$

and it follows from $(\ddagger)$ that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \le \varepsilon$$

for $0 < |x - x_0| < \delta$. Thus $F'(x_0) = f(x_0)$.

Second Part of the Fundamental Theorem of Calculus. Let $a < b$ be real numbers and $f(x)$ be a real-valued function on the bounded closed interval $[a, b]$ such that $f'(x)$ at every point of $[a, b]$ and as a function of x the function $f'(x)$ is continuous on $[a, b]$. Then

$$\int_a^b f' = f(b) - f(a).$$

Its proof is as follows. For $x \in [a, b]$ define

$$F(x) = \int_a^x f'.$$

The function $F(x) - (f(x) - f(a))$ is continuous on $[a, b]$ and by the first part of the Fundamental Theorem of Calculus has zero derivative at every point of (a, b) and hence is constant on $[a, b]$ from the Mean-Value Theorem. Since $F(x) - (f(x) - f(a))$ vanishes at $x = a$, it follows that $F(x) - (f(x) - f(a))$ is identically zero on $[a, b]$. The vanishing of its value at $x = b$ yields

$$\int_a^b f' = f(b) - f(a).$$